Palestine Journal of Mathematics

Vol. 6(Special Issue: II) (2017) , 327–332

Some Properties of α-Sasakian manifolds

Amit Sil

Communicated by Ayman Badawi

MSC 2010 Classifications: 53 C 25.

Keywords and phrases: α-Sasakian manifold, CL-transformation, ϕ-Ricci symmetry, locally ϕ-Ricci symmetry, Killing vector field, Conformal Killing vector field.

.

Abstract
The object of the present paper is to study CL-transformations on α- Sasakian manifolds. ϕ-Ricci symmetry of such manifolds has been considered. Conditions for a conformal Killing vector field to be Killing on such manifolds have been deduced

1 Introduction

Contact geometry is an important branch of modern mathematics. It has evolved from the work of Christian Huygen. It has important applications in many branches of physical science like geometric optics, thermodynamics, relativity cosmology and string theory.

α- Sasakian manifolds are a class of contact manifolds. Several authors have studied α-Sasakian manifolds [1][3].

In 1963, Tashiro and Tachibana [8] introduce a transformation called CL-transformation on a Sasakian manifold under which C-loxodrome remains invariant. A loxodrome is a curve on the unit sphere that intersects the meridians at a fixed angle and the C-loxodrome was mainly used in navigation and usually called rhumb lines. A C-loxodrome is a loxodrome cutting geodesic trajectories of the characteristic vector field ξ of the Sasakian manifold with constant angle. In [7] the authors have studied CL-transformation on Kenmotsu manifolds. In this paper we would like to study CL-transformations on α- Sasakian manifolds.

Symmetry of a manifold is an important geometric property. It has been defined by several authors in several ways. In the paper [5], the authors have introduced the notion of ϕ-Ricci symmetry for Sasakian manifolds. In this paper, we would like to study ϕ-Ricci symmetry for α-Sasakian manifolds. Condition for a vector field to be Killing on a Kenmotsu manifold has been studied in the paper [6]. In this paper we are interested to find conditions for a vector field to be Killing in α- Sasakian manifolds. The present paper is organized as follows. After the introduction, we give some preliminaries in Section 2. In Section 3, we have studied α-Sasakian manifolds admitting infinitesimal CL-transformation. In Section 4, we have investigate the condition of ϕ-Ricci symmetry for three dimensional α-Sasakian manifolds. In last Section, we have obtained the condition for a conformal Killing vector field to be Killing in three dimensional α-Sasakian manifolds.

2 Preliminaries

Let M be an almost contact metric manifold of dimension n, equipped with an almost contact metric structure (ϕ, ξ, η, g) consisting of a (1,1) tensor field ϕ, a vector field ξ, a one form η and a Riemannian metric g which satisfy [2]

$$\phi^2 X = -X + \eta(X)\xi, \quad \eta(\xi) = 1, \quad \eta(\phi X) = 0, \quad \phi\xi = 0 \tag{2.1}$$

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \quad \eta(X) = g(X, \xi) \tag{2.2}$$

[0]The author is supported by Rajiv Gandhi National Fellowship, India, Grant No. F1-17.1/2015-16/RGNF-2015-17-SC-WES-11891

for all $X, Y \in \chi(M)$. An almost contact metric manifold $M(\phi, \xi, \eta, g)$ is said to be α-Sasakian manifold if the following conditions hold:

$$(\nabla_X \phi)Y = \alpha(g(X,Y)\xi - \eta(Y)X) \tag{2.3}$$

$$\nabla_X \xi = -\alpha\phi X, \quad (\nabla_X \eta)Y = \alpha g(X, \phi Y) \tag{2.4}$$

holds for some smooth function α on M.

In an α-Sasakian manifold of dimension three, the following relations hold:

$$R(X,Y)\xi = \alpha^2[\eta(Y)X - \eta(X)Y] + (Y\alpha)\phi X - (X\alpha)\phi Y \tag{2.5}$$

$$R(\xi, X)Y = \alpha^2[g(X,Y)\xi - \eta(Y)X] + g(X, \phi Y)(grad\alpha) + (Y\alpha)\phi X \tag{2.6}$$

$$\eta(R(X,Y)Z) = \alpha^2[g(Y,Z)\eta(X) - g(X,Z)\eta(Y)] + (X\alpha)g(X,Z) - (Y\alpha)g(Y,Z) \tag{2.7}$$

$$S(X,\xi) = \alpha^2(n-1)\eta(X) - (\phi X)\alpha \tag{2.8}$$

$$S(\xi,\xi) = \alpha^2(n-1) \tag{2.9}$$

$$QX = (\frac{r}{2} - \alpha^2)X - (\frac{r}{2} - 3\alpha^2)\eta(X)\xi + \eta(X)\phi grad\alpha - (\phi X)\alpha\xi \tag{2.10}$$

for all $X, Y, Z \in \chi(M)$, where R is the Riemannian Curvature tensor, S is the Ricci tensor and Q is the Ricci operator.

From [4] putting $\beta = 0$, we get the curvature tensor of three dimensional α-Sasakian manifolds as follows:

$$\begin{aligned} R(X,Y)Z &= (\frac{r}{2} - 2\alpha^2)(g(Y,Z)X - g(X,Z)Y) - g(Y,Z)[(\frac{r}{2} - 3\alpha^2)\eta(X)\xi \\ &- \eta(X)\phi grad\alpha + (\phi X)\alpha\xi] + g(X,Z)[(\frac{r}{2} - 3\alpha^2)\eta(Y)\xi \\ &- \eta(Y)\phi grad\alpha + (\phi Y)\alpha\xi] - [(\phi Z)\alpha\eta(Y) + (\phi Y)\alpha\eta(Z) \\ &+ (\frac{r}{2} - 3\alpha^2)\eta(Y)\eta(Z)]X \\ &+ [(\phi Z)\alpha\eta(X) + (\phi X)\alpha\eta(Z) + (\frac{r}{2} - 3\alpha^2)\eta(X)\eta(Z)]Y \end{aligned} \tag{2.11}$$

3 Infinitesimal CL-transformation on α- Sasakian manifolds

Definition 3.1: A vector field V on α- Sasakian manifold M is said to be an infinitesimal CL-transformation [9] if it satisfies

$$L_V\{^h_{ji}\} = \rho_j\delta^h_i + \rho_i\delta^h_j + \gamma(\eta_j\phi^h_i + \eta_i\phi^h_j) \tag{3.1}$$

for a certain constant γ, where ρ_i are the components of an 1-form ρ, L_V denotes the Lie derivative with respect to V and $\{^h_{ji}\}$is the christoffel symbol of the Riemannian metric g.

Theorem 3.1: If a non cosymplectic α-Sasakian manifolds with α as constant admits infinitesimal CL-transformation V, then $L_V\phi Z$ is orthogonal to ξ, where Z is any arbitrary vector field on the manifold.

Proof. It is known from [10] that

$$L_V R^h_{kji} = \nabla_K L_V\{^h_{ji}\} - \nabla_j L_V\{^h_{ki}\} \tag{3.2}$$

Substituting (3.1) into (3.2) and then using (2.3) and (2.4), we obtain

$$\begin{aligned}(L_VR)(X,Y)Z &= (\nabla_X\rho)(Z)Y-(\nabla_Y\rho)(Z)X+\gamma[(2\alpha g(\phi Y,X)\phi Z\\ &- \alpha g(\phi X,Z)\phi Y+\alpha g(\phi Y,Z)\phi X)\\ &+ (\alpha(2\eta(X)Y-2\eta(Y)X))\eta(Z)\\ &+ (\alpha(g(X,Z)\eta(Y)-g(Y,Z)\eta(X)))\xi]\end{aligned} \tag{3.3}$$

for any vector fields X, Y and Z on M.
Operating η to (3.3), we get,

$$\begin{aligned}\eta((L_VR)(X,Y)Z) &= (\nabla_X\rho)(Z)\eta(Y)-(\nabla_Y\rho)(Z)\eta(X)\\ &+ \gamma\alpha(g(X,Z)\eta(Y)-g(Y,Z)\eta(X))\end{aligned} \tag{3.4}$$

Taking Lie derivative of (2.7) with respect to V and using (3.4) and then replacing X and Y to Y and ξ respectively we get,

$$\begin{aligned}(L_V\eta)R(Y,\xi)Z &= (2\alpha\alpha'^2L_V\eta(Z)\eta(Y)+\alpha^2L_V\eta(Y)\eta(Z)\\ &- (2\alpha\alpha'+\gamma\alpha-\alpha''^2-\alpha')L_Vg(Y,Z)-\alpha'L_V\eta(Z)\\ &- \alpha''\eta(Z)-(\nabla_Y\rho)(Z)+(\nabla_\xi\rho)(Z)\eta(Y)\end{aligned} \tag{3.5}$$

Taking Lie derivative of (2.11) with respect to V and operating η and then replacing X and Y to Y and ξ we get,

$$\begin{aligned}\eta((L_VR)(Y,\xi)Z) &= (2\alpha\alpha'+\eta(L_V(\phi grad\alpha)))\eta(Y)\eta(Z)\\ &- (\alpha^2\eta(L_V\xi)+2\alpha\alpha'+\eta(L_V(\phi grad\alpha)))g(Y,Z)\\ &- \alpha^2L_Vg(Y,Z)+\alpha^2L_V\eta(Z)\eta(Y)+\alpha^2\eta(Z)\eta(L_VY)\\ &- \alpha'\eta(L_VY)+\alpha'L_V\eta(Y)+\alpha'\eta(Y)\eta(L_V\xi)\end{aligned} \tag{3.6}$$

Again operating η to (2.11) and taking Lie derivative with respect to V and then replacing X and Y to Y and ξ we get,

$$\begin{aligned}L_V\eta(R(Y,\xi)Z) &= 2\alpha\alpha'^2L_V\eta(Y)\eta(Z)+\alpha^2\eta(Y)L_V\eta(Z)\\ &- 2\alpha\alpha'^2L_Vg(Y,Z)\end{aligned} \tag{3.7}$$

Now subtracting (3.6) from (3.7) and then using (3.5) we get,

$$\begin{aligned}&(2\alpha\alpha'^2L_V\eta(Z)\eta(Y)-(2\alpha\alpha'+\gamma\alpha-\alpha'')g(Y,Z)\\ &\quad- (\alpha^2-\alpha')L_Vg(Y,Z)-\alpha'L_V\eta(Z)-\alpha''\eta(Z)\\ &\quad- (\nabla_Y\rho)(Z)+(\nabla_\xi\rho)(Z)\eta(Y)\\ &\quad= (\alpha^2\eta(L_V\xi)\\ &\quad+ \eta(L_V(\phi grad\alpha)))g(Y,Z)\\ &\quad- \eta(L_V(\phi grad\alpha))\eta(Y)\eta(Z)\\ &\quad- \alpha^2\eta(Z)\eta(L_VY)\\ &\quad+ \alpha'\eta(L_VY)-\alpha'L_V\eta(Y)\\ &\quad- \alpha'\eta(Y)\eta(L_V\xi)\end{aligned} \tag{3.8}$$

Interchanging Y and Z in (3.8) and subtracting it from (3.8) and then replacing Y to ξ we get,

$$\alpha^2(L_V\eta)(Z)+(\nabla_Z\rho)\xi-(\nabla_\xi\rho)(\xi)\eta(Z)+\alpha^2\eta(Z)\eta(L_V\xi)=0 \tag{3.9}$$

Replacing Z by ϕZ in (3.9), we get,

$$\alpha^2(L_V\eta(\phi Z) - \eta(L_V\phi Z)) + \nabla_{\phi Z}\rho(\xi) - \rho(\nabla_{\phi Z}\xi) = 0 \tag{3.10}$$

Now choosing $\rho = \eta$ in (3.10) and using (2.4) we get,
$\alpha^2(-\eta(L_V\phi Z) - \eta(-\alpha\phi^2 Z) = 0$
i.e. $\alpha^2\eta(L_V\phi Z) = 0$
If $\alpha \neq 0$,then $\eta(L_V\phi Z) = 0$
i.e., we can say $L_V\phi Z$ is perpendicular to ξ.

4 ϕ-Ricci symmetric α-Sasakian manifolds of dimension three

Definition 4.1: A three dimensional α-Sasakian manifold will be called ϕ- Ricci symmetric if $\phi^2(\nabla_W Q)X = 0$, for any vector field W, X on the manifolds. Here Q is Ricci operator. The notion of ϕ-Ricci symmetry was introduced in the paper [5].
Theorem 4.1: A three-dimensional α-Sasakian manifolds with α as constant is locally ϕ-Ricci symmetric if and only if the scalar curvature r is constant.
Proof. Using (2.10), we get,

$$\begin{aligned}(\nabla_W Q)X &= (\frac{dr(W)}{2} - 2\alpha d\alpha(W))X - (\frac{dr(W)}{2} - 6\alpha d\alpha(W))\eta(X)\xi \\ &- (\frac{r}{2} - 3\alpha^2)(\nabla_W\eta)(X)\xi - (\frac{r}{2} - 3\alpha^2)\eta(X)\nabla_W\xi + (\nabla_W\eta)(X)\phi grad\alpha \\ &+ \eta(X)\nabla_W(\phi grad\alpha) - \nabla_W((\phi X)\alpha)\xi \\ &- ((\phi X)\alpha)\nabla_W\xi + (\phi\nabla_W X)\alpha\xi \end{aligned} \tag{4.1}$$

Applying ϕ^2 on both sides of (4.1), we get,

$$\begin{aligned}\phi^2((\nabla_W Q)(X)) &= (\frac{dr(W)}{2} - 2\alpha d\alpha(W))\phi^2 X - (\frac{r}{2} - 3\alpha^2)\eta(X)\phi^2(\nabla_W\xi) \\ &+ (\nabla_W\eta)(X)\phi^2(\phi grad\alpha) + \eta(X)\phi^2(\nabla_W(\phi grad\alpha)) \\ &- ((\phi X)\alpha)\phi^2(\nabla_W\xi)\end{aligned} \tag{4.2}$$

Now from (2.1), we get,

$$\begin{aligned}\phi^2((\nabla_W Q)(X)) &= (\frac{dr(W)}{2} - 2\alpha d\alpha(W))(-X + \eta(X)\xi) \\ &- (\frac{r}{2} - 3\alpha^2)\eta(X)\phi^2(\nabla_W\xi) - ((\phi X)\alpha)\phi^2(\nabla_W\xi) \\ &- (\nabla_W\eta)(X)\phi grad\alpha + \eta(X)\phi^2(\nabla_W(\phi grad\alpha))\end{aligned} \tag{4.3}$$

Choosing the vector field orthogonal to ξ, we get,
$\phi^2((\nabla_W Q)X) = -(\frac{dr(W)}{2} - 2\alpha d\alpha(W))X - ((\phi X)\alpha)\phi^2(\nabla_W\xi) - (\nabla_W\eta)(X)\phi grad\alpha$
This completes the proof of the theorem.

5 Condition for a conformal Killing vector field to be Killing

Definition 5.1: A vector field on an α-Sasakian manifolds is said to be conformal Killing if $L_X g = fg$, where f is a function on the manifold. The vector field X is said to be Killing if $L_X g = 0$.
Theorem 5.1: If a conformal Killing vector field X on a α-Sasakian manifold is orthogonal to ξ, then X is Killing.
Proof. Let the vector field X be conformal Killing vector field on an α-Sasakian manifolds.

So,

$$(L_X g)(Y,Z) = fg(Y,Z) \tag{5.1}$$

From (2.4), we get $\nabla_\xi \xi = 0$. So the integral curves are geodesics and we have from (5.1), by putting $Y = Z = \xi$

$f = (L_X g)(\xi, \xi)$

Now,

$$\begin{aligned}(L_X g)(\xi,\xi) &= L_X g(\xi,\xi) - g(L_X\xi,\xi) - g(\xi, L_X\xi) \\ &= -2g(L_X\xi,\xi) \\ &= -2g(\nabla_X\xi - \nabla_\xi X, \xi) \\ &= -2g(-\alpha\phi X,\xi) + 2g(\nabla_\xi X,\xi) \\ &= 2\alpha g(X,\phi\xi) + 2g(\nabla_\xi X,\xi) \\ &= 2g(\nabla_\xi X,\xi) \qquad (since, \phi\xi = 0, so \quad g(X,0) = 0)\end{aligned}$$

Again, $2\nabla_\xi g(X,\xi) = 2g(\nabla_\xi X,\xi)$

$$f = (L_X g)(\xi,\xi) = 2g(\nabla_\xi X,\xi) = 2\nabla_\xi(g(X,\xi)) \tag{5.2}$$

Now if X is orthogonal to ξ, f=0 and hence $L_X g = 0$ i.e., X is a Killing vector field .

Hence the theorem is proved.

Theorem 5.2: If V is a vector field on a α-Sasakian manifolds provided α as constant satisfying $L_V R = 0$, then V is Killing.

Proof. Let V be a vector field on the α-Sasakian manifold such that $L_V R = 0$.

Now, in a Riemannian manifold we have,

$g(R(W,X)Y,Z) + g(R(W,X)Z,Y) = 0$

Taking Lie derivative of the above identity along V, we get,

$$(L_V g)(R(W,X)Y,Z) + (L_V g)(R(W,X)Z,Y) = 0 \tag{5.3}$$

Putting $W = Y = Z = \xi$ in (5.3) we get,

$$(L_V g(R(\xi,X)\xi,\xi)) + (L_V g)(R(\xi,X)\xi,\xi) = 0 \tag{5.4}$$

Now putting $Y = \xi$ in (2.6) and using it in (5.4), we get,

$$\alpha^2(L_V g)(X,\xi) = \eta(X)\alpha^2(L_V g)(\xi,\xi) \tag{5.5}$$

Again putting $W = Y = \xi$ in (5.3) and using (2.6) we get,

$$\begin{aligned} g(X,Z)\alpha^2(L_V g)(\xi,\xi) - \alpha^2(L_V g)(X,Z) &+ (L_V g)((\xi\alpha)\phi X, Z) \\ &+ g(X,\phi Z)(L_V g)(grad\alpha,\xi) \\ &= 0 \end{aligned} \tag{5.6}$$

If α is constant, then from (5.6) we get,

$$(L_V g) = (L_V g)(\xi,\xi)g \tag{5.7}$$

From (2.9), we have, $S(\xi,\xi) = \alpha^2(n-1)$

We know that $L_V R = 0$ implies $L_V S = 0$

Now applying L_V on $S(\xi,\xi)$, we get,

$$S(L_V\xi,\xi) = 0 \tag{5.8}$$

Again from (2.8), we have,

$$S(X,\xi) = \alpha^2(n-1)\eta(X) - (\phi X)\alpha \tag{5.9}$$

From (5.8) and (5.9), we have, $L_V\xi = 0$.
Hence $g(L_V\xi, \xi) = 0$.
Thus $(L_V g)(\xi, \xi) = 0$.
So, from (5.2) we get $f = 0$.
Thus the vector field V is Killing.
This completes the proof of the theorem.

References

[1] C. S. Bagewadi and G. Inagalahalli, Ricci solitons on Lorentzian Sasakian manifolds, Acta Mathematica, Academiae Paedagogicae Nyiregyhaziensis, 28(2012), 59-68.

[2] D. E. Blair, Riemannain geometry of contact and symplectic manifolds, Lecture notes on mathematics, Springer (1976)

[3] L. Das, "Second order parallel tensor on α-Sasakian manifold", Acta Mathematica, Academiae Paedagogicae Nyiregyhaziensis, 23(2007), 65-69.

[4] U.C. De and M.M. Tripathi, Ricci tensor in three dimensional trans-Sasakian manifolds, Kyungpook Math. J.,43(2003), 247-255.

[5] U.C. De and A. Sarkar, On ϕ-Ricci symmetric Sasakian manifolds, Proc. Jangjeon Math. Soc. 11(2008), 47-52.

[6] A. De, Some theorems on Kenmotsu manifolds, Bull. Math.Analysis and Applications, 3(2010), 1-6.

[7] A.A. Shaikh, F.R. Al-Solamy and H. Ahmad, Some transformations on Kenmotsu manifolds, SUT J.Math., 49(2013), 109-119.

[8] Y. Tashiro and S. Tachibana, On Fubinian and C-Fubinian manifold, Kodai Math Sem.Rep., 15(1963), 176-183.

[9] K. Takamatsu and H. Mizusawa, On infinitesimal CL-transformations of compact normal contact metric spaces, Sci.Rep.Neigata Univ., Ser. A, 3(1966), 31-40.

[10] K. Yano, Differential geometry on complex and almost complex spaces, Pergamon Press, 1965.

Author information

Amit Sil, Department of Mathematics, University of Kalyani, Kalyani- 741235, West-Bengal, India.
E-mail: amitsil666@gmail.com

Received: January 15, 2016.

Accepted: September 15, 2016.