Palestine Journal of Mathematics

Vol. 7(Special Issue: I, 2018) , 137–142

Generalized derivations acting as homomorphisms or anti-homomorphisms on Lie ideals

Gurninder S. Sandhu and Deepak Kumar

Communicated by Ayman Badawi

MSC 2010 Classifications: Primary 16W25, 16N60; Secondary 13N15, 15A27.

Keywords and phrases: Prime ring, Semiprime ring, Lie ideal, Generalized derivation.

Abstract Let R be a 2-torsion free semiprime ring with center $Z(R)$ and L be a non-zero square closed Lie ideal of R. A mapping $F : R \to R$ is said to be a generalized derivation of R if for all $u, v \in R$, $F(u+v) = F(u) + F(v)$ and $F(uv) = F(u)v + ud(v)$, where d is a derivation of R. In this note, we prove that if F acts as a homomorphism or as an anti-homomorphism on L, then d maps R into $Z(R)$. Also, we study the prime ring case in more general settings and consequently extend a theorem of Rehman [18].

1 Introduction

All through this paper, R denotes an associative ring with char$(R) \neq 2$ and center $Z(R)$. Recall that a ring R in which 0 is a prime ideal is called a prime ring and if R has no non-zero nilpotent ideal then it is called a semiprime ring. For any $x, y \in R$, we denote the commutator $xy - yx$ by $[x, y]$. By a Lie ideal of R, we mean an additive subgroup L of R such that $[L, R] \subseteq L$. Evidently, every ideal of R is a Lie ideal but converse is not true. A Lie ideal L is said to be square closed if $u^2 \in L$ for all $u \in L$. An additive mapping $d : R \to R$ is called a derivation of R if $d(xy) = d(x)y + xd(y)$ for all $x, y \in R$. For a fixed $a \in R$, the function $\phi_a : x \mapsto [a, x]$ is called an inner derivation associated with a, which is a well-known example of a derivation. For some $a, b \in R$, $\psi : x \mapsto ax + xb$ is said to be a generalized inner derivation of R. Now we see that $\psi(xy) = \psi(x)y + x\phi_b(y)$, where ϕ_b is the inner derivation of R associated with b. Brešar [7] observed these computations and thereafter introduced the notion of the generalized derivation. Let $F : R \to R$ be an additive mapping such that $F(xy) = F(x)y + xd(y)$ for all $x, y \in R$. Then F is called a generalized derivation R associated with a derivation d. In [14], Hvala developed a remarkable algebraic theory of generalized derivations.

Next, we consider a generalized derivation $F : R \to R$ such that $F(xy) = F(x)F(y)$ or $F(xy) = F(y)F(x)$ for all $x, y \in R$. Then F is said to be a generalized derivation acts as a homomorphism or as an anti-homomorphism on R. Bell and Kappe [6] studied these type of derivations very first time on prime rings. Precisely, they proved the following theorem:

Let R be a prime ring and U a nonzero right ideal of R. If d is a derivation of R, which acts as a homomorphism or as an anti-homomorphism on U, then $d = 0$.

Many authors extended this result in several ways, for up-to-date discussions we refer the reader to [1], [2], [3], [4], [5], [9], [17], [18], [21] and references therein. In this note, we shall prove the following theorems:

Theorem 1.1. *Let R be a 2-torsion free semiprime ring, L a nonzero square-closed Lie ideal of R. Suppose that R admits a generalized derivation (F, d).*

(i) If F acts as a homomorphism on L, then $d(R) \subseteq Z(R)$.

(ii) If F acts as an anti-homomorphism on L, then $d(R) \subseteq Z(R)$.

Theorem 1.2. *Let R be a 2-torsion free prime ring, L a nonzero square-closed Lie ideal of R and $m, n \geq 1$ are fixed integers. Suppose R admits a generalized derivation (F, d).*

(i) If $F(x^m y^n) = F(x^m)F(y^n)$ for all $x, y \in L$, then $d = 0$ or $L \subseteq Z(R)$.

(ii) If $F(x^m y^n) = F(y^n)F(x^m)$ for all $x, y \in L$, then $d = 0$ or $L \subseteq Z(R)$.

2 Preliminaries Results

The the commutator identities: $[x, yz] = y[x, z] + [x, y]z, [xy, z] = x[y, z] + [x, z]y$ and the following facts are useful in the main section:

Lemma 2.1. [[13], COROLLARY 2.1] *Let R be a 2-torsion free semiprime ring, L a Lie ideal of R such that $L \nsubseteq Z(R)$ and let $a, b \in L$. (i) If $aLa = (0)$, then $a = 0$. (ii) If $aL = (0)$ (or $La = (0)$), then $a = 0$. (iii) If L is square-closed and $aLb = (0)$, then $ab = 0$ and $ba = 0$.*

Lemma 2.2. [[20], LEMMA 2.5] *Let R be a 2-torsion free semiprime ring, L a Lie ideal of R such that $L \nsubseteq Z(R)$. If L is square-closed then there exist a nonzero ideal $M = R[L, L]R$ of R such that $2M \subseteq L$.*

Lemma 2.3. [[16], REMARK 2.1] *Let R be a ring, L a square-closed Lie ideal of R. Then $2R[L, L] \subseteq L$ and $2[L, L]R \subseteq L$.*

Lemma 2.4. *Let R be a 2-torsion free semiprime ring and L be a nonzero Lie ideal of R. Then $C_R(L) = Z(R)$.*

Proof. Clearly, $Z(R) \subseteq C_R(L)$. It is easy to see that $C_R(L)$ is both a Lie ideal and a subring of R. Since $C_R(L)$ can not contain a nonzero ideal of R, in light of Herstein [[12], Lemma 1.3] $C_R(L) \subseteq Z(R)$. Hence, $C_R(L) = Z(R)$. □

Lemma 2.5. [[19], THEOREM 3.1] *Let d is a derivation of a 2-torsion free semiprime ring R and L be a square-closed Lie ideals of R. If d is centralizing on L, then d maps R into $Z(R)$.*

3 Main Results

The following propositions can be considered as independent results in themselves.

Proposition 3.1. *Let R be a 2-torsion free semiprime ring and L be a non-zero square-closed Lie ideal of R. If R admits a generalized derivation (F, d) which is centralizing on L, then $d(R) \subseteq Z(R)$.*

Proof. By hypothesis, we have $[u, F(u)] \in Z(R)$ for all $u \in L$. Linearizing this relation w.r.t.u, we get $[u, F(v)] + [v, F(u)] \in Z(R)$ where $u, v \in L$. For some $r \in R$, we substitute $[v, r]$ for u and get $[[v, r], F(v)] + [v, [F(v), r]] + [v, [v, d(r)]] \in Z(R)$. That is,

$$[v, [F(v), r]] + [F(v), [r, v]] + [v, [v, d(r)]] \in Z(R). \tag{3.1}$$

By Jacobi's identity we must have

$$[v, [F(v), r]] + [F(v), [r, v]] + [r, [v, F(v)]] = 0. \tag{3.2}$$

Combining Eq. (3.1) and (3.2) and using our hypothesis, we get $[[d(r), v], v] \in Z(R)$ for each $v \in L$ and $r \in R$. It can be written as $[\phi_{d(r)}(v), v] \in Z(R)$, where $\phi_{d(r)} : R \to R$ stands for the inner derivation of R associated with element $d(r)$. In view of Lemma 2.5, we find that $\phi_{d(r)}(R) \subseteq Z(R)$ i.e.; $[d(r), s] \subseteq Z(R)$ for all $r, s \in R$. By simple substitutions, we obtain $d(R) \subseteq Z(R)$, as desired. □

Proposition 3.2. *Let R be a 2-torsion free semiprime ring and L a non-zero Lie ideal of R. If R admits a derivation d such that $d(L) = (0)$, then $d(R) \subseteq Z(R)$.*

Proof. By assumption, $d(u) = 0$ for all $u \in L$. Replacing u by $[u, r]$, where $r \in R$, we get $d([u, r]) = [u, d(r)] = 0$. Replacing r by rs in the last expression, we get $d(r)[u, s] + [u, r]d(s) = 0$. In particular, we get

$$d(R)[L, L] = (0) \tag{3.3}$$

That means, $d(r)[u, v] = 0$ for all $u, v \in L$ and $r \in R$. Replacing r by ru, we obtain $d(r)u[u, v] = 0$. By Filippis et al. [[10], Corollary 1.4], we get $d(R)[L, R] = (0)$ and $[d(R), L] = (0)$. In view of Lemma 2.4, we get the conclusion. □

3.1 Proof of Theorem 1.1

(i) If possible, let us assume that $L \not\subseteq Z(R)$. By hypothesis, we have $F(xy) = F(x)F(y)$ for any $x, y \in L$. Substitute $2yz$ for y, where $z \in L$, we find

$$F(xyz) = F(x)F(yz) = F(x)F(y)z + F(x)yd(z) \tag{3.4}$$

On the other hand, we have

$$F(xyz) = F(xy)z + xyd(z) \tag{3.5}$$

Comparing (3.4), (3.5) and using our hypothesis, we get

$$(F(x) - x)yd(z) = 0 \tag{3.6}$$

Replace x by $2xz$ in (3.6), we get $(F(xz) - xz)yd(z) = 0$ where $x, y, z \in L$. On expanding the relation, we get

$$xd(z)yd(z) + (F(x) - x)zyd(z) = 0 \tag{3.7}$$

Replace y by $2zy$ in (3.6) and we have

$$(F(x) - x)zyd(z) = 0 \tag{3.8}$$

On subtraction (3.8) from (3.7), we obtain

$$xd(z)yd(z) = 0 \tag{3.9}$$

for all $x, y, z \in L$. Replace x by $2xz$ and y by $2yz$ in (3.9), we obtain

$$xzd(z)yzd(z) = 0. \tag{3.10}$$

Again, we substitute $2zy$ for y in (3.9) and right multiply it by z, we find

$$xd(z)zyd(z)z = 0 \tag{3.11}$$

Substracting (3.10) from (3.11), we get

$$x[d(z), z]y[d(z), z] = 0 \tag{3.12}$$

for all $x, y, z \in L$. Replace y by $4yx$ in (3.12), we get $2x[d(z), z]L2x[d(z), z] = (0)$ for all $x, z \in L$. In light of Lemma 2.1, we obtain $x[d(z), z] = 0$ for all $x, z \in L$. Again utilizing Lemma 2.1, we find $[d(z), z] = 0$ for all $z \in L$. Hence, Lemma 2.5 completes the proof.

(ii) If possible assume that $L \not\subseteq Z(R)$. By hypothesis,

$$F(xy) = F(y)F(x) \quad \text{for all } x, y \in L. \tag{3.13}$$

Replace x by $2xy$ in (3.13), we obtain $2F(xy^2) = 2F(y)F(xy)$. Using 2-torsion freeness of R, we get

$$F(xy^2) = F(y)F(xy)$$

$$F(xy)y + xyd(y) = F(y)F(x)y + F(y)xd(y)$$

Using (3.13), we get $(xy - F(y)x)d(y) = 0$ for all $x, y \in L$. By Lemma 2.2, we have $2M \subseteq L$. Putting $x = 2m$, we obtain

$$(my - F(y)m)d(y) = 0 \tag{3.14}$$

Replace m by $F(z)m$ in (3.14), where $z \in L$, we get

$$(F(z)my - F(y)F(z)m)d(y) = 0 \tag{3.15}$$

On multiplying (3.14) by $F(z)$ from left side, we get

$$(F(z)my - F(z)F(y)m)d(y) = 0 \tag{3.16}$$

Subtracting (3.15) from (3.16), we obtain $(F(z)F(y) - F(y)F(z))md(y) = 0$ for all $y, z \in L$ and $m \in M$. Again using our hypothesis, we get

$$F([y, z])md(y) = 0 \tag{3.17}$$

Replace z by $2zy$ in (3.17), we get $F([y, zy])md(y) = 0$ for any $y, z \in L$ and $m \in M$.

$$F([y, z])ymd(y) + [y, z]d(y)md(y) = 0 \quad \text{for all } y, z \in L \text{ and } m \in M. \tag{3.18}$$

Replace m by ym in (3.17) and subtract from (3.18), we obtain $[y, z]d(y)md(y) = 0$. Since $R[L, L] \subseteq M$ so we substitute $r[y, z]$ instead of m, where $r \in R$ and $y, z \in L$, we get $[y, z]d(y)r[y, z]d(y) = 0$. That is, $[y, z]d(y)R[y, z]d(y) = (0)$ where $y, z \in L$. Semiprimeness of R yields that

$$[y, z]d(y) = 0 \quad \text{for all } y, z \in L. \tag{3.19}$$

Linearizing the above relation we get

$$[x, z]d(y) = -[y, z]d(x) \tag{3.20}$$

Replace z by $2zu$ in (3.19), where $u \in L$, we get $2[y, zu]d(y) = 0$ for all $y, z, u \in L$. Since R is 2-torsion free so we left with $[y, z]ud(y) = 0$. By Lemma 2.3, we substitute $2r[x, z]$ in place of u, where $r \in R$ and $x, z \in L$ in the last expression and obtain $[y, z]r[x, z]d(y) = 0$. Using (3.20), we obtain that $[y, z]r[y, z]d(x) = 0$. Replacing r by $rd(x)$, we get $[y, z]d(x)R[y, z]d(x) = (0)$. Since R is semiprime ring, so we get $[y, z]d(x) = 0$. Again application of Lemma 2.2 implies that $[m, m_1]d(x) = 0$, where $x \in L$ and $m, m_1 \in M$. Substituting $d(x)m$ for m in the last expression and using it we get $[d(x), m_1]md(x) = 0$. From this, it easily follows that $[d(x), m_1]M[d(x), m_1] = (0)$ for each $x \in L$ and $m_1 \in M$. Since every ideal of a semiprime ring is a semiprime ring itself, we get $[d(x), m_1] = 0$ for all $x \in L$ and $m_1 \in M$. Now, as $R[L, L] \subseteq M$ so we put $m = r[y, z]$ in the last relation, where $r \in R$ and $y, z \in L$, we find $[d(x), r[y, z]] = 0$. On expanding this expression and using the fact that $[L, L] \subseteq M$ we obtain $[d(x), r][y, z] = 0$ for all $x, y, z \in L$ and $r \in R$. Now, replace y by y^2 in the last equation, we get $[d(x), r]y[y, z] = 0$ for all $x, y, z \in L$ and $r \in R$. In view of corollary 1.4 in [10], we find

$$[d(x), r][y, s] = 0 \quad \text{for all } \ x, y \in L, r, s \in R. \tag{3.21}$$

For any $p \in R$, replace s by sp in (3.21), we obtain $[d(x), r]s[y, p] = 0$ for all $x, y \in L$ and $r, s, p \in L$. In particular, we have $[d(x), x]R[d(x), x] = (0)$ for all $x \in L$. Since R is semiprime ring, we find $[d(x), x] = 0$ for all $x \in L$. Hence the conclusion follows from Lemma 2.5.

Corollary 3.3. *Let R be a 2-torsion free prime ring and L a nonzero square-closed Lie ideal of R. Suppose $F : R \to R$ be a generalized derivation associated with a derivation d.*

(i) If F acts as a homomorphism on L, then either $d = 0$ or $L \subseteq Z(R)$.

(ii) If F acts as an anti-homomorphism on L, then either $d = 0$ or $L \subseteq Z(R)$.

Proof. By Theorem 1.1, we obtain $d(R)[L, R] = (0)$ i.e.; $d(r)[x, s] = 0$ for any $r, s \in R$ and $x \in L$. Replacing r by $r_1 r$, where $r_1 \in R$, we get $d(r_1)R[x, r] = (0)$. By primeness of R we have either $d(r_1) = 0$ or $[x, r] = 0$. We set $A = \{r \in R : d(r) = 0\}$ and $B = \{r \in R : [x, r] = 0\}$, where $x \in L$. It is easy to see that both A and B are subgroups of $(R, +)$ and $R = A \cup B$. By Brauer's trick, we have either $A = R$ or $B = R$. Therefore, either $d = 0$ or $L \subseteq Z(R)$. But $L \nsubseteq Z(R)$, so we must have $d = 0$. □

3.2 Proof of Theorem 1.2

If possible, assume that $L \nsubseteq Z(R)$. Let us consider first $F(x^m y^n) = F(x^m)F(y^n)$ for all $x, y \in L$. By Lemma 2.2, every noncentral Lie ideal L of R contains a nonzero ideal $I = 2R[L, L]R$ of R. And therefore L contains a nonzero ideal say $I = 2R[L, L]R$ of R. With this, our assumption yields $F(x^m y^n) = F(x^m)F(y^n)$ for all $x, y \in I$. Let the set $A_1 = \{x^m : x \in I\}$ and G_1 be the additive subgroup of R generated by A_1 and G_2 be the additive subgroup generated by the set $A_2 = \{y^n : x \in I\}$. By hypothesis, we have

$$F(uv) - F(u)F(v) = 0 \quad \text{for all } \ u \in G_1, \ \ v \in G_2.$$

By Chuang [8], either $G_1 \subseteq Z(R)$ or char$(R) = 2$ and R satisfies s_4 identity, unless G_1 contains a noncentral Lie ideal L_1 of R. As we assumed R is of characteristic different from 2 and if $G_1 \subseteq Z(R)$ i.e.; $x^m \in Z(R)$ for all $x \in I$. By Lee [15], I_1, I_2, R and U satisfies the same differential identities, thus we find $x^m \in Z(R)$ for all $x \in R$. Then a well-known result of Herstein [11] forces R to be commutative, a contradiction to our assumption.

Therefore, G_1 contains a noncentral ideal L_1 of R. Then, we have

$$F(uv) - F(u)F(v) = 0 \text{ for all } u \in L_1, \ v \in G_2.$$

Similarly, there exists a noncentral Lie ideal L_2 of G_2 such that

$$F(uv) - F(u)F(v) = 0 \text{ for all } u \in L_1, \ v \in L_2.$$

In view of Herstein [[12], pg. 4-5], there exists a nonzero ideal I_1 such that $0 \neq [I_1, R] \subseteq L_1$. Similarly, there exists a nonzero ideal I_2 such that $0 \neq [I_2, R] \subseteq L_2$. Thus, we obtain $F(uv) - F(u)F(v) = 0$ for all $u \in [I_1, I_1]$ and $v \in [I_2, I_2]$. In light of Lee [15], I_1, I_2, R and U satisfies the same differential identities. So, we find $F(uv) - F(u)F(v) = 0$ for all $u \in [R, R]$. Clearly, $[R, R]$ is a nonzero Lie ideal of R. Therefore, by case 1 of Theorem 1.2 in [17], we get either $d = 0$ or $[R, R] \subseteq Z(R)$. The latter case implies commutativity of R, which is not possible. Hence, $d = 0$.

In case, $F(x^m y^n) = F(y^n)F(x^m)$ for all $x, y \in L$. Analogously as above, we can obtain the same conclusions.

Acknowledgement: We would like to express our sincere gratitude to the anonymous referee(s) for their helpful suggestions, which increase the readability of the paper.

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Author information

Gurninder S. Sandhu and Deepak Kumar, Department of Mathematics, Punjabi University, Patiala, Punjab 147001, INDIA.
E-mail: `gurninder_rs@pbi.ac.in`