

ADDITIVE MAPS PRESERVING ZERO-PRODUCTS ON TRIANGULAR RINGS

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Abstract Let A and B be two unital rings and let M be a unital (A, B) -bimodule such that M is faithful as a left A -module and also as a right B -module. Consider the triangular ring $\mathcal{T} = \begin{pmatrix} A & M \\ 0 & B \end{pmatrix}$ and let 1 be its identity. It is shown that for a surjective additive mapping $\Phi : \mathcal{T} \rightarrow \mathcal{T}$, Φ is a ring isomorphism if and only if $\Phi(1) = 1$ and $\Phi(x)\Phi(y) = 0 \Leftrightarrow xy = 0$ for all $x, y \in \mathcal{T}$.

1 Introduction

Throughout this paper, all rings considered are associative and have identity. Let A and B be two rings. Recall that a left A -module (resp., a right B -module) M is said to be *faithful* if for any $a \in A$, $aM = \{0\}$ (resp., for any $b \in B$, $Mb = \{0\}$) implies $a = 0$ (resp., $b = 0$).

Let $\Phi : A \rightarrow B$ be an additive mapping. We say that Φ preserves zero-products in both directions if Φ satisfies the following condition:

$$(*) : x_1x_2 = 0 \text{ if and only if } \Phi(x_1)\Phi(x_2) = 0 \text{ for all } x_1, x_2 \in A.$$

Let A and B be two unital rings and let M be a unital (A, B) -bimodule which is faithful as a left A -module as well as a right B -module. The set

$$Tri(A, M, B) = \left\{ \begin{pmatrix} a & m \\ 0 & b \end{pmatrix} : a \in A, m \in M, b \in B \right\}$$

is a unital associative ring under the usual matrix operations. Each ring which is isomorphic to $Tri(A, M, B)$ is called a *triangular ring*. It is well known that upper triangular matrix rings and block upper triangular matrix rings are triangular rings.

Our research was motivated by the following results. Many research works have been done on linear maps preserving the spectrum [1], square-zero matrices [5] and zero-products [2, 3, 4, 5, 6] on many kinds of algebras. For example, from [2, Corollary 4.3], it follows that if \mathcal{A} and \mathcal{B} are unital rings such that \mathcal{A} contains a noncentral idempotent and \mathcal{B} is a prime ring, then any bijective mapping $h : \mathcal{A} \rightarrow \mathcal{B}$ such that $h(1) = 1$ and h satisfies the property $x_1x_2 = 0 \Rightarrow h(x_1)h(x_2) = 0$ for all $x_1, x_2 \in \mathcal{A}$ is a ring isomorphism.

The aim of this paper is to prove that for a triangular ring \mathcal{T} , any surjective additive mapping $\Phi : \mathcal{T} \rightarrow \mathcal{T}$ that preserves identity and zero-products in both directions is a ring isomorphism.

2 The results

The following is our main result.

Theorem 2.1. *Let $\mathcal{T} = Tri(A, M, B)$ be a triangular ring. Let 1 be the identity of \mathcal{T} and let $\Phi : \mathcal{T} \rightarrow \mathcal{T}$ be a surjective additive mapping. Then $\Phi(1) = 1$ and Φ preserves zero-products in both directions if and only if Φ is bijective and $\Phi(xy) = \Phi(x)\Phi(y)$ for all $x, y \in \mathcal{T}$.*

To prove this theorem, we need the following lemmas. We begin with the following trivial one.

Lemma 2.2. *Let \mathcal{A} and \mathcal{B} be two rings and let $\Phi : \mathcal{A} \rightarrow \mathcal{B}$ be an additive mapping. Then $\Phi(-x) = -\Phi(x)$ for all x in \mathcal{A} .*

The following result is presumably well known (see, for example, [6, Lemma 2.1]) but is included for completeness.

Lemma 2.3. *Let \mathcal{A} and \mathcal{B} be two rings with identities $1_{\mathcal{A}}$ and $1_{\mathcal{B}}$, respectively. Let $\Phi : \mathcal{A} \rightarrow \mathcal{B}$ be an additive mapping such that $\Phi(1_{\mathcal{A}}) = 1_{\mathcal{B}}$ and Φ preserves zero-products in both directions. Then:*

- (a) Φ is injective.
- (b) $\Phi(u) = \Phi(u)^2$ for all idempotents u in \mathcal{A} .

Proof. (a) Let $x, x' \in \mathcal{A}$ such that $\Phi(x) = \Phi(x')$. Then $\Phi(x) + \Phi(-x') = 0$ by Lemma 2.2. As Φ is an additive mapping, we have $\Phi(x - x') = 0$. So $\Phi(x - x')\Phi(1_{\mathcal{A}}) = 0$. Since Φ preserves zero-products in both directions, it follows that $x - x' = (x - x')1_{\mathcal{A}} = 0$. Therefore Φ is injective.

(b) Let u be an idempotent in \mathcal{A} . Since $u(1_{\mathcal{A}} - u) = 0$, it follows that $\Phi(u)\Phi(1_{\mathcal{A}} - u) = 0$. As $\Phi(1_{\mathcal{A}}) = 1_{\mathcal{B}}$, we have $\Phi(u) = \Phi(u)^2$ by Lemma 2.2. This completes the proof. \square

The next lemma is a straightforward generalization of [6, Lemma 2.2]. We notice that their proofs are similar in spirit.

Lemma 2.4. *Let \mathcal{A} and \mathcal{B} be two rings with identities $1_{\mathcal{A}}$ and $1_{\mathcal{B}}$, respectively. Let $\Phi : \mathcal{A} \rightarrow \mathcal{B}$ be an additive mapping such that $\Phi(1_{\mathcal{A}}) = 1_{\mathcal{B}}$ and $\Phi(x)\Phi(y) = 0$ for all $x, y \in \mathcal{A}$ with $xy = 0$. Then $\Phi(uxv) = \Phi(u)\Phi(x)\Phi(v)$ for all x in \mathcal{A} and all idempotents u, v in \mathcal{A} .*

Proof. Let x be an element in \mathcal{A} and let u be an idempotent in \mathcal{A} . Note that $(1_{\mathcal{A}} - u)ux = u(1_{\mathcal{A}} - u)x = 0$. By hypothesis, we have $\Phi(1_{\mathcal{A}} - u)\Phi(ux) = 0$ and $\Phi(u)\Phi((1_{\mathcal{A}} - u)x) = 0$. As $\Phi(1_{\mathcal{A}}) = 1_{\mathcal{B}}$, we conclude that $\Phi(ux) = \Phi(u)\Phi(ux)$ and $\Phi(u)\Phi(x) = \Phi(u)\Phi(ux)$ (see Lemma 2.2). It follows that

$$\Phi(ux) = \Phi(u)\Phi(x)$$

for all x in \mathcal{A} and all idempotents u in \mathcal{A} . In the same manner we can see that

$$\Phi(xv) = \Phi(x)\Phi(v)$$

for all x in \mathcal{A} and all idempotents v in \mathcal{A} . Thus,

$$\Phi(uxv) = \Phi(u)\Phi(xv) = \Phi(u)\Phi(x)\Phi(v)$$

for all x in \mathcal{A} and all idempotents u, v in \mathcal{A} . This proves the lemma. \square

Proof of Theorem 2.1. The sufficiency is straightforward. The necessity will be organized in a sequence of claims. We will use the following notation

$e = \begin{pmatrix} 1_{\mathcal{A}} & 0 \\ 0 & 0 \end{pmatrix}$ and $f = 1 - e = \begin{pmatrix} 0 & 0 \\ 0 & 1_{\mathcal{B}} \end{pmatrix}$. Here 1 , $1_{\mathcal{A}}$ and $1_{\mathcal{B}}$ are identities of \mathcal{T} , \mathcal{A} and \mathcal{B} , respectively. It is easily seen that $f\mathcal{T}e = 0$ and so any element $t \in \mathcal{T}$ may be represented as

$$t = (e + f)t(e + f) = ete + etf + ftf.$$

Also, note that $e\mathcal{T}e$ and $f\mathcal{T}f$ are subrings of \mathcal{T} and $e\mathcal{T}f$ is an $(e\mathcal{T}e, f\mathcal{T}f)$ -bimodule which is faithful as a left $e\mathcal{T}e$ -module and also as a right $f\mathcal{T}f$ -module.

Assume that $\Phi(1) = 1$ and Φ preserves zero-products in both directions. By Lemma 2.3(a), Φ is bijective. It is easy to check that $\Phi^{-1} : \mathcal{T} \rightarrow \mathcal{T}$ is an additive mapping preserving identity and zero-products in both directions. Let $g, h \in \mathcal{T}$ such that $g = \Phi^{-1}(e)$ and $h = \Phi^{-1}(f)$. From Lemma 2.3(b), it follows that $g^2 = g$ and $h^2 = h$. Since $fe = 0$, we have $\Phi^{-1}(f)\Phi^{-1}(e) = 0$. Hence $hg = 0$.

Claim 1. We have $xg = gxg$ and $hx = h x h$ for all $x \in \mathcal{T}$.

Let $x \in \mathcal{T}$ and let $y = \Phi(x) \in \mathcal{T}$. Then $\Phi^{-1}(y) = x$. Note that

$$ye = (e + f)ye = eye + fye = eye.$$

From Lemma 2.4, it follows that

$$\Phi^{-1}(ye) = \Phi^{-1}(eye) = \Phi^{-1}(e)\Phi^{-1}(y)\Phi^{-1}(e). \tag{1}$$

Applying again Lemma 2.4 and using the fact $\Phi^{-1}(1) = 1$, we get

$$\Phi^{-1}(ye) = \Phi^{-1}(1ye) = \Phi^{-1}(1)\Phi^{-1}(y)\Phi^{-1}(e) = \Phi^{-1}(y)\Phi^{-1}(e). \tag{2}$$

Therefore,

$$\Phi^{-1}(y)\Phi^{-1}(e) = \Phi^{-1}(e)\Phi^{-1}(y)\Phi^{-1}(e). \tag{3}$$

That is, $xg = gxg$. Similarly, we can show that $hx = h x h$.

Claim 2. For all $x, t \in \mathcal{T}$, we have

$$\Phi(xgth) = \Phi(x)\Phi(gth) \tag{4}$$

and

$$\Phi(gthx) = \Phi(gth)\Phi(x). \tag{5}$$

Let x and t be elements in \mathcal{T} . Since $hg = 0$, $g + gth$ is an idempotent in \mathcal{T} . Replacing u by 1 and v by $g + gth$ in Lemma 2.4, we get

$$\Phi(x(g + gth)) = \Phi(x)\Phi(g + gth).$$

Hence,

$$\Phi(xg) + \Phi(xgth) = \Phi(x)\Phi(g) + \Phi(x)\Phi(gth).$$

As $\Phi(xg) = \Phi(x)\Phi(g)$ (see Lemma 2.4), we have $\Phi(xgth) = \Phi(x)\Phi(gth)$.

Similarly, replacing u by $g + gth$ and v by 1 in Lemma 2.4, we obtain $\Phi(gthx) = \Phi(gth)\Phi(x)$.

Claim 3. We have $[\Phi(xy) - \Phi(x)\Phi(y)]e\mathcal{T}f = 0$ for all $x, y \in \mathcal{T}$.

Let x, y and t be elements in \mathcal{T} . As Φ is surjective, there exists $s \in \mathcal{T}$ such that $\Phi(s) = t$. Replacing x by xy and t by s in Eq. (4), we have

$$\Phi(xygsh) = \Phi(xy)\Phi(gsh). \tag{6}$$

On the other hand, it follows from claim 1 and Eq. (4) that

$$\begin{aligned} \Phi(xygsh) &= \Phi(xgygsh) \\ &= \Phi(x)\Phi(gygsh) \\ &= \Phi(x)\Phi(ygsh) \\ &= \Phi(x)\Phi(y)\Phi(gsh). \end{aligned} \tag{7}$$

Combining Eqs. (6) and (7), we conclude that

$$[\Phi(xy) - \Phi(x)\Phi(y)]\Phi(gsh) = 0.$$

By Lemma 2.4, we have

$$\Phi(gsh) = \Phi(g)\Phi(s)\Phi(h) = etf.$$

Hence,

$$[\Phi(xy) - \Phi(x)\Phi(y)]etf = 0.$$

This is our claim.

To complete the proof of Theorem 2.1, let x and y be elements in \mathcal{T} . Since $e^2 = e$, we have $[e(\Phi(xy) - \Phi(x)\Phi(y))e]e\mathcal{T}f = 0$ by claim 3. Since $e\mathcal{T}f$ is a faithful left $e\mathcal{T}e$ -module, we have

$$e(\Phi(xy) - \Phi(x)\Phi(y))e = 0. \quad (8)$$

Similarly, by claim 1 and Eq. (5), we can see that

$$etf[\Phi(xy) - \Phi(x)\Phi(y)] = 0$$

for all $x, y, t \in \mathcal{T}$. As $f^2 = f$, we obtain

$$e\mathcal{T}f[f(\Phi(xy) - \Phi(x)\Phi(y))f] = 0.$$

Since $e\mathcal{T}f$ is a faithful right $f\mathcal{T}f$ -module, we get

$$f(\Phi(xy) - \Phi(x)\Phi(y))f = 0 \quad (9)$$

Now by Lemma 2.4 and Eqs. (4) and (5), we have

$$\begin{aligned} e\Phi(xy)f &= \Phi(g)\Phi(xy)\Phi(h) \\ &= \Phi(gxyh) \\ &= \Phi(gx1yh) \\ &= \Phi(gx(g+h)yh) \\ &= \Phi(gxgyh) + \Phi(gxhyh) \\ &= \Phi(gx)\Phi(gyh) + \Phi(gxh)\Phi(yh) \\ &= e\Phi(x)e\Phi(y)f + e\Phi(x)f\Phi(y)f \\ &= e\Phi(x)(e+f)\Phi(y)f \\ &= e\Phi(x)\Phi(y)f. \end{aligned}$$

This implies that

$$e[\Phi(xy) - \Phi(x)\Phi(y)]f = 0. \quad (10)$$

Combining Lemma 2.2 with Eqs. (8), (9) and (10), we conclude that

$$\begin{aligned} \Phi(xy) - \Phi(x)\Phi(y) &= e[\Phi(xy) - \Phi(x)\Phi(y)]e \\ &\quad + e[\Phi(xy) - \Phi(x)\Phi(y)]f \\ &\quad + f[\Phi(xy) - \Phi(x)\Phi(y)]f \\ &= 0. \end{aligned}$$

Consequently, $\Phi(xy) = \Phi(x)\Phi(y)$ for all $x, y \in \mathcal{T}$. This completes the proof. \square

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