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Difference cordial labeling of some special graphs

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Abstract Let G be a (p, q) graph. Let f be a map from $V(G)$ to $\{1, 2, \ldots, p\}$. For each edge xy, assign the label $|f(x) - f(y)|$. f is called a difference cordial labeling if f is a one to one map and $|e_f(0) - e_f(1)| \leq 1$ where $e_f(1)$ and $e_f(0)$ denote the number of edges labeled with 1 and not labeled with 1 respectively. A graph with a difference cordial labeling is called a difference cordial graph. In this paper, we investigate the difference cordial labeling behavior of Dragon, $C_n^{(2)}$, windmill graph $K_n^{(m)}$, caterpillar and some standard graphs.

1 Introduction

Graphs considered here are simple and undirected. Throughout this paper p and q respectively denote the order and size of the graph G. The union of two graphs G_1 and G_2 is the graph $G_1 \cup G_2$ with $V(G_1 \cup G_2) = V(G_1) \cup V(G_2)$ and $E(G_1 \cup G_2) = E(G_1) \cup E(G_2)$. A graph labeling is an assignment of integers to the vertices or edges, or both, subject to certain conditions. Labeled graphs serve as useful models for a broad range of applications such as: astronomy, circuit design, communication network addressing and models for constraint programming over finite domains [1]. In [3], Ponraj et al. introduced a new labeling called difference cordial labeling in [3]. In [3, 4, 5, 6, 7], difference cordial labeling behavior of several graphs like path, cycle, complete graph, complete bipartite graph, bistar, wheel, web and some more standard graphs have been investigated. Seoud and Salman [8], studied the difference cordial labeling behavior of some families of graphs and they are ladder, triangular ladder, grid, step ladder and two sided step ladder graphs etc. In this paper we examine the difference cordial labeling behavior of dragon, $C_n^{(2)}$, windmill graph $K_n^{(m)}$, one point union of even paths, jelly fish graphs, caterpillar, lobsters. Let x be any real number. Then $\lfloor x \rfloor$ stands for the largest integer less than or equal to x and $\lceil x \rceil$ stands for the smallest integer greater than or equal to x. Terms not defined here are used in the sense of Harary [2].

2 Difference cordial labeling

Definition 2.1. Let G be a (p, q) graph. Let $f : V(G) \to \{1, 2, \ldots, p\}$ be a bijection. For each edge uv, assign the label $|f(u) - f(v)|$. f is called a difference cordial labeling if f is $1-1$ and $|e_f(0) - e_f(1)| \leq 1$ where $e_f(1)$ and $e_f(0)$ denote the number of edges labeled with 1 and not labeled with 1 respectively. A graph with a difference cordial labeling is called a difference cordial graph.

Theorem 2.2. *Let H be any (p, q) difference cordial graph. Let G be a graph obtained from H by attaching C_4 at every vertex of H. Then G is difference cordial.*

Proof. Let $V(H) = \{u_i : 1 \leq i \leq p\}$ and let $V(G) = V(H) \cup \{v_1^i, v_2^i, v_3^i : 1 \leq i \leq p\}$ and $E(G) = E(H) \cup \{u_i v_1^i, v_1^i v_2^i, v_2^i v_3^i, v_3^i u_i : 1 \leq i \leq p\}$. Clearly, the order and size of G are $4p$ and $4p + q$ respectively. Let f be a difference cordial labeling of H.

Case 1. $f(u_1) \neq p$. Define, $g : V(G) \to \{1, 2, 3, \ldots, 4p\}$ as follows:

$$\begin{aligned} g(u_i) &= f(u_i) && 1 \le i \le p \\ g(v_1^i) &= p + 3i - 2 && 1 \le i \le p \\ g(v_2^i) &= p + 3i - 1 && 1 \le i \le p \\ g(v_3^i) &= p + 3i && 1 \le i \le p. \end{aligned}$$

Case 2. $f(u_1) = p$.
Assign the labels to the vertices u_i $(1 \le i \le p)$ and v_1^j, v_2^j, v_3^j $(2 \le i \le p)$ as in case 1. Then assign the labels $p + 1$, $p + 3$ and $p + 2$ to the vertices v_1^1, v_2^1 and v_3^1 respectively. In both cases, each C_4^i $(1 \le i \le p)$ contribute two edges with label 0 and two edges with label 1. This implies g is difference cordial. □

Theorem 2.3. *[3] Any path is difference cordial.*

We now investigate the graph dragon. Dragon $C_m@P_n$ is obtained from the cycle C_m and the path P_n by identifying the end vertex of the path to any vertex of the cycle C_m.

Theorem 2.4. *Dragons are difference cordial.*

Proof. Let C_m be the cycle $u_1u_2 \ldots u_mu_1$ and P_n be the path $v_1v_2 \ldots v_n$. Let the dragon $C_m@P_n$ be obtained from C_m and P_n by identifying u_1 and v_1. Clearly, $C_m@P_n - \{u_1u_2\} \cong P_{m+n-1}$. By Theorem 2.3, P_{m+n-1} is difference cordial. Let f be the corresponding difference cordial labeling. With this labeling

$$e_f(0) = \begin{cases} \frac{m+n-3}{2} & m+n-1 \text{ is even} \\ \frac{m+n-2}{2} & m+n-1 \text{ is odd} \end{cases} \quad e_f(1) = \begin{cases} \frac{m+n-1}{2} & m+n-1 \text{ is even} \\ \frac{m+n-2}{2} & m+n-1 \text{ is odd} \end{cases}$$

Since the label of the edge u_1u_2 is 0, $P_{m+n-1} + \{u_1u_2\}$ is also difference cordial. □

The graph $C_n^{(t)}$ denote the one-point union of t cycles of length n.

Theorem 2.5. *$C_n^{(2)}$ is difference cordial.*

Proof. Let $u_1u_2 \ldots u_nu_1$ and $v_1v_2 \ldots v_nv_1$ be the first and second copies of C_n. Let $u_{\lceil \frac{n+1}{2} \rceil} = v_1$. Define a map $f : V(C_n^{(2)}) \to \{1, 2, \ldots, 2n - 1\}$ as follows:

$$\begin{aligned} f(u_i) &= i && 1 \le i \le \left\lceil \tfrac{n+1}{2} \right\rceil \\ f(u_{\lceil \frac{n+1}{2} \rceil + i}) &= n + 2i && 1 \le i \le \left\lfloor \tfrac{n-1}{2} \right\rfloor \\ f(v_i) &= \left\lceil \tfrac{n+1}{2} \right\rceil + i - 1 && 2 \le i \le \left\lceil \tfrac{n+2}{2} \right\rceil \\ f(v_{\lceil \frac{n+2}{2} \rceil + i}) &= n + 1 + 2i && 1 \le i \le \left\lfloor \tfrac{n-1}{2} \right\rfloor \end{aligned}$$

Since $e_f(0) = e_f(1) = n$, f is a difference cordial labeling of $C_n^{(2)}$. □

Theorem 2.6. *[3] K_n is difference cordial iff $n \le 4$.*

Theorem 2.7. *[3] If G is a (p, q) difference cordial graph, then $q \le 2p - 1$.*

The windmill graph $K_n^{(m)}$ $(n > 3)$ consists of m copies of K_n with a vertex in common.

Theorem 2.8. *The windmill graph $K_n^{(m)}$ $(n > 3)$ is difference cordial iff $n = 4$ and $m \le 2$.*

Proof. Since $K_n^{(1)} \cong K_n$, by theorem 2.6, $K_n^{(m)}$ $(n > 3)$ is difference cordial iff $n = 4$. Now $K_n^{(m)}$ consists of $mn - m + 1$ vertices and $\frac{mn(n-1)}{2}$ edges. Suppose $K_n^{(m)}$ is difference cordial. Then by theorem 2.7, $\frac{mn(n-1)}{2} \le 2(mn - m + 1) - 1. \Rightarrow 2 \ge m(n-1)(n-4) \ge 2(n-1)(n-4)$. This is true only when $n = 4$. For $n = 4$, clearly, $e_f(1) \le 2m + 2$ and $e_f(0) \ge q - e_f(1) \ge 4m - 2$. Hence $e_f(0) - e_f(1) \ge 2m - 4$. This gives $m = 2$. The difference cordial labeling of $K_4^{(2)}$ is given in Figure 2.

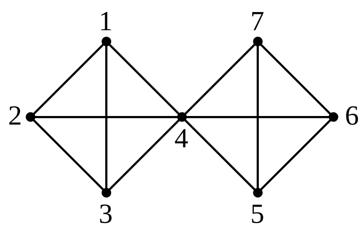

Figure 2

□

Theorem 2.9. *One point union of even paths is difference cordial.*

Proof. Let the identified vertex be u. Consider any path of length k. Take any one neighbor of u. Let it be u_1. Label u_1 by 1. Then assign the label 2 to the neighbor u_2 of u_1. Now consider the neighbor u_3 of u_2. Label u_3 by label of u_1 plus two, that is 4. Then assign the label "label of $u_3 - 1$" to u_4. Proceed like this until we reach the pendent vertex. Then we move to the next path. Let v_i be the neighbor of v_{i-1}. Label v_1, v_2 by $k+1$ and $k+2$ respectively. Then label v_3, v_4 by $(k+2)+2$ and $(k+2)+1$ respectively and so on. Then we move to the next path. Continuing in this way, finally we label the vertex u by p. Clearly the above vertex labeling is a difference cordial labeling.

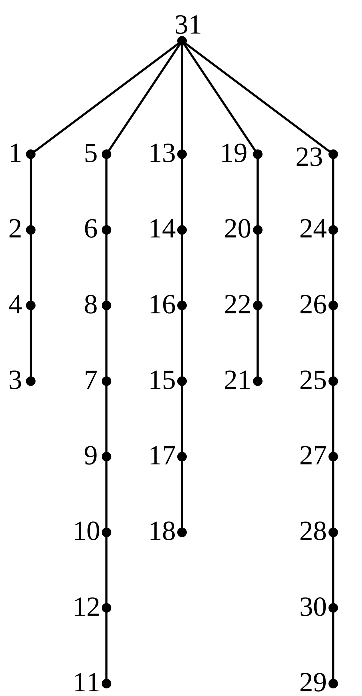

Figure 3

□

A caterpillar is a tree with the property that the removal of its pendant vertices results in a path. A caterpillar T is a tree with a path $P_n : u_1u_2 \dots u_n$ called spine with leaves (pendent vertices) known as feet attached to the vertices of the spine by edges known as legs. It is noted that every spine vertex u_i is attached to x_i (possibly zero) number of leaves b_{ij} $(1 \le j \le x_i,\ 1 \le i \le n)$. The caterpillar T is denoted as $S(x_1, x_2, \dots, x_n)$.

Theorem 2.10. *The caterpillar $S(x_1, x_2, \dots, x_n)$ with $x_1+x_2+\dots+x_n > 3n+2$ is not difference cordial.*

Proof. Suppose $S(x_1, x_2, \ldots, x_n)$ is difference cordial. Clearly, every spine vertex u_i contributes at most two edges with label 1. It follows that $e_f(1) \leq 2n$. Now $e_f(0) \geq q - 2n$. Then $e_f(0) - e_f(1) \geq q - 4n \geq x_1 + x_2 + \ldots + x_n + n - 1 - 4n > 1$, a contradiction. □

The lobster $LS(m, n)$ is obtained from a path P_n and a collection of stars $K_{1,m}$ where each vertex of P_n is joined to the central vertex of exactly one star.

Theorem 2.11. *If $m \leq 4$, then the lobster $LS(m, n)$ is difference cordial.*

Proof. Let P_n be the path $u_1u_2 \ldots u_n$ and v_i $(1 \leq i \leq n)$ be the central vertex of $K_{1,m}$.
Case 1. $m = 1$. Let w_i $(1 \leq i \leq n)$ be the pendent vertices. Label the vertices of P_n as in theorem 2.3 and define

$$\begin{aligned} f(v_i) &= n + 2i - 1 & 1 \leq i \leq n \\ f(w_i) &= n + 2i & 1 \leq i \leq n. \end{aligned}$$

Clearly, the above vertex labeling is a difference cordial labeling.
Case 2. $m = 2$.
Let x_i, y_i $(1 \leq i \leq n)$ be the pendent vertices. Define an injecctive map $f : V(LS(2, n)) \to \{1, 2, \ldots, 4n\}$ by

$$\begin{aligned} f(u_i) &= 4i & 1 \leq i \leq n \\ f(v_i) &= 4i - 2 & 1 \leq i \leq n \\ f(x_i) &= 4i - 3 & 1 \leq i \leq n \\ f(y_i) &= 4i - 1 & 1 \leq i \leq n. \end{aligned}$$

Since $e_f(1) = 2n$ and $e_f(0) = 2n - 1$, f is a difference cordial labeling.
Case 3. $m = 3$.
Let x_i, y_i, z_i $(1 \leq i \leq n)$ be the pendent vertices. Label the vertices of the path as in theorem 2.3 and define

$$\begin{aligned} f(v_i) &= n + 3i - 1 & 1 \leq i \leq n \\ f(x_i) &= n + 3i - 2 & 1 \leq i \leq n \\ f(y_i) &= n + 3i & 1 \leq i \leq n \\ f(z_i) &= 4n + i & 1 \leq i \leq n. \end{aligned}$$

Clearly the above vertex labeling is a difference cordial labeling.
Case 4. $m = 4$.
Let w_i, x_i, y_i, z_i $(1 \leq i \leq n)$ be the pendent vertices. Define $f : V(LS(4, n)) \to \{1, 2, \ldots, 6n\}$ by

$$\begin{aligned} f(u_i) &= i & 1 \leq i \leq n \\ f(v_i) &= n + 3i - 1 & 1 \leq i \leq n \\ f(w_i) &= n + 3i - 2 & 1 \leq i \leq n \\ f(x_i) &= n + 3i & 1 \leq i \leq n \\ f(y_i) &= 4n + i & 1 \leq i \leq n \\ f(z_i) &= 5n + i & 1 \leq i \leq n. \end{aligned}$$

Since $e_f(0) = 3n$ and $e_f(1) = 3n - 1$, f is a difference cordial labeling. □

Theorem 2.12. *Let $G_{m,n}$ be a graph obtained from the wheel W_m and a path P_n by replacing each edge of the path by a rim edge of the wheel.*

Proof. Let $V(G_{m,n}) = \{v_i^j : 1 \leq i \leq m,\ 1 \leq j \leq n\} \cup \{u_j : 1 \leq j \leq n\}$ and $E(G_{m,n}) = \{u_jv_i^j, v_i^jv_{(i+1) \pmod m}^j : 1 \leq i \leq m,\ 1 \leq j \leq n\}$. Let $v_m^j = v_1^{j+1}$ $(1 \leq i \leq n - 1)$. Define a one-one map $f : V(G_{m,n}) \to \{1, 2, \ldots, mn - m + 1\}$ by $f(v_1^1) = 2$, $f(u_1) = 1$,

$$\begin{aligned} f(v_i^j) &= n(j - 1) + 1 + i & 1 \leq j \leq n,\ 2 \leq i \leq m \\ f(u_j) &= m(j - 1) + 2 & 2 \leq j \leq n. \end{aligned}$$

Since $e_f(1) = e_f(0) = m(n-1)$, f is a difference cordial labeling of $G_{m,n}$. □

Theorem 2.13. *Let G_1 and G_2 be the (p_1, q_1) and (p_2, q_2) difference cordial graphs respectively. Then $G_1 \cup G_2$ is difference cordial if q_1 and q_2 are not odd simultaneously.*

Proof. Let f and g respectively be the difference cordial labeling of G_1 and G_2. Let $V(G_1) = \{u_i : 1 \le i \le p_1\}$ and $V(G_2) = \{v_i : 1 \le i \le p_2\}$.
Case 1. Both q_1 and q_2 are even.
In this case $e_f(0) = e_f(1)$ and $e_g(0) = e_g(1)$. Define an injective map $f : V(G_1 \cup G_2) \to \{1, 2, \ldots, p_1 + p_2\}$ by

$$\begin{aligned} h(u_i) &= f(u_i) & 1 \le i \le p_1 \\ h(v_i) &= p_1 + g(v_i) & 1 \le i \le p_2. \end{aligned}$$

Hence $e_h(0) = e_f(0) + e_g(0)$ and $e_h(1) = e_f(1) + e_g(1)$. Therefore $e_h(0) = e_h(1)$. Hence h is a difference cordial labeling of $G_1 \cup G_2$.
Case 2. q_1 is even and q_2 is odd.
In this case $e_f(0) = e_f(1)$ and $e_g(0) = e_g(1)+1$ or $e_g(1) = e_g(0)+1$. Let h be a vertex labeling as in case 1. Then $|e_h(0) - e_h(1)| = 1$. Hence h is a difference cordial labeling of $G_1 \cup G_2$.
Case 3. q_1 is odd and q_2 is even.
Similar to case 2. □

Jelly fish graphs $J(m, n)$ obtained from a cycle $C_4 : v_1v_2v_3v_4v_1$ by joining v_1 and v_3 with an edge and appending m pendent edges to v_2 and n pendent edges to v_4.

Theorem 2.14. *The Jelly fish graphs $J(m, n)$ are difference cordial iff $m + n \le 6$.*

Proof. Suppose $m+n > 6$ and f is a difference cordial labeling of $J(m, n)$. Obviously $e_f(1) \le 5$. Then $e_f(0) \ge q - 5$. This implies $e_f(0) - e_f(1) > 1$, a contradiction. Suppose $m + n \le 6$. The difference cordial labeling of $J(m, n)$ is shown in figure 4.

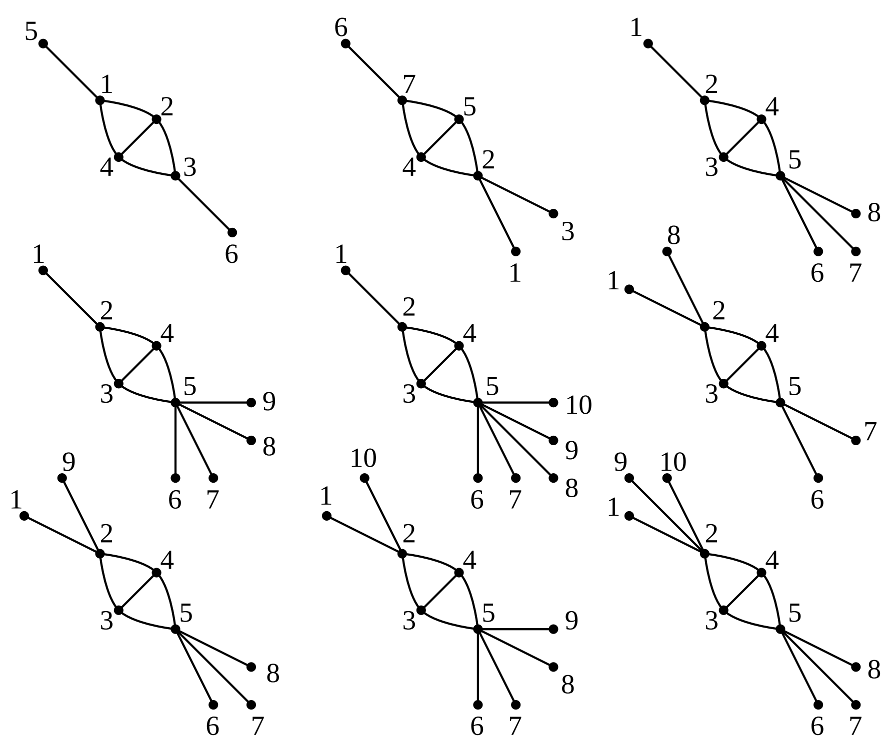

Figure 4

□

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