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On $\mathcal{SWGC}$-projective and $\mathcal{SWGC}$-injective Modules

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Abstract An S-module M (resp., R-module N) is called $\mathcal{SWGC}$-projective (resp., $\mathcal{SWGC}$-injective) if there exists a $\mathrm{Hom}_S(-, \mathcal{P}_C(S))$ exact exact complex (resp., $\mathrm{Hom}_R(\mathcal{I}_C(R), -)$ exact exact complex

$$\mathbb{P} := \cdots \longrightarrow P \xrightarrow{d} P \xrightarrow{d} P \xrightarrow{d} \cdots$$

of $\mathcal{P}_C(S)$-projective (resp., $\mathcal{I}_C(R)$-injective) modules such that $M \cong \mathrm{Im}d$ (resp., $N \cong \mathrm{Im}d$), where C is a semidualizing $(S\text{-}R)$-bimodule. It will be shown that an S-module M (resp., R-module N) is $\mathcal{SWGC}$-projective (resp., $\mathcal{SWGC}$-injective) if and only if $\mathrm{Ext}^{\geq 1}_{\mathcal{P}_C(S)}(M, \mathcal{P}_C(S))$(resp., $\mathrm{Ext}^{\geq 1}_{\mathcal{I}_C(R)}(\mathcal{I}_C(R), N)$) vanishes and there exists a short exact sequence $0 \longrightarrow M \longrightarrow C \otimes_R P \longrightarrow M \longrightarrow 0$ (resp., $0 \longrightarrow N \longrightarrow \mathrm{Hom}_S(C, I) \longrightarrow N \longrightarrow 0$), where P (resp., I) is R-projective (resp., S-injective) module. Then we show that, with respect to the mentioned short exact sequences, $\{\mathrm{Ext}^i_{\mathcal{P}_C(S)}(-, M)\}_{i\geq 0}$ (resp., $\{\mathrm{Ext}^i_{\mathcal{I}_C(R)}(N, -)\}_{i\geq 0}$) become strongly connected sequence of functors, and by using it, we prove that a $\mathcal{SWGC}$-projective (resp., $\mathcal{SWGC}$-injective) module of finite $\mathcal{P}_C(S)$-projective (resp., $\mathcal{I}_C(R)$-injective) dimension is C-projective (resp., C-injective). Finally, over Noetherian rings, a characterization of finitely generated $\mathcal{SWGC}$-projective modules with respect to the class $\mathcal{F}_C(S)$ is investigated.

1 Introduction

Throughout, unless stated otherwise, R and S will be associative rings with 1 and all modules will be unitary. In [1], Auslander and Bridger introduced the notion of Gorenstein dimension, for finitely generated modules over a Noetherian ring R, and explored several properties of modules of finite Gorenstein dimension, where the name of Gorenstein dimension comes back to the fact that over a local ring $(R, \mathfrak{m}, k)$ the following statements are equivalent:

- R is Gorenstein;
- G-dim$(M) < \infty, \forall\, M$ finitely generated R-module;
- G-dim$(k) < \infty$;

at which G-dim(X), for an R-module X, denotes the Gorenstein dimension of X.

Later, Enochs and Jenda [4], introduced the class of Gorenstein injective, projective and flat modules and related dimensions and characterized these invariants in terms of vanishing of extension and torsion functors. Especially, in [6], Enochs et al. proved that, whenever $(R, \mathfrak{m})$ is a local Cohen-Macaulay ring admitting a dualizing module then the Bass (resp., Auslander) class, is the class of modules of finite Gorenstein injective (resp., Gorenstein projective) dimension.

Bennis and Mahdu [2], introduced the concept of an Strongly Gorenstein injective, projective and flat module and provided some new characterizations of Gorenstein injective, projective and flat modules.

Takahashi and White, [11, Theorem 3.2, 3.3], proved that vanishing of $\{\mathrm{Ext}^i_{\mathcal{P}_C(S)}(M, -)\}_{i\geq n}$ (resp., $\{\mathrm{Ext}^i_{\mathcal{I}_C(R)}(-, N)\}_{i\geq n}$) measures finiteness of $\mathcal{P}_C(S)$-pd(M) (resp., $\mathcal{I}_C(R)$-id(N)), where $\mathcal{P}_C(S)$-pd(M) (resp., $\mathcal{I}_C(R)$-id(N)) stands for the $\mathcal{P}_C(S)$-projective (resp., $\mathcal{I}_C(R)$-injective) dimension of an S-module M (resp., R-module N). In this paper, we shall introduce the concept of an $\mathcal{SWGC}$-projective module M (resp., $\mathcal{SWGC}$-injective module N) and recognize these modules in terms of vanishing of $\mathrm{Ext}^{\geq 1}_{\mathcal{P}_C(S)}(M, \mathcal{P}_C(S))$ (resp., $\mathrm{Ext}^{\geq 1}_{\mathcal{I}_C(R)}(\mathcal{I}_C(R), N)$. Using these characterizations it is proved that a $\mathcal{SWGC}$-projective module M (resp., $\mathcal{SWGC}$-injective module N) of finite $\mathcal{P}_C(S)$-projective (resp., $\mathcal{I}_C(R)$-injective) dimension is a C-projective (resp., C-injective) module. From this, by taking $C = R = S$, some well-known results of Enochs and Jenda (see [5, Proposition 10.2.3 and 10.1.2]), are concluded as special cases. For definitions concerning the functors $\{\mathrm{Ext}^i_{\mathcal{P}_C(S)}(M, -)\}_{i\geq 0}$ and $\{\mathrm{Ext}^i_{\mathcal{I}_C(R)}(-, N)\}_{i\geq 0}$, see Remark 3.5.

2 Preliminaries

In this section we bring the facts and definitions, which will be used in the sequel. Note that by an $(S\text{-}R)$-bimodule X, denoted by ${}_SX_R$, we mean a left S-module and a right R-module such that for all $s \in S, x \in X$ and $r \in R$ we have $(sx)r = s(xr)$. To avoid confusion, a right R-module will be denoted by R^{op}-module. Also, the symbols ${}_RM$ and N_R mean that M is an R-module and N is an R^{op}-module. Recall that an R-module M is said to admits a degreewise finite projective resolution if there exists a projective resolution P of M such that each component P_i of P is finitely generated.

Remark 2.1. (1) Consider modules ${}_SM$, ${}_SN_R$ and ${}_RF$. It is easy to see that, if M is a finitely presented S-module and F is a flat R-module, then the mapping $\nu_{_{MNF}} : \mathrm{Hom}_S(M, N) \otimes_R F \to \mathrm{Hom}_S(M, N \otimes_R F)$, where for each $\psi \in \mathrm{Hom}_S(M, N)$, $f \in F$ and $m \in M$, $\nu_{_{MNF}}(\psi \otimes f)(m) = \psi(m) \otimes f$, is a natural equivalence of (contravariant) functors. If M is an $(S\text{-}R)$-bimodule, then $\nu_{_{MNF}}$ is an R-isomorphism, which in turn implies the R-isomorphism

$$\mathrm{Ext}^i_S(M, N) \otimes_R F \cong \mathrm{Ext}^i_S(M, N \otimes_R F),$$

provided that M admits a degreewise finite S-projective resolution.

(2) Now, consider modules M_R, ${}_SN_R$ and ${}_SI$. Again, it is easy to see that, if M is a finitely presented R^{op}-module and I is an injective S-module, then the mapping $\mu_{_{MNI}} : M \otimes_R \mathrm{Hom}_S(N, I) \to \mathrm{Hom}_S(\mathrm{Hom}_{R^{\mathrm{op}}}(M, N), I)$, where for each $\varphi \in \mathrm{Hom}_{R^{\mathrm{op}}}(M, N)$, $\phi \in \mathrm{Hom}_S(N, I)$ and $m \in M$, $\mu_{_{MNI}}(m \otimes \phi)(\varphi) = \phi(\varphi(m))$, is an equivalence of (covariant) functors. If M is an $(S\text{-}R)$-bimodule, then $\mu_{_{MNI}}$ is an S-isomorphism, which in turn implies the S-isomorphism

$$\mathrm{Tor}^R_i(M, \mathrm{Hom}_S(N, I)) \cong \mathrm{Hom}_S(\mathrm{Ext}^i_{R^{\mathrm{op}}}(M, N), I),$$

provided that M admits a degreewise finite R^{op}-projective resolution.

Definition 2.2. An $(S\text{-}R)$-bimodule ${}_SC_R$ is semidualizing if:

- ${}_SC$ (resp., C_R) admits a degreewise finite S-projective (resp., R^{op}-projective) resolution,
- the natural homothety maps ${}_SS_S \to \mathrm{Hom}_{R^{\mathrm{op}}}(C, C)$ and ${}_RR_R \to \mathrm{Hom}_S(C, C)$ are isomorphisms, and
- $\mathrm{Ext}^{\geqslant 1}_S(C, C) = \mathrm{Ext}^{\geqslant 1}_{R^{\mathrm{op}}}(C, C) = 0$.

Throughout, $C = {}_SC_R$ denotes a semidualizing $(S\text{-}R)$-bimodule.

Definition 2.3. The *Bass* class with respect to C, denoted by $\mathcal{B}_C(S)$, consists of all S-modules M such that

(i) $\mathrm{Ext}^{\geqslant 1}_S(C, M) = \mathrm{Tor}^R_{\geqslant 1}(C, \mathrm{Hom}_S(C, M)) = 0$;

(ii) the natural map $\nu_{_{CCM}} : C \otimes_R \mathrm{Hom}_S(C, M) \longrightarrow M$ is an isomorphism.

The *Auslander* class with respect to C, denoted by $\mathcal{A}_C(R)$, consists of all R-modules M such that

(i) $\mathrm{Tor}^R_{\geqslant 1}(C, M) = \mathrm{Ext}^{\geqslant 1}_S(C, C \otimes_R M) = 0$;

(ii) the natural map $\mu_{_{CCM}} : M \longrightarrow \mathrm{Hom}_S(C, C \otimes_R M)$ is an isomorphism.

Definition 2.4. An S-module (resp., R-module) is said to be C-flat, C-projective (resp., C-injective) if it is isomorphic to $C\otimes_R F$, $C\otimes_R P$ (resp., $\mathrm{Hom}_S(C,I)$) for some R-flat, R-projective (resp., S-injective) module, F, P (resp., I), respectively. The class of C-flat, C-projective and C-injective modules will be denoted by $\mathcal{F}_C(S), \mathcal{P}_C(S)$ and $\mathcal{I}_C(R)$, respectively; i.e.,

$$\mathcal{F}_C(S) := \{C \otimes_R F : F \textit{ is } R\textit{-flat}\},$$
$$\mathcal{P}_C(S) := \{C \otimes_R P : P \textit{ is } R\textit{-projective}\},$$
$$\mathcal{I}_C(R) := \{\mathrm{Hom}_S(C, I) : I \textit{ is } S\textit{-injective}\}.$$

Remark 2.5. By Remark 2.1, it is easily seen that the Auslander class $\mathcal{A}_C(R)$ (resp., Bass class $\mathcal{B}_C(S)$) contains R-flat (resp., S-injective) modules. Since the mappings $C\otimes_R(-) : \mathcal{A}_C(R) \to \mathcal{B}_C(S)$ and $\mathrm{Hom}_S(C,(-)) : \mathcal{B}_C(S) \to \mathcal{A}_C(R)$ constitute equivalence between the categories $\mathcal{A}_C(R)$ and $\mathcal{B}_C(R)$, then we have the containments $\mathcal{P}_C(S) \subseteq \mathcal{F}_C(S) \subseteq \mathcal{B}_C(S)$ and $\mathcal{I}_C(R) \subseteq \mathcal{A}_C(R)$.

Definition 2.6. Let M be an R-module and let $\mathcal{F}$ be a class of R-modules. A linear map $\varphi : F \to M$ where $F \in \mathcal{F}$ is called an $\mathcal{F}$-precover of M if for each $F' \in \mathcal{F}$ the mapping $\mathrm{Hom}_R(id_{F'}, \varphi) : \mathrm{Hom}_R(F', F) \to \mathrm{Hom}_R(F', M)$ is surjective. A precover is called a cover in case that for every endomorphism $f \in \mathrm{End}_R(F)$, the equality $\varphi = \varphi \circ f$ implies that f is an automorphism of F. Dually, one can define preenvelope and envelope. The class $\mathcal{F}$ is said to be precovering, covering, preenveloping, enveloping, if every R-module has an $\mathcal{F}$-precover, $\mathcal{F}$-cover, $\mathcal{F}$-preenvelope, $\mathcal{F}$-envelope, respectively (see [5, Definition 5.1.1]).

Definition 2.7. Let $\mathcal{F}$ be a class of R-modules and let M be an R-module. A complex $\mathbb{X}$ is said to be $\mathrm{Hom}(-,\mathcal{F})$ exact if for all $F \in \mathcal{F}$ the complex $\mathrm{Hom}(\mathbb{X}, F)$ is exact. The complex $\mathbb{X}$ is said to be $\mathrm{Hom}(\mathcal{F}, -)$ exact if for all $F \in \mathcal{F}$ the complex $\mathrm{Hom}(F, \mathbb{X})$ is exact. By a left $\mathcal{F}$-resolution of M we mean a $\mathrm{Hom}(\mathcal{F}, -)$ exact complex $\cdots \longrightarrow F_1 \longrightarrow F_0 \longrightarrow M \longrightarrow 0$ (not necessarily exact) where $F_i \in \mathcal{F}$. By a right $\mathcal{F}$-resolution of M we mean a $\mathrm{Hom}(-,\mathcal{F})$ exact complex $0 \longrightarrow M \longrightarrow F^0 \longrightarrow F^1 \longrightarrow \cdots$ (not necessarily exact) where $F_i \in \mathcal{F}$ (see [5, Definition 8.1.2]).

Definition 2.8. Let $\mathcal{F}$ be a precovering class of R-modules an let M be an R-module. The $\mathcal{F}$-projective dimension of M, denoted by $\mathcal{F}\text{-pd}({}_RM)$, is

$$\mathcal{F}\text{-pd}({}_RM) = \inf\{\sup\{n \mid F_n \neq 0\} \mid F \text{ is a left } \mathcal{F}\text{-resolution of } M\}.$$

Dually, $\mathcal{G}$-injective dimension, denoted by $\mathcal{G}\text{-id}({}_RM)$, for a preenveloping class $\mathcal{G}$, is defined. For a precovering (resp., preenveloping) class $\mathcal{F}$ (resp., $\mathcal{G}$) the class of modules with finite $\mathcal{F}$-projective (resp., $\mathcal{F}$-injective) dimension will be denoted by $\overline{\mathcal{F}\text{-pd}}$ (resp., $\overline{\mathcal{G}\text{-id}}$).

Theorem 2.9. *Let ${}_SC_R$ be a semidualizing module.*

(i) The class $\mathcal{F}_C(S)$ (resp., $\mathcal{P}_C(S)$) is covering (resp., precovering) on the category of S-modules and is closed under direct sum and direct summand.

(ii) The class $\mathcal{I}_C(R)$ is enveloping on the category of R-modules and is closed under direct product and direct summand.

(iii) The class $\mathcal{A}_C(R)$ contains R-modules of finite $\mathcal{I}_C(R)$-injective dimension and the class $\mathcal{B}_C(S)$ contains S-modules of finite $\mathcal{F}_C(S)$-projective dimension and finite $\mathcal{P}_C(S)$-projective dimension.

Remark 2.10. Let $\mathcal{F}$ be a class of R-modules. In general an $\mathcal{F}$-precover ($\mathcal{F}$-preenvelope) need not to be surjective (injective). It is easily seen that if $\mathcal{F}$ is precovering (preenveloping) and containing projective modules (injective module) then an $\mathcal{F}$-precover ($\mathcal{F}$-preenvelope) is surjective (injective). By Theorem 2.9 we know that, on the category of S-modules, $\mathcal{F}_C(S)$ is precovering. Indeed, a careful reading of the proof of [7, Proposition 5.10] shows that if $\alpha : F \longrightarrow \mathrm{Hom}_S(C, M)$ is a flat precover of $\mathrm{Hom}_S(C, M)$, that exists by [3, Theorem 3], then the composition

$$C \otimes_S F \xrightarrow{\mathrm{id}_C \otimes \alpha} C \otimes_R \mathrm{Hom}_S(C, M) \xrightarrow{\mu_{CCM}} M$$

is an $\mathcal{F}_C(S)$-precover of M. Therefore, in case that the natural homomorphism $\mu_{CCM} : C \otimes_S \mathrm{Hom}_S(C, M) \longrightarrow M$ is a surjection, we will have a surjective $\mathcal{F}_C(S)$-precover. Similarly, if $\mu_{CCM} : C \otimes_S \mathrm{Hom}_S(C, M) \longrightarrow M$ is a surjection then we will have a surjective $\mathcal{P}_C(S)$-precover. Concerning $\mathcal{I}_C(R)$-preenvelopes, again by Theorem 2.9, we know that the class $\mathcal{I}_C(R)$ is preenveloping. Actually, for an R-module N, if $\beta : C \otimes_S N \longrightarrow E$ is the injective hull of $C \otimes_S N$, then the composition

$$N \xrightarrow{\nu_{CCN}} \mathrm{Hom}_S(C, C \otimes_S N) \xrightarrow{\mathrm{Hom}(\mathrm{id}_C, \alpha)} \mathrm{Hom}_S(C, E)$$

is an $\mathcal{I}_C(R)$-preenvelope of N. Therefore, if $\nu_{CCN} : N \longrightarrow \mathrm{Hom}_R(C, C \otimes_S N)$ is an injection, then any $\mathcal{I}_C(R)$-preenvelope of N will be an injection. This means that an $\mathcal{F}_C(S)$ or $\mathcal{P}_C(S)$-precover (resp., $\mathcal{I}_C(R)$-preenvelope) of an element of the Auslander class $\mathcal{A}_C(S)$ (resp., Bass class $\mathcal{B}_C(R)$) is a surjection (resp., an injection).

3 The Results

Definition 3.1. (1) A complete C-projective resolution of an S-module M is a $\mathrm{Hom}_S(-, \mathcal{P}_C(S))$ exact exact complex $\cdots \longrightarrow P_{i-1} \xrightarrow{d_{i-1}} P_i \xrightarrow{d_i} P_{i+1} \xrightarrow{d_{i+1}} P_{i+2} \longrightarrow \cdots$ of C-projective modules P_i, such that $M \cong \mathrm{Im}d_0$. We will call M strongly weak Gorenstein C-projective (abbreviated as $\mathcal{SWGC}$-projective) if it has a complete C-projective resolution such that $P_i = P_{i+1}$ and $d_i = d_{i+1}$, for all $i \in \mathbb{Z}$.

(2) A complete C-injective resolution of an R-module N is a $\mathrm{Hom}_R(\mathcal{I}_C(R), -)$ exact exact complex $\cdots \longrightarrow I_{i-1} \xrightarrow{d_{i-1}} I_i \xrightarrow{d_i} I_{i+1} \xrightarrow{d_{i+1}} I_{i+2} \longrightarrow \cdots$ of C-injective modules I_i, such that $N \cong \mathrm{Im}d_0$. We will call N strongly weak Gorenstein C-injective (abbreviated as $\mathcal{SWGC}$-injective) if it has a complete C-injective resolution such that $I_i = I_{i+1}$ and $d_i = d_{i+1}$, for all $i \in \mathbb{Z}$.

Now, we are going to examine the behaviour of $\mathcal{SWGC}$-projective (resp., $\mathcal{SWGC}$-injective) class with respect to direct sum (resp., direct product). Recall that, for an R-module M, the injective envelope of M is denoted by $E_R(M)$.

Proposition 3.2. *Let $\{M_i\}_{i\in I}$ be a family of $\mathcal{SWGC}$-projective (resp., $\mathcal{SWGC}$-injective) modules. Then, $\coprod_{i\in I} M_i$ (resp., $\prod_{i\in I} M_i$) is $\mathcal{SWGC}$-projective (resp., $\mathcal{SWGC}$-injective). Furthermore, if ${}_SS$ is an Artinian ring and the injective hull of each simple S-module is finitely generated, then the direct sum of an arbitrary family of $\mathcal{SWGC}$-injective modules is again $\mathcal{SWGC}$-injective.*

Proof. By Theorem 2.9, the classes $\mathcal{P}_C(S)$ and $\mathcal{I}_C(R)$ are closed under direct sum and direct product, respectively. Then, by [5, Proposition 1.2.6 and 1.2.7], the first assertion is obvious. Now, let $\{N_i\}_{i\in I}$ be a family of $\mathcal{SWGC}$-injective R-modules. For each $i \in I$, there exists an S-injective module E_i and an exact sequence

$$\mathbb{I}_{N_i} : \cdots \xrightarrow{d_i} \mathrm{Hom}_S(C, E_i) \xrightarrow{d_i} \mathrm{Hom}_S(C, E_i) \xrightarrow{d_i} \cdots,$$

such that $\mathrm{Hom}_R(\mathrm{Hom}_S(C, E), \mathbb{I}_{N_i})$ is an exact complex, for each S-injective module E. By Remark 2.1(2) and hom-tensor adjoint isomorphism, we have the following isomorphisms:

$$\begin{aligned} \mathrm{Hom}_R(\mathrm{Hom}_S(C, E), \mathrm{Hom}_S(C, E_i)) &\cong \mathrm{Hom}_S(C \otimes_R \mathrm{Hom}_S(C, E), E_i) \\ &\cong \mathrm{Hom}_S(E, E_i). \end{aligned} \tag{3.1}$$

By [5, page 16 exercise 2 and Theorem 3.1.17],

$$\coprod_{i\in I} \mathbb{I}_{N_i} : \cdots \xrightarrow{\coprod d_i} \mathrm{Hom}_S(C, \coprod_{i\in I} E_i) \xrightarrow{\coprod d_i} \mathrm{Hom}_S(C, \coprod_{i\in I} E_i) \xrightarrow{\coprod d_i} \cdots$$

is an exact complex of C-injective modules and $\coprod_{i\in I} N_i = \ker(\coprod d_i)$. Let E be an injective S-module. By [9, Theorem 6.6.4], there exists a family $\{S_j\}_{j\in J}$ of simple S-modules such that

$E = \coprod_{j\in J} E_S(S_j)$. Now,

$$\begin{aligned}\operatorname{Hom}_R\big(\operatorname{Hom}_S(C,E),\operatorname{Hom}_S(C,\coprod_{i\in I}E_i)\big) &\cong \operatorname{Hom}_S\big(C\otimes_R \operatorname{Hom}_S(C,E),\coprod_{i\in I}E_i\big)\\ &\cong \operatorname{Hom}_S\big(\coprod_{j\in J}E_S(S_j),\coprod_{i\in I}E_i\big)\\ &\cong \prod_{j\in J}\operatorname{Hom}_S\big(E_S(S_j),\coprod_{i\in I}E_i\big)\\ &\cong \prod_{j\in J}\coprod_{i\in I}\operatorname{Hom}_S\big(E_S(S_j),E_i\big)\end{aligned}$$

where the first, second and fourth isomorphisms are true by hom-tensor adjoint isomorphism, Remark 2.1(2) and [5, page 16 exercise 2], respectively. So, by the isomorphism (3.1), exactness of $\operatorname{Hom}_R(\operatorname{Hom}_S(C,E),\mathbb{I}_{N_i})$ and the above isomorphism, it is concluded that $\coprod_{i\in I}\mathbb{I}_{N_i}$ is a complete $\mathcal{I}_C(R)$-resolution of $\coprod_{i\in I}N_i$ and we are done. □

Recall that a ring W is called a V-ring if each simple W-module is W-injective.

Corollary 3.3. *If one of the following statement hold*

(i) S is a commutative Artinian ring.

(ii) S is a commutative quasi-Frobenius ring.

(iii) $S = KG$, where G is a finite Abelian group and K is an arbitrary field.

(iv) S is an Artinian V-ring.

then the direct sum of every family of $\mathcal{SWGC}$-injective S-modules is $\mathcal{SWGC}$-injective.

Proof. First assume that S is a commutative Artinian ring. If E is an injective S-module then, by [5, Theorem 3.3.10], we have $E \cong \coprod_{\mathfrak{n}_i\in \mathrm{Max}(R)} E_S(S/\mathfrak{n}_i)^{(\Lambda_i)}$, for some index set Λ_i. By [5, Theorem 3.4.1 and Corollary 2.3.24], $E_S(S/\mathfrak{n})$ is finitely generated, for each maximal ideal $\mathfrak{n}$. Hence, in this case, the result follows by Proposition 3.2. If G is a finite Abelian group then, by [12, Proposition 4.2.6], KG is a commutative quasi-Frobenius ring. Since quasi-Frobenius rings are Artinian, then (2) and (3) steam from (1). In case (4), by [9, Theorem 6.6.4], for each injective module E, there exists a family $\{S_i\}_{i\in I}$ of simple S-modules such that $E \cong \coprod_{i\in I} E_S(S_i) \cong \coprod_{i\in I} S_i$. Therefore, the result follows from Proposition 3.2. □

Now, we are going to give an example of an R-module which is simultaneously a $\mathcal{SWGC}$-injective and $\mathcal{SWGC}$-projective R-module, while it is neither C-injective nor C-projective. i.e; we have the inclusion $\mathcal{P}_C(S) \subsetneq$ the class of $\mathcal{SWGC}$-projectives and $\mathcal{I}_C(R) \subsetneq$ the class of $\mathcal{SWGC}$-injectives. Recall that a ring R is called n-Gorenstein if it is left and right Noetherian and $\mathrm{id}({}_RR) \leq n$ and $\mathrm{id}(R_R) \leq n$.

Example 3.4. Let R be a 1-Gorenstein ring, and let n a natural integer. Assume that x is a central R-regular element. Set $R_n := \frac{R}{Rx^n}$ and $X_{n,2n} := \frac{Rx^n}{Rx^{2n}}$. Then (as R_{2n}-module) $X_{n,2n}$ is $\mathcal{SWGC}$-injective and $\mathcal{SWGC}$-projective, while it is neither C-projective nor C-injective.

To see why this is true, consider that, the second change of rings theorem for the injective dimension, [8, Theorem 205], implies that R_n is quasi-Frobenius. Therefore, by [5, Theorem 9.1.10], an R_n-module is projective, if and only if it is injective, if and only if it is flat. Consider the exact sequence

$$\mathbb{P}_{n,2n} := \cdots \xrightarrow{x^n} R_{2n} \xrightarrow{x^n} R_{2n} \xrightarrow{x^n} R_{2n} \xrightarrow{x^n} \cdots.$$

of R_{2n}-injective (and so R_{2n}-projective) modules. As mentioned above if M is either R_{2n}-injective or R_{2n}-projective then the complexes

$$\operatorname{Hom}_{R_{2n}}(M,\mathbb{P}_{n,2n}) \text{ and } \operatorname{Hom}_{R_{2n}}(\mathbb{P}_{n,2n},M)$$

are exact. Therefore, $X_{n,2n}$ is simultaneously a $\mathcal{SWGC}$-injective and $\mathcal{SWGC}$-projective R_{2n}-module. However, it is easily seen that $X_{n,2n}$ is not an R_{2n}-projective (and so neither an R_{2n}-injective nor R_{2n}-flat) module.

Remark 3.5. By Theorem 2.9, on the category of S-modules, the class $\mathcal{P}_C(S)$ is precovering. Therefore, for every S-module M, there exists an R-projective module P and an S-module homomorphism $\varphi : C \otimes_R P \longrightarrow M$ such that, for every C-projective module Q, the induced map $\mathrm{Hom}_S(Q, C \otimes_R P) \longrightarrow \mathrm{Hom}_S(Q, M)$ is surjective. This means that, for every S-module M, one can construct a complex of C-projective modules Q_i,

$$\mathbb{Y}_M : \cdots \longrightarrow Q_1 \longrightarrow Q_0 \longrightarrow M \longrightarrow 0$$

such that $\mathrm{Hom}_S(Q, \mathbb{Y}_M)$ is exact for each C-projective modules Q; i.e., $\mathbb{Y}_M$ is a left $\mathcal{P}_C(S)$-resolution of M. It is easy to see that if $\mathbb{X}_M$ is another left $\mathcal{P}_C(S)$-resolution of M then we have a chain map $f : \mathbb{Y}_M \to \mathbb{X}_M$ and any two such chain maps are homotopic. This gives rise to the well-defined cohomology modules $\mathrm{Ext}^i_{\mathcal{P}_{C(S)}}(M, L)$, for all S-modules M and L. Again, by Theorem 2.9, on the category of R-modules, the class $\mathcal{I}_C(R)$ is enveloping. Consequently, for an arbitrary R-module N one can construct a complex

$$\mathbb{I}_N : 0 \longrightarrow N \longrightarrow I^0 \longrightarrow I^1 \longrightarrow \cdots$$

of C-injective modules I^i such that $\mathrm{Hom}_R(\mathbb{I}_N, J)$ is exact, for each C-injective module J; i.e., $\mathbb{I}_N$ is a right $\mathcal{I}_C(R)$-resolution of N. Then, as mentioned above, for all R-modules T and N, we have the well-defined cohomology modules $\mathrm{Ext}^i_{\mathcal{I}_{C(R)}}(T, N)$.

The following lemma was proved in [11, Theorem 4.1], in case that $R = S$ is a commutative ring. For completeness we include the proof in our non-commutative situation $C = {}_SC_R$.

Lemma 3.6. *Let M, L be S-modules and let N, T be R-modules. There exist isomorphisms:*

(i) $\mathrm{Ext}^i_{\mathcal{P}_{C(S)}}(M, L) \cong \mathrm{Ext}^i_R(\mathrm{Hom}_S(C, M), \mathrm{Hom}_S(C, L))$ *and*

(ii) $\mathrm{Ext}^i_{\mathcal{I}_{C(R)}}(T, N) \cong \mathrm{Ext}^i_S(C \otimes_R T, C \otimes_R N)$.

Proof. First we prove (1). By Theorem 2.9, the class $\mathcal{P}_C(S)$ is precovering. So let $\mathcal{P} : \cdots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow M \longrightarrow 0$ be a left $\mathcal{P}_C(S)$-resolution of M, where $P_i = C \otimes_R Q_i$ for some projective R-module Q_i. By Remark 2.5, $\mathrm{Hom}_S(C, \mathcal{P}) : \cdots \longrightarrow Q_1 \longrightarrow Q_0 \longrightarrow \mathrm{Hom}_S(C, M) \longrightarrow 0$ is a projective resolution of $\mathrm{Hom}_S(C, M)$. Then

$$\begin{aligned}
\mathrm{Ext}^i_R(\mathrm{Hom}_S(C, M), \mathrm{Hom}_S(C, L)) &\cong \mathrm{H}^i\big(\mathrm{Hom}_R(\mathrm{Hom}_S(C, \mathcal{P}), \mathrm{Hom}_S(C, L))\big)\\
&\cong \mathrm{H}^i\big(\mathrm{Hom}_S(C \otimes_R \mathrm{Hom}_S(C, \mathcal{P}), L)\big)\\
&\cong \mathrm{H}^i\big(\mathrm{Hom}_S(\mathcal{P}, L)\big)\\
&\cong \mathrm{Ext}^i_{\mathcal{P}_{C(S)}}(M, L),
\end{aligned}$$

where the second and third isomorphisms are true by hom-tensor adjoint isomorphism and Remark 2.5, respectively. To prove (2) note that, again by Theorem 2.9, the class $\mathcal{I}_C(R)$ is preenveloping. If $\mathcal{E} : 0 \longrightarrow N \longrightarrow I^0 \longrightarrow I^1 \longrightarrow \cdots$ is a right $\mathcal{I}_C(R)$-resolution of N, where $I^i = \mathrm{Hom}_S(C, E^i)$ for some injective S-module E^i then, hom-tensor adjoint isomorphism and Remark 2.5, implies that $C \otimes_R \mathcal{E} : 0 \longrightarrow C \otimes_R N \longrightarrow E^0 \longrightarrow E^1 \longrightarrow \cdots$ is an injective resolution of $C \otimes_R N$. Therefore,

$$\begin{aligned}
\mathrm{Ext}^i_S(C \otimes_R T, C \otimes_R N) &\cong \mathrm{H}^i\big(\mathrm{Hom}_S(C \otimes_R T, C \otimes_R \mathcal{E})\big)\\
&\cong \mathrm{H}^i\big(\mathrm{Hom}_R(T, \mathrm{Hom}_S(C, C \otimes_R \mathcal{E}))\big)\\
&\cong \mathrm{H}^i\big(\mathrm{Hom}_R(T, \mathcal{E})\big)\\
&\cong \mathrm{Ext}^i_{\mathcal{I}_{C(R)}}(T, N)),
\end{aligned}$$

where the second and third isomorphisms are true by hom-tensor adjoint isomorphism and Remark 2.5, respectively. □

Definition 3.7. Let $\mathcal{X}$ be a class of R-modules. The i^{th} associated left orthogonal class of $\mathcal{X}$, denoted by ${}^{i\perp}\mathcal{X}$, is defined as

$$ {}^{i\perp}\mathcal{X} = \{N \in R{-}mod \,|\, \operatorname{Ext}^i_R(N, X) = 0 \text{ for all } X \in \mathcal{X}\}. $$

Also, the i^{th} associated right orthogonal class, denoted by $\mathcal{X}^{\perp i}$, is defined as

$$ \mathcal{X}^{\perp i} = \{N \in R{-}mod \,|\, \operatorname{Ext}^i_R(X, N) = 0 \text{ for all } X \in \mathcal{X}\}. $$

The next theorem provides some necessary an sufficient conditions for an S-module M to be $\mathcal{SWGC}$-projective in terms of vanishing of the cohomology modules $\{\operatorname{Ext}^i_{\mathcal{P}_{C(S)}}(M, \mathcal{P}_C(S))\}_{i\geq 1}$.

Theorem 3.8. *For an S-module M consider the following statements:*

(i) M is $\mathcal{SWGC}$-projective;

(ii) there exists a C-projective module, $C \otimes_R P$ say, such that the sequence $0 \longrightarrow M \longrightarrow C \otimes_R P \longrightarrow M \longrightarrow 0$ is exact and $\operatorname{Ext}^{\geq 1}_{\mathcal{P}_{C(S)}}(M, Q) = 0$ for each C-projective module Q;

(iii) there exists a C-projective module, $C \otimes_R P$ say, such that the sequence $0 \longrightarrow M \longrightarrow C \otimes_R P \longrightarrow M \longrightarrow 0$ is exact and $\operatorname{Ext}^{\geq 1}_{\mathcal{P}_{C(S)}}(M, Q) = 0$ whenever, Q is C-projective or $\operatorname{id}({}_R\operatorname{Hom}_S(C, Q)) < \infty$;

(iv) there exists a C-projective module, $C \otimes_R P$ say, such that the sequence $0 \longrightarrow M \longrightarrow C \otimes_R P \longrightarrow M \longrightarrow 0$ is exact and $\operatorname{Hom}_S(-, Q)$ leaves it exact whenever, Q is C-projective or $\operatorname{id}({}_R\operatorname{Hom}_S(C, Q)) < \infty$;

(v) there exists a C-projective module, $C \otimes_R P$ say, such that the sequence $0 \longrightarrow M \longrightarrow C \otimes_R P \longrightarrow M \longrightarrow 0$ is exact and for each C-projective module Q the sequence

$$ 0 \longrightarrow \operatorname{Hom}_S(M, Q) \longrightarrow \operatorname{Hom}_S(C \otimes_R P, Q) \longrightarrow \operatorname{Hom}_S(M, Q) \longrightarrow 0 $$

is exact, too.

Then $(1) \Leftrightarrow (5)$ *and if* $\operatorname{pd}({}_S C)$ *is finite, then* (1)-(5) *are equivalent.*

Proof. $(1) \Rightarrow (5)$ is obvious.

$(5) \Rightarrow (1)$ From the short exact sequence $0 \longrightarrow M \longrightarrow C \otimes_R P \longrightarrow M \longrightarrow 0$ we have the following commutative diagram:

$$
\begin{array}{ccccccccccc}
 & & 0 & & 0 & & 0 & & 0 & & \\
 & & & \searrow\ M \nearrow & & & & \searrow\ M \nearrow & & & \\
 & & & \nearrow \quad \searrow & & & & \nearrow \quad \searrow & & & \\
\mathbb{P}_{\mathbb{M}}: & \cdots \to & C \otimes_R P & \longrightarrow & C \otimes_R P & \longrightarrow & C \otimes_R P & \longrightarrow & C \otimes_R P & & \to \cdots \\
 & \nearrow & & & & \searrow \quad \nearrow & & & & \searrow & \\
 & M & & & & M & & & & & M \\
\nearrow & & & & \nearrow & & \searrow & & & & \searrow \\
0 & & & & 0 & & 0 & & & & 0
\end{array}
$$

Let Q be a C-projective R-module. Applying $\operatorname{Hom}_S(-, Q)$ to the above commutative diagram and using our assumption we get that $\operatorname{Hom}_S(\mathbb{P}_M, Q)$ is exact. Therefore, M is a $\mathcal{SWGC}$-projective module.

$(1) \Rightarrow (2)$ Let Q be a C-projective module. By definition M has a complete $\mathcal{P}_C(S)$-resolution

$$ \cdots \xrightarrow{d} C \otimes_R P \xrightarrow{d} C \otimes_R P \xrightarrow{d} C \otimes_R P \xrightarrow{d} \cdots \tag{3.2} $$

such that $M = \ker d$. From the short exact sequence

$$ 0 \longrightarrow M \longrightarrow C \otimes_R P \longrightarrow M \longrightarrow 0 \tag{3.3} $$

we have the following long exact sequence:

$$
\begin{aligned}
\cdots \to \operatorname{Ext}^i_S(C, C \otimes_R P) \longrightarrow \operatorname{Ext}^i_S(C, M) &\longrightarrow \operatorname{Ext}^{i+1}_S(C, M) \\
&\longrightarrow \operatorname{Ext}^{i+1}_S(C, C \otimes_R P) \to \cdots .
\end{aligned}
$$

By Remark 2.5 we have $\mathcal{P}_C(S) \subseteq \mathcal{B}_C(S)$ and so, for each natural integer i, we have $\mathrm{Ext}_S^{\geq 1}(C, C \otimes_R P) = 0$. This means that $\mathrm{Ext}_S^{i+1}(C, M) \cong \mathrm{Ext}_S^{i}(C, M)$, for all $i \geq 1$. Since $\mathrm{pd}({}_SC)$ is finite, then $\mathrm{Ext}_S^{i}(C, M)$ vanishes for large values of i and so $\mathrm{Ext}_S^{\geq 1}(C, M) = 0$. Therefore, from the short exact sequence (3.2), we obtain the following short exact sequence:

$$0 \longrightarrow \mathrm{Hom}_S(C, M) \longrightarrow \mathrm{Hom}_S(C, C \otimes_R P) \longrightarrow \mathrm{Hom}_S(C, M) \longrightarrow 0.$$

This means that $C \otimes_R P \longrightarrow M \longrightarrow 0$ is a surjective $\mathcal{P}_C(S)$-percover of M and so

$$\cdots \xrightarrow{\ d\ } C \otimes_R P \xrightarrow{\ d\ } C \otimes_R P \xrightarrow{\ d\ } M \longrightarrow 0$$

is a left $\mathcal{P}_C(S)$-resolution of M. Since, for an arbitrary C-projective module Q, $\mathrm{Hom}_S((3.2), Q)$ is an exact complex then $\mathrm{Ext}_{\mathcal{P}_C(S)}^{\geq 1}(M, Q) = 0$, as desired.

(2) $\Rightarrow$ (3) As discussed above $\mathrm{Ext}_S^{\geq 1}(C, M) = 0$. Therefore, for an arbitrary projective R-module T, from the short exact sequence (3.3) we obtain the following short exact sequence

$$0 \to \mathrm{Hom}_S(C \otimes_R T, M) \to \mathrm{Hom}_S(C \otimes_R T, C \otimes_R P) \to \mathrm{Hom}_S(C \otimes_R T, M) \to 0.$$

By Theorem 2.9, $\mathcal{P}_C(S)$ is precovering. Hence, by [5, Theorem 8.2.3], we have the following long exact sequence:

$$\begin{aligned} \cdots \to \mathrm{Ext}^{i}_{\mathcal{P}_C(S)}(C \otimes_R P, Q) \longrightarrow \mathrm{Ext}^{i}_{\mathcal{P}_C(S)}(M, Q) &\longrightarrow \mathrm{Ext}^{i+1}_{\mathcal{P}_C(S)}(M, Q) \\ &\longrightarrow \mathrm{Ext}^{i+1}_{\mathcal{P}_C(S)}(C \otimes_R P, Q) \to \cdots . \end{aligned}$$

Therefore, for each $i \geq 1$,

$$\begin{aligned} \mathrm{Ext}^{i}_{\mathcal{P}_C(S)}(M, Q) &\cong \mathrm{Ext}^{i+1}_{\mathcal{P}_C(S)}(M, Q) \\ &\cong \mathrm{Ext}^{i+1}_{R}(\mathrm{Hom}_S(C, M), \mathrm{Hom}_S(C, Q)) \end{aligned}$$

where the last isomorphism is true by Lemma 3.6. Since $\mathrm{id}({}_R\mathrm{Hom}_S(C, Q))$ is finite, then the last modules vanish for large values of i. Hence, $\mathrm{Ext}^{\geq 1}_{\mathcal{P}_C(S)}(M, Q) = 0$ and we are done.

(3) $\Rightarrow$ (4) Form the short exact sequence (3.3) we have the commutative diagram

$$\begin{array}{ccccccccc} & & C \otimes_R \mathrm{Hom}_S(C, M) & \longrightarrow & C \otimes_R P & \longrightarrow & C \otimes_R \mathrm{Hom}_S(C, M) & \longrightarrow & 0 \\ & & \big\downarrow {\scriptstyle \mu_{CCM}} & & \big\| & & \big\downarrow {\scriptstyle \mu_{CCM}} & & \\ 0 & \longrightarrow & M & \longrightarrow & C \otimes_R P & \longrightarrow & M & \longrightarrow & 0 \end{array}$$

which implies that μ_{CCM} is surjective and so, by snake lemma, μ_{CCM} will be an isomorphism. Therefore, by Remark 2.10, we deduce that $\mathrm{Ext}^{0}_{\mathcal{P}_C(S)}(M, N) \cong \mathrm{Hom}_S(M, N)$, for every R-module N. By the proof of the implication (2) $\Rightarrow$ (3) we can write long exact sequence for the contravariant functors $\{\mathrm{Ext}^{i}_{\mathcal{P}_C(S)}(-, Q)\}_{i \geq 0}$ with respect to the short exact sequences of the form $0 \longrightarrow M \longrightarrow C \otimes_R P \longrightarrow M \longrightarrow 0$, where P is a projective R-module and we are done.

(4) $\Rightarrow$ (5) Evident. □

Corollary 3.9. *Assume that S is a left Noetherian ring and $\mathrm{id}({}_SS) \leq n$ for some non-negative integer n. If $\mathrm{fd}({}_SC)$ is finite, then, for an S-module M, the statements (1)-(5) of Theorem 3.8 are equivalent.*

Proof. By [5, Proposition 9.1.2], we have $\mathrm{pd}({}_SC)$ is finite. Now, the proof proceeds as it was done in the proof of Theorem 3.8. □

The following Corollary provides some class of S-modules at which the statements (1)-(5) of Theorem 3.8 are equivalent for a semidualizing module C.

Corollary 3.10. *Let M be an S-module. If $\mathrm{id}({}_SM)$ is finite or $M \in \mathcal{P}_C(S)^{\perp 1}$ then the statements (1)-(5) of Theorem 3.8 are equivalent for M.*

Proof. According to the implication $(1) \Rightarrow (2)$ in the proof of Theorem 3.8, for all $i \geq 1$, we have $\mathrm{Ext}^i_S(C, M) \cong \mathrm{Ext}^{i+1}_S(C, M)$. In both cases, our assumptions imply that $\mathrm{Ext}^{\geq 1}_S(C, M) = 0$ which, by Theorem 2.9 and [5, Theorem 8.2.3], allows us to write long exact sequence for the contravariant functors $\{\mathrm{Ext}^i_{\mathcal{P}_{C(S)}}(-, N)\}_{i\geq 0}$ with respect to the short exact sequences of the form $0 \longrightarrow M \longrightarrow C \otimes_R P \longrightarrow M \longrightarrow 0$, where P is a projective R-module. Now, the proof proceed as it was done in the proof of Theorem 3.8. □

Corollary 3.11. *Every C-projective module is $\mathcal{SWGC}$-projective. In particular, C is a $\mathcal{SWGC}$-projective module.*

Proof. First, we show that each C-projective module belongs to the class $\mathcal{P}_C(S)^{\perp i}$, for each natural integer i. Let P, Q be arbitrary projective R-modules and choose K in a way that $Q \oplus K \cong S^{(\Lambda)}$. Then

$$\begin{aligned} \mathrm{Ext}^i_S(C \otimes_R Q, C \otimes_R P) \oplus \mathrm{Ext}^i_S(C \otimes_R K, C \otimes_R P) &\cong \mathrm{Ext}^i_S(C \otimes_S S^{(\Lambda)}, C \otimes_R P) \\ &\cong \prod_{\lambda\in\Lambda} \mathrm{Ext}^i_S(C, C \otimes_R P) \\ &\cong 0, \end{aligned}$$

where the last equality is true by Remark 2.5. Now, the result follows from the split short exact sequence

$$0 \longrightarrow C \otimes_R P \longrightarrow (C \otimes_R P) \oplus (C \otimes_R P) \longrightarrow C \otimes_R P \longrightarrow 0,$$

and Corollary 3.10, as desired. □

The following Theorem is a generalization of the fact that "*A Gorenstein projective module of finite projective dimension is projective*" (see [5, Proposition 10.2.3]).

Theorem 3.12. *A $\mathcal{SWGC}$-projective module is C-projective if and only if its $\mathcal{P}_C(S)$-projective dimension is finite. In other words, the equality $\mathcal{SWGC}(S) \bigcap \overline{\mathcal{P}_C(S)} = \mathcal{P}_C(S)$ holds where $\mathcal{SWGP}(S)$ is the class of $\mathcal{SWGC}$-projective modules.*

Proof. Let M be a $\mathcal{SWGC}$-projective S-module such that $\mathcal{P}_C(S)$-pd(M) is finite. By Theorem 3.8, there exists the short sequence $0 \longrightarrow M \longrightarrow C \otimes_R P \longrightarrow M \longrightarrow 0$ where P is a projective R-module. By Theorem 2.9, $M \in \mathcal{B}_C(S)$ and so $\mathrm{Ext}^{\geq 1}_S(C, M) = 0$. Therefore, as discussed in the implication $(2) \Rightarrow (3)$ of the proof of Theorem 3.8, for an arbitrary S-module N, we have the following long exact sequence:

$$\begin{aligned} \cdots \longrightarrow \mathrm{Ext}^i_{\mathcal{P}_{C(S)}}(C \otimes_R P, N) \longrightarrow \mathrm{Ext}^i_{\mathcal{P}_{C(S)}}(M, N) &\longrightarrow \mathrm{Ext}^{i+1}_{\mathcal{P}_{C(S)}}(M, N) \\ &\longrightarrow \mathrm{Ext}^{i+1}_{\mathcal{P}_{C(S)}}(C \otimes_R P, N) \longrightarrow \cdots. \end{aligned}$$

As $\mathrm{Ext}^{\geq 1}_{\mathcal{P}_{C(S)}}(C \otimes_R P, N) = 0$, then $\mathrm{Ext}^i_{\mathcal{P}_{C(S)}}(M, N) \cong \mathrm{Ext}^{i+1}_{\mathcal{P}_{C(S)}}(M, N)$, for all $i \geq 1$. Since $\mathcal{P}_C(S)$-pd(M) is finite, then $\mathrm{Ext}^i_{\mathcal{P}_{C(S)}}(M, N) = 0$ for large values of i. This implies that $\mathrm{Ext}^{\geq 1}_{\mathcal{P}_{C(S)}}(M, N) = 0$. As discussed in the implication $(3) \Rightarrow (4)$ of Theorem 3.8, we have $\mathrm{Ext}^0_{\mathcal{P}_{C(S)}}(M, N) \cong \mathrm{Hom}_S(M, N)$. Thinking of the fact that N is an arbitrary S-module, we deduce that $0 \longrightarrow M \longrightarrow C \otimes_R P \longrightarrow M \longrightarrow 0$ splits. Now, Theorem 2.9(1), implies that M is C-projective, as desired. □

The next theorem characterizes the $\mathcal{SWGC}$-injectivity of an R-module N in terms of vanishing of cohomology modules $\{\mathrm{Ext}^i_{\mathcal{I}_{C(R)}}(\mathcal{I}_C(R), N)\}_{i\geq 1}$.

Theorem 3.13. *For an R-module N consider the following statements:*

(i) N is $\mathcal{SWGC}$-injective;

(ii) there exists a C-injective module, $\mathrm{Hom}_S(C, I)$ say, such that the sequence $0 \longrightarrow N \longrightarrow \mathrm{Hom}_S(C, I) \longrightarrow N \longrightarrow 0$ is exact and $\mathrm{Ext}^{\geq 1}_{\mathcal{I}_{C(R)}}(J, N) = 0$ for each C-injective module J;

(iii) there exists a C-injective module, $\mathrm{Hom}_S(C,I)$ *say, such that the sequence* $0 \longrightarrow N \longrightarrow \mathrm{Hom}_S(C,I) \longrightarrow N \longrightarrow 0$ *is exact and* $\mathrm{Ext}^{\geq 1}_{\mathcal{I}_C(R)}(J,N) = 0$ *whenever J is C-injective or* $\mathrm{pd}({}_SC \otimes_R J) < \infty$*;*

(iv) there exists a C-injective module, $\mathrm{Hom}_S(C,I)$ *say, such that the sequence* $0 \longrightarrow N \longrightarrow \mathrm{Hom}_S(C,I) \longrightarrow N \longrightarrow 0$ *is exact and* $\mathrm{Hom}_S(J,-)$ *leaves it exact whenever, J is C-injective or* $\mathrm{pd}({}_SC \otimes_R J) < \infty$*;*

(v) there exists a C-injective module, $\mathrm{Hom}_S(C,I)$ *say, such that the sequence* $0 \longrightarrow N \longrightarrow \mathrm{Hom}_S(C,I) \longrightarrow N \longrightarrow 0$ *is exact and for each C-injective module J the sequence*

$$0 \longrightarrow \mathrm{Hom}_R(J,N) \longrightarrow \mathrm{Hom}_R(J,\mathrm{Hom}_S(C,I)) \longrightarrow \mathrm{Hom}_R(J,N) \longrightarrow 0$$

is exact, too.

Then $(1) \Leftrightarrow (5)$ *and if* $\mathrm{fd}(C_R)$ *is finite, then* (1)-(5) *are equivalent.*

Proof. $(1) \Rightarrow (5)$ Is obvious.

$(5) \Rightarrow (1)$ It proceeds as the implication $(5) \Rightarrow (1)$ in the proof of Theorem 3.8.

$(1) \Rightarrow (2)$ By definition N has a complete $\mathcal{I}_C(R)$-resolution

$$\cdots \xrightarrow{d} \mathrm{Hom}_S(C,I) \xrightarrow{d} \mathrm{Hom}_S(C,I) \xrightarrow{d} \mathrm{Hom}_S(C,I) \xrightarrow{d} \cdots \tag{3.4}$$

such that $N \cong \ker(d)$. From the short exact sequence

$$0 \longrightarrow N \longrightarrow \mathrm{Hom}_R(C,I) \longrightarrow N \longrightarrow 0 \tag{3.5}$$

we have the following long exact sequence:

$$\cdots \to \mathrm{Tor}^R_{i+1}(C,\mathrm{Hom}_R(C,I)) \longrightarrow \mathrm{Tor}^R_{i+1}(C,N) \longrightarrow \mathrm{Tor}^R_i(C,N) \\ \longrightarrow \mathrm{Tor}^R_i(C,\mathrm{Hom}_R(C,I)) \to \cdots.$$

By Remark 2.5, we have $\mathrm{Tor}^R_{\geq 1}(C,\mathrm{Hom}_S(C,I)) = 0$. Therefore form the above long exact sequence it is concluded that $\mathrm{Tor}^R_{i+1}(C,N) \cong \mathrm{Tor}^R_i(C,N)$. Since $\mathrm{fd}(C_R)$ is finite then $\mathrm{Tor}^R_i(C,N)$ vanishes for large values of i and so $\mathrm{Tor}^R_{\geq 1}(C,N) = 0$. Let $J = \mathrm{Hom}_S(C,E)$ be a C-injective R-module. Then, from the short exact sequence (3.5) we obtain the short exact sequence $0 \to C \otimes_R N \to C \otimes_R I \to C \otimes_R N \to 0$, which leads (by injectivity of E and hom-tensor adjoint isomorphism) to the short exact sequence $0 \to \mathrm{Hom}_R(N,J) \to \mathrm{Hom}_R(I,J) \to \mathrm{Hom}_R(N,J) \to 0$. This means that $0 \to N \to I$ is an $\mathcal{I}_C(R)$-preenvelope of N and so $0 \longrightarrow N \xrightarrow{\text{inc}} I \xrightarrow{d} I \xrightarrow{d} \cdots$ is a right $\mathcal{I}_C(R)$-resolution of N. Since $\mathrm{Hom}_R(J,(3.4))$ is an exact complex, then $\mathrm{Ext}^{\geq 1}_{\mathcal{I}_C(R)}(J,N) = 0$, as desired.

$(2) \Rightarrow (3)$ Let $J = \mathrm{Hom}_S(C,E)$ be a C-injective R-module. For an arbitrary R-module A and for large values of i, by finiteness of $\mathrm{fd}(C_R)$, we have $\mathrm{Tor}^R_i(C,A) = 0$. By [5, Theorem 3.2.1], we have

$$\mathrm{Ext}^i_R(A,J) \cong \mathrm{Ext}^i_R(A,\mathrm{Hom}_S(C,E)) \cong \mathrm{Hom}_S(\mathrm{Tor}^R_i(C,A),E).$$

Therefore, $\mathrm{Ext}^i_R(A,J)$ vanishes for large values of i, which in turn implies that $\mathrm{id}({}_RJ)$ is finite. Now, from the short exact sequence (3.5), we have the following long exact sequence:

$$\cdots \to \mathrm{Ext}^i_R(\mathrm{Hom}_{_S}(C,I),J) \to \mathrm{Ext}^i_R(N,J) \to \mathrm{Ext}^{i+1}_R(N,J) \\ \to \mathrm{Ext}^{i+1}_R(\mathrm{Hom}_{_S}(C,I),J) \to \cdots.$$

For an arbitrary natural integer i we have

$$\mathrm{Ext}^i_R(\mathrm{Hom}_S(C,I),J) \cong \mathrm{Hom}_S\Big(\mathrm{Tor}^R_i(C,\mathrm{Hom}_S(C,I)),E\Big) \cong 0,$$

where the first isomorphisms is true by [5, Theorem 3.2.1] and the second equality holds by Remark 2.5. Therefore, $\mathrm{Ext}^i_R(N,J) \cong \mathrm{Ext}^{i+1}_R(N,J)$ for all integers $i \geq 1$. As $\mathrm{id}({}_RJ)$ is finite, then $\mathrm{Ext}^{\geq 1}_R(N,J) = 0$. Now, by [5, Theorem 8.2.5], we have the following long exact sequence:

$$\cdots \to \mathrm{Ext}^i_{\mathcal{I}_C(R)}(J, \mathrm{Hom}_S(C,I)) \to \mathrm{Ext}^i_{\mathcal{I}_C(R)}(J,N) \to \mathrm{Ext}^{i+1}_{\mathcal{I}_C(R)}(J,N) \to \mathrm{Ext}^{i+1}_{\mathcal{I}_C(R)}(J, \mathrm{Hom}_S(C,I)) \to \cdots$$

Since $\mathrm{Ext}^{\geq 1}_{\mathcal{I}_C(R)}(J, \mathrm{Hom}_S(C,I)) = 0$, then $\mathrm{Ext}^i_{\mathcal{I}_C(R)}(J,N) \cong \mathrm{Ext}^{i+1}_{\mathcal{I}_C(R)}(J,N)$, for each $i \geq 1$. By Lemma 3.6, $\mathrm{Ext}^i_{\mathcal{I}_C(R)}(J,N) \cong \mathrm{Ext}^i_S(C \otimes_R J, C \otimes_R N)$ and by assumption $\mathrm{Ext}^i_S(C \otimes_R J, C \otimes_R N)$ vanishes for large values of i, then $\mathrm{Ext}^{\geq 1}_{\mathcal{I}_C(R)}(J,N) = 0$, as desired.

(3) $\Rightarrow$ (4) As proved in the implication (2) $\Rightarrow$ (3) from the short exact sequence (3.5) we have the short exact sequence

$$0 \longrightarrow C \otimes_R N \longrightarrow I \longrightarrow C \otimes_R N \longrightarrow 0,$$

that leads to the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & N & \longrightarrow & \mathrm{Hom}_S(C,I) & \longrightarrow & N & \longrightarrow & 0 \\ & & \downarrow \nu_{CCN} & & \| & & \nu_{CCN} \downarrow & & \\ 0 & \longrightarrow & \mathrm{Hom}_S(C, C \otimes_R N) & \longrightarrow & \mathrm{Hom}_S(C,I) & \longrightarrow & \mathrm{Hom}_S(C, C \otimes_R N) & & \end{array}$$

which in turn implies that ν_{CCN} is an injection and so, by snake lemma, ν_{CCN} will be an isomorphism. Then, by Remark 2.10, we deduce that $\mathrm{Ext}^0_{\mathcal{I}_C(R)}(J,N) \cong \mathrm{Hom}_R(J,N)$. By implication (2) $\Rightarrow$ (3), we can write long exact sequence for the covariant functors $\{\mathrm{Ext}^i_{\mathcal{I}_C(R)}(N,-)\}_{i \geq 0}$, with respect to the short exact sequence (3.5). This makes every thing obvious.

(4) $\Rightarrow$ (5) It is Obvious. □

The following corollary provides some class of R-modules that, for an arbitrary semidualizing module C, the statements (1)-(5) of Theorem 3.13 are equivalent.

Corollary 3.14. *Let N be an R-module. If $N \in {}^{1\perp}\mathcal{I}_C(R)$ or $\mathrm{pd}({}_RN)$ is finite then the statements (1)-(5) of Theorem 3.13 are equivalent for N.*

Proof. Let I be a C-injective R-module. For each C-injective module J, from the short exact sequence $0 \longrightarrow N \longrightarrow I \longrightarrow N \longrightarrow 0$ we get the following long exact sequence:

$$\cdots \to \mathrm{Ext}^i_R(I,J) \to \mathrm{Ext}^i_R(N,J) \to \mathrm{Ext}^{i+1}_R(N,J) \to \mathrm{Ext}^{i+1}_R(I,J) \to \cdots.$$

By [5, Theorem 3.2.1] and Remark 2.5, it is concluded that $\mathrm{Ext}^{\geq 1}_R(I,J) = 0$ and so, for all $i \geq 1$, $\mathrm{Ext}^i_R(N,J) \cong \mathrm{Ext}^{i+1}_R(N,J)$. In both cases our assumption implies that $\mathrm{Ext}^{\geq 1}_S(N,I) = 0$, at which, by Theorem 2.9(2) and [5, Theorem 8.2.5], allows us to write long exact sequences for the covariant functors $\{\mathrm{Ext}^i_{\mathcal{I}_C(R)}(N,-)\}_{i \geq 0}$ with respect to the short exact sequences of the form $0 \longrightarrow N \longrightarrow \mathrm{Hom}_S(C,E) \longrightarrow N \longrightarrow 0$, where E is an S-injective module. Now, the result follows as in the proof of Theorem 3.13. □

Corollary 3.15. *Every C-injective module is $\mathcal{SWGC}$-injective.*

Proof. Let E and E' be S-injective modules. By [5, Theorem 3.2.1] and Remark 2.5, we have $\mathrm{Ext}^{\geq 1}_R(\mathrm{Hom}_S(C,E), \mathrm{Hom}_S(C,E')) = 0$. So, $\mathcal{I}_C(R) \subseteq {}^{1\perp}\mathcal{I}_C(R)$. Now, for an S-injective module I, the result follows from the split short exact sequence

$$0 \longrightarrow \mathrm{Hom}_S(C,I) \longrightarrow \mathrm{Hom}_S(C,I) \oplus \mathrm{Hom}_S(C,I) \longrightarrow \mathrm{Hom}_S(C,I) \longrightarrow 0$$

and Corollary 3.14, as desired. □

The following theorem is a generalization of the fact that "*A Gorenstein injective module of finite injective dimension is injective*" (see [5, Proposition 10.1.2]).

Theorem 3.16. *A $\mathcal{SWGC}$-injective module is C-injective if and only if its $\mathcal{I}_C(R)$-injective dimension is finite. In other words, the equality $\mathcal{SWGC}(R)\bigcap\overline{\mathcal{I}_C(R)} = \mathcal{I}_C(R)$ holds where $\mathcal{SWGC}(R)$ is the class of $\mathcal{SWGC}$-injective modules.*

Proof. Let N be a $\mathcal{SWGC}$-injective R-module such that $\mathcal{I}_C(R)$-id(N) is finite. By Theorem 3.13, there exists an S-injective module I such that the sequence

$$0 \longrightarrow N \longrightarrow \mathrm{Hom}_S(C, I) \longrightarrow N \longrightarrow 0 \tag{3.6}$$

is exact. By Theorem 2.9, we have $N \in \mathcal{A}_C(R)$ and so $0 \longrightarrow C \otimes_R N \longrightarrow I \longrightarrow C \otimes_R N \longrightarrow 0$ is exact. Therefore, for an arbitrary S-injective module E, by the hom-tensor adjoint isomorphism, we get the short exact sequence

$$0 \longrightarrow \mathrm{Hom}_R(N, \mathrm{Hom}_S(C, E)) \longrightarrow \mathrm{Hom}_S(I, E) \longrightarrow \mathrm{Hom}_R(N, \mathrm{Hom}_S(C, E)) \longrightarrow 0,$$

where, by using the short exact sequence (3.6) and the fact that $\mathcal{I}_C(R) \subseteq^{1\perp}\mathcal{I}_C(R)$, implies that $\mathrm{Ext}^1_R(N, \mathrm{Hom}_S(C, E)) = 0$. By the proof of Corollary 3.14, for an arbitrary R-module T and for each natural integer i, we have $\mathrm{Ext}^i_{\mathcal{I}_C(R)}(T, N) \cong \mathrm{Ext}^{i+1}_{\mathcal{I}_C(R)}(T, N)$. Since $\mathcal{I}_C(R)$-id(N) is finite, then $\mathrm{Ext}^i_{\mathcal{I}_C(R)}(T, N)$ vanishes for large values of i. Thus $\mathrm{Ext}^{\geq 1}_{\mathcal{I}_C(R)}(T, N) = 0$. It was discussed in the implication $(3) \Rightarrow (4)$ of Theorem 3.13, that $\mathrm{Ext}^0_{\mathcal{I}_C(R)}(T, N) \cong \mathrm{Hom}_R(T, N)$. Since T is arbitrary, then it is deduced that the short exact sequence $0 \longrightarrow N \longrightarrow \mathrm{Hom}_S(C, I) \longrightarrow N \longrightarrow 0$ splits. Therefore, by Theorem 2.9(2), we conclude that N is C-injective. □

In Theorem 3.8 it is proved that an S-module M is $\mathcal{SWGC}$-projective if and only if all of the cohomology modules $\{\mathrm{Ext}^i_{\mathcal{P}_C(S)}(M, \mathcal{P}_C(S))\}_{i\geq 1}$ vanish. It is natural to ask what is the result of vanishing of cohomology modules $\{\mathrm{Ext}^i_{\mathcal{P}_C(S)}(M, \mathcal{F}_C(S))\}_{i\geq 1}$. The next theorem explore this question under some circumferences.

Theorem 3.17. *Let $S = R$ be Noetherian rings and let M be an S-module. If* pd$({}_SM)$ *or* id$({}_SC)$ *is finite, then the following statements are equivalent:*

(i) M is a finitely generated $\mathcal{SWGC}$-projective module;

(ii) there exists a finitely generated projective module P, such that the sequence

$$0 \longrightarrow M \longrightarrow C \otimes_S P \longrightarrow M \longrightarrow 0$$

is exact and $\mathrm{Ext}^{\geq 1}_{\mathcal{P}_C(S)}(M, C) = 0$*;*

(iii) there exists a finitely generated projective module P, such that the sequence

$$0 \longrightarrow M \longrightarrow C \otimes_S P \longrightarrow M \longrightarrow 0$$

is exact and $\mathrm{Ext}^{\geq 1}_{\mathcal{P}_C(S)}(M, C \otimes_S F) = 0$*, for all flat modules F.*

Proof. By Theorem 3.8, there exists an S-projective module P such that the sequence

$$0 \longrightarrow M \longrightarrow C \otimes_S P \longrightarrow M \longrightarrow 0 \tag{3.7}$$

is exact. Since M is a finitely generated module then, by exactness of (3.7), left exactness of $\mathrm{Hom}_S(C, -)$ and Remark 2.1, one deduces that $P \cong \mathrm{Hom}_S(C, C \otimes_S P)$ is finitely generated. By Lazard's Theorem [10, Theorem 5.40], there exists a family $\{F_i\}_{i\in I}$, of finitely generated free modules, such that $F \cong \varinjlim_{i\in I} F_i$. Then,

$$\begin{aligned}
\mathrm{Ext}^{\geq 1}_{\mathcal{P}_C(S)}(M, C \otimes_S F) &\cong \mathrm{Ext}^{\geq 1}_S(\mathrm{Hom}_S(C, M), \mathrm{Hom}_S(C, C \otimes_S F)) \\
&\cong \mathrm{Ext}^{\geq 1}_S(\mathrm{Hom}_S(C, M), \varinjlim_{i\in I} \mathrm{Hom}_S(C, C \otimes_S F_i)) \\
&\cong \varinjlim_{i\in I} \mathrm{Ext}^{\geq 1}_S(\mathrm{Hom}_S(C, M), \mathrm{Hom}_S(C, C \otimes_S F_i)) \\
&\cong \varinjlim_{i\in I} \mathrm{Ext}^{\geq 1}_{\mathcal{P}_C(S)}(M, C \otimes_S F_i),
\end{aligned}$$

where the first isomorphism is true by Lemma 3.6, the second and third by [5, Lemma 3.1.16]. Now, by the proof of Theorem 3.8, every thing is evident. □

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