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On (n, p)-clean commutative rings and n-almost clean rings

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Abstract Let R be a commutative ring with identity. A ring R is called n-Clean if every element of R can be written as a sum of idempotent and n units in R. The class of n-clean rings contains Clean rings (ie every element can be written as a sum of a unit and an idempotent). This notion of n-cleaness first appared in [14]. We say that a ring R is almost clean if every element can be written as a sum of a unit and an regular element. In this paper, we introduce the new notion of (n, p)-Clean rings and n-almost clean rings. Next, we investigate some properties of such rings, and then generate new and original families of rings with these properties.

1 Introduction

Throughout this paper, all rings are commutative with unity. For a ring R, $Id(R)$, $U(R)$, $Rad(R)$ and $Reg(R)$ are denote the set of idempotents of R , the set of units of R, the Jacobson radical of R and the set of regular elements of R, respectively. Nicholson [9] defined a ring R to be clean if every element of R can be written as a sum of a unit and an idempotent. Recently, this class of rings is studied extensively in literatures see for example, [8], [13] and [1]. According to Xiao and Tong [14], a ring R is called n-clean if every element can be written as a sum of n-units and an idempotent. Following McGovern [4], we say that a ring R is almost clean if, for each $x \in R$, x can be written as $x = r + e$ where $r \in Reg(R)$ and $e \in Idem(R)$. Almost clean rings have been studied in [12]. In [12], it is shown that a commutative Rickart ring is almost clean. Up to date, almost cleanness has been considered mostly for commutative rings. In [10], various classes of almost clean rings that are not necessarily commutative are given. In this paper, we introduce the new notion of (n, p)-Clean rings and we extend some results on n-clean rings to (n, p)-clean rings. Next we generalize the notion of almost clean rings.

2 (n, p)-Clean rings

Definition 2.1. Let n and p two positive integers $(p \geq 2)$. A ring R is said (n, p)-clean if every element $a \in R$ can be written in the form $a = u_1 + u_2 + ... + u_n + x$ where $u_i \in U(R)$ $(i = 1, ...n)$ and $x^p = x$.

Note that clean rings are (1,2)-clean rings and n-clean rings are (n,2)-clean rings. However, for $(p \geq 2)$, (n, p)-clean rings need not be clean rings, as shown by the following example.

Example 2.2. $\mathbb{Z}_{15}$ is a (1,3)-clean which is not clean.

Proof. One can easily verified that $U(\mathbb{Z}_{15}) = \{\overline{1}, \overline{2}\,\overline{4}, \overline{7}, \overline{8}, \overline{11}, \overline{14}\}$, $Id(\mathbb{Z}_{15}) = \{\overline{0}, \overline{1}\}$ and $\{x \in \mathbb{Z}_{15}/x^3 = x\}$=$\{\overline{0}, \overline{1}, \overline{4}, \overline{5}, \overline{6}, \overline{9}, \overline{10}, \overline{11}, \overline{14}\}$. Hence each $a \in \mathbb{Z}_{15}$ can be written in the form $a = u + x$ where $u \in U(\mathbb{Z}_{15})$ and $x^3 = x$. Consequently $\mathbb{Z}_{15}$ is a (1,3)-clean which is not clean. □

Proposition 2.3. *Let n and p two positive integers $(p \geq 2)$. Then a ring R is (n, p)-clean if and only if every element $a \in R$ has the form $a = u_1 + u_2 + ... + u_n - x$ where $u_i \in U(R)$ $(i = 1, ...n)$ and $x^p = x$.*

Proof. Let $a \in R$. Since R is (n, p)-clean, we have $-a = v_1 + v_2 + ... + v_n + x$ where $v_i \in U(R)$ $(i = 1, ...n)$ and $x^p = x$. Hence, $a = u_1 + u_2 + ... + u_n - x$ where $u_i = -v_i \in U(R)$

$(i = 1, ...n)$ and $x^p = x$. Conversely, let $a \in R$. Then $-a = u_1 + u_2 + ... + u_n - x$ where $u_i \in U(R)$ $(i = 1, ...n)$ and $x^p = x$. Hence, $a = (-u_1) + (-u_{2)} + ... + (-u_n) + x$ where $(-u_i) \in U(R)$ $(i = 1, ...n)$ and $x^p = x$. Thus R is (n, p)-clean. □

It is known that homomorphic images of n-clean rings are n-clean (see [14]). For (n, p)-clean rings we have the following:

Proposition 2.4. *Let n and p two positive integers* $(p \geq 2)$. *Then every homomorphic image of an* (n, p)*-clean ring is* (n, p)*-clean . In particular, every homomorphic image of a n-clean ring is n-clean.*

Proof. Let R be an (n, p)-clean ring and let $\varphi : R \to S$ be a ring epimorphism. Let $x \in S$. Then $x = \varphi(y)$ for some $y \in R$. Since R is an (n, p)-clean , then $y = u_1 + u_2 + ... + u_n + x$ where $u_i \in U(R)$ $(i = 1, ...n)$ and $x^p = x$. That is, $x = \varphi(y) = \varphi(u_1) + \varphi(u_2) + ... + \varphi(u_n) + \varphi(x)$. Since φ is an epimorphism, we then have that $\varphi(u_i) \in U(S)$ and $\varphi(x)^p = \varphi(x)$. It follows that $S = \varphi(R)$ is an (n, p)-clean. □

We now consider direct products. For (n, p)-clean rings, we have the following:

Proposition 2.5. *Let n and p two positive integers* $(p \geq 2)$. *The direct product ring* $\prod_{i \in I} R_i$ *is* (n, p)*-clean if and only if each* R_i *is* (n, p)*-clean.*

Proof. One direction immediately follows from proposition 2.4 (since R_i is a homomorphic image of $\prod_{i \in I} R_i$ (via the natural projection :$\Pi_i : \prod_{i \in I} R_i \to R_i$) . Conversely, suppose that each R_i is an (n, p)-clean ring. Let $a = (a_i) \in \prod_{i \in I} R_i$. Then for each i, $a_i = u_{i1} + u_{i2} + ... + u_{in} + x_i$ where $u_{ij} \in U(R_i)$ $(j = 1, ...n)$ and $x_i^p = x_i$. Thus, $a = (a_i) = (u_{i1}) + (u_{i2}) +, ... + (u_{in}) + (x_i)$ with $(u_{ij} \in U(\prod_{i \in I} R_i)$ for $(j = 1, ...n)$ and $(x_i)^p = (x_i)$. Hence, $\prod_{i \in I} R_i$ is (n, p)-clean. □

Polynomial rings over (n, p)-clean rings are not necessarily (n, p)-clean. For example, the ring $\mathbb{Z}_2$ is $(1, 2)$- clean but the polynomial ring $\mathbb{Z}_2[x]$ is not $(1, 2)$-clean. However, power series ring over n -weakly clean rings are (n, p)-clean as shown in the following:

Proposition 2.6. *Let n and p two positive integers* $(p \geq 2)$. *Then the power series ring* $R[[x]]$ *is* (n, p)*-clean if and only if* R *is* (n, p)*-clean.*

Proof. Suppose that $R[[x]]$ is (n, p)-clean. Then it follows by the isomorphism $R \cong R[[x]]/(x)$ and Proposition 2.4 that R is an (n, p)-clean ring. Conversely, suppose that R is (n, p)-clean. Let $f = \sum_{i=0}^{+\infty} r_i x^i \in R[[x]]$. Since R is (n, p)-clean, we have that $a_0 = u_1 + u_2 + ... + u_n + e$ where $u_i \in U(R)$ $(i = 1, ...n)$ and $e^p = e$. Then $f = (u_1 + r_1 x + r_2 x^2 + ...) + u_2 + ... + u_n + e$ where $(u_1 + r_1 x + r_2 x^2 + ...) \in U(R[[x]])$, $u_i \in U(R[[x]])$ $(i = 2, ...n)$ and $e \in R \subseteq R[[x]]$. Thus, $R[[x]]$ is an (n,p)-clean ring. □

For more examples of (n,p)-clean rings, we consider the method of trivial ring extension. Let R be a ring and E an R-module. The trivial ring extension of R by E is the ring $R \propto E$ with product with $r \in R$ and $e \in E$, under coordinatewise addition and under an adjusted defined by $(r, e)(r', e') = (rr', re' + r'e)$ for all $r, r' \in R$, $e, e' \in E$.

Theorem 2.7. *Consider n and p two positive integers* $(p \geq 2)$. *Let* R *be a ring and* E *an* R*-module. Then* $R \propto E$ *is (n,p)-clean if and only if R is an (n,p)-clean ring.*

Proof. Note that $R \cong R \propto E/0 \propto E$ is a homomorphic image of $R \propto E$. Hence if $R \propto E$ is (n, p)-clean, so by Proposition 2.4, R is (n, p)-clean. Conversely, racall that $1_{R \propto E} = (1, 0)$ and observe that if $u \in U(R)$, then $(u, e) \in U(R \propto E)$ for each $e \in E$ and if $x^p = x$ for each $x \in R$, then $(x, 0)^p = (x^p, 0) = (x, 0)$ in $R \propto E$. Hense if $a \in R$ with $a = u_1 + u_2 + ... + u_n + x$ where $u_i \in U(R)$ $(i = 1, ...n)$ and $x^p = x$, then for $e \in E$, $(a, e) = (u_1 + u_2 + ... + u_n + x, e) = (u_1, e) + (u_2, 0) + (u_3, 0) + ... + (x, 0)$ where $(u_1, e) \in U(R \propto E)$, $(u_i, 0) \in U(R \propto E)$ $(i = 2, ...n)$ and $(x, 0)^p = (x, 0)$. Thus if R is (n,p)-clean, so is $R \propto E$. □

Let A and B be two rings with unity, let J be an ideal of B and let $f : A \to B$ be a ring homomorphism. In this setting, we can consider the following subring of $A \times B$:

$$A \bowtie^f J := \{(a, f(a)+j) \mid a \in A, j \in J\}$$

called *the amalgamation of A with B along J with respect to f*. This construction has been introduced and studied in [3, 4], and it is a generalization of the amalgamated duplication of a ring along an ideal (introduced and studied in [5, 6]). Moreover, other classical constructions (such as the $A+XB[X]$, $A+XB[[X]]$, and the $D+M$ constructions) can be studied as particular cases of the amalgamation ([3, Examples 2.5 and 2.6]).

Theorem 2.8. *Consider n and p two positive integers ($p \geq 2$). Let $f : A \to B$ be a ring homomorphism and J an ideal of B such that $f(u)+j$ is invertible (in B) for each $u \in U(A)$ and $j \in J$. Then $A \bowtie^f J$ is (n,p)-clean if and only if A is an (n,p)-clean ring.*

Proof. Suppose that $A \bowtie^f J$ is (n,p)-clean. Then it follows by the isomorphism $A \cong A \bowtie^f J/(\{0\}) \times J$ and Proposition 2.4 that A is an (n,p)-clean ring. Conversely, assume that A is clean and $f(u)+j$ is invertible (in B) for each $u \in U(A)$ and $j \in J$. Consider $(a,j) \in A \times J$. Since A is clean, $a = u_1 + u_2 + ... + u_n + x$ where $u_i \in U(R)$ $(i = 1, ...n)$ and $x^p = x$. Moreover, $f(u_1)+j$ is invertible in B. Then, there exists $v \in B$ such that $(f(u_1)+j)v = 1$. Hence,

$$\begin{aligned}
(f(u_1)+j)(f(u_1^{-1}) - vf(u_1^{-1})j) &= f(u_1)f(u_1^{-1}) + jf(u_1^{-1}) - (f(u_1)+j)vf(u_1^{-1})j \\
&= 1 + jf(u_1^{-1}) - f(u_1^{-1})j \\
&= 1
\end{aligned}$$

Thus, $(u_1, f(u_1)+j)$ is invertible in $A \bowtie^f J$ (since $(u_1, f(u_1)+j)(u_1^{-1}, f(u_1^{-1}) - vf(u_1^{-1})j) = (1,1)$). Hence,

$$\begin{aligned}
(a, f(a)+j) &= (u_1 + u_2 + ... + u_n + x, f(u_1 + u_2 + ... + u_n + x) + j) \\
&= (u_1, f(u_1)+j) + (u_2, f(u_2)) + ... + (u_n, f(u_n)) + (x, f(x))
\end{aligned}$$

where $(u_1, f(u_1)+j) \in U(R \propto E)$, $(u_i, f(u_i) \in U(R \propto E)$ $(i = 2, ...n)$ and $(x, f(x))^p = (x, f(x))$. Consequently, $A \bowtie^f J$ is clean. □

Corollary 2.9. *Let $f : A \to B$ be a ring homomorphism and J an ideal of B such that $J \subseteq Rad(B)$. Then $A \bowtie^f J$ is (n,p)- clean if and only if A is(n,p)-clean.*

Corollary 2.10. *Let $A \subset B$ be an extension of commutative rings and $X := \{X_1, X_2, ..., X_n\}$ a finite set of indeterminates over B. Set the subring $A + XB[[X]] := \{h \in B[[X]] \mid h(0) \in A\}$ of the ring of power series $B[[X]]$. Then, $A + XB[[X]]$ is (n,p)-clean if and only if A (n,p)-is clean.*

Proof. By [3, Example 2.5], $A + XB[[X]]$ is isomorphic to $A \bowtie^\sigma J$, where $\sigma : A \hookrightarrow B[[X]]$ is the natural embedding and $J := XB[[X]]$. It is well known that $Rad(B[[X]]) = \{g \in B[[X]] \mid g(0) \in Rad(A)\}$. Thus, $J \subseteq Rad(B[[x]]$. Hence, by Corollary 2.9, $A \bowtie^\sigma J$ is (n,p)-clean if and only if A is (n,p)-clean. Thus, we have the desired result. □

Corollary 2.11. *Let T be a ring and $J \subseteq Rad(T)$ an ideal of T and let D be a subring of T such that $J \cap D = (0)$. The ring $D + J$ is (n,p)-clean if and only if D is (n,p)-clean.*

Proof. By [3, Proposition 5.1 (3)], $D + J$ is isomorphic to the ring $D \bowtie^\iota J$ where $\iota : D \hookrightarrow T$ is the natural embedding. Thus, by Corollary 2.9, $D + J$ is (n,p)-clean if and only if D is (n,p)-clean. □

3 n-almost clean rings

Definition 3.1. Let n be an integer $(n \geq 2)$. A ring R is said n-almost clean if every element $a \in R$ can be written in the form $a = r + x$ where $r \in Reg(R)$ and $x^n = x$.

Note that an almost clean ring is an n-almost clean ring for each $n \geq 2$. However, for $n \geq 3$, n-almost clean rings need not be almost clean, as shown by the following example.

Example 3.2. $\mathbb{Z}_{15}$ is a 3-almost clean which is not almost clean.

Proof. It is easily to see that $Id(\mathbb{Z}_{15}) = \{\overline{0}, \overline{1}\}$ and $Reg(\mathbb{Z}_{15}) = \{\overline{1}, \overline{2}, \overline{4}, \overline{7}, \overline{8}, \overline{11}, \overline{13}, \overline{14}\}$.
We have $\overline{6}$, $\overline{6} - \overline{1}$ is not in $Reg(\mathbb{Z}_{15})$. Thus $\mathbb{Z}_{15}$ is not an almost clean ring. While $\{x \in \mathbb{Z}_{15}/x^3 = x\}$=$\{\overline{0}, \overline{1}, \overline{4}, \overline{5}, \overline{6}, \overline{9}, \overline{10}, \overline{11}, \overline{14}\}$ and $\overline{3} = \overline{2} + \overline{1}$, $\overline{5} = \overline{4} + \overline{1}$, $\overline{6} = \overline{2} + \overline{4}$, $\overline{9} = \overline{13} + \overline{11}$, and $\overline{12} = \overline{14} + \overline{11}$,. Hence each $a \in \mathbb{Z}_{15}$ can be written in the form $a = r + x$ where $r \in Reg(\mathbb{Z}_{15})$ and $x^3 = x$. Consequently $\mathbb{Z}_{15}$ is 3-almost clean. □

Remark 3.3. From [2, Example 2.9] it is clear that an homomorphic image of an n-almost clean ring is not necessary an n-almost clean ring.

Proposition 3.4. *A direct product rings $\prod_{i\in I} R_i$ is n-almost clean if and only if each R_i is n-almost clean.*

Proof. Let $i \in I$ and consider $a \in R_i$. The element $(0, .., a, 0, ..., 0)$, which has 0 is all j'th place with $i \neq j$, can be written as $(0, .., a, 0, ..., 0) = (r_j)_j + (x_j)_j$ with $r_j)_j \in Reg(R)$ and $(x_j)_j^n = (x_j)_j$. Since $Reg(R) = \prod Reg(R_j)$, then $r_j \in Reg(R_j)$ for each j, . On the other hand, it is clear that $x_j^n = x_j$. Thus, $a = r_i + x_i$ where $r_i \in Reg(R_i)$ and $x_i^n = x_i$. Consequently, R_i is n-almost clean. Conversely, suppose that each R_i is n-almost clean. Let $(a_i)_i \in \prod_{i\in I} R_i$. Write $a_i = r_i + x_i$ where $r_i \in Reg(R_i)$ and $x_i^n = x_i$. Then, $(a_i)_i = (r_i)_i + (x_i)_i$ where $(r_i)_i \in \prod_{i\in I} Reg(R_i)$ and $(x_i)_i^n = (x_i)_i$. So, $\prod_{i\in I} R_i$ is an n-almost clean ring. □

Proposition 3.5. *If R a commutative ring is n-almost clean, then the power series ring $R[[x]]$ is also n-almost clean.*

Proof. Suppose that R is n-almost clean. Let $f \in R[[x]]$, so $f = f_0 + h$ where $f_0 \in R$ and $h \in \langle\{x\}\rangle$. Write $f_0 = r + e$ where $r \in Reg(R)$ and $e^n = e$. Then $f = (r + h) + e$ where $r + h \in Reg(R[[x]])$ and $e \in R \subseteq R[[x]]$. Thus $R[[x]]$ is n-almost clean. □

Given aring R, we set $\sqrt[n]{1} = \{x \in R/x^n = 1\}$. Racall that a ring R is called indecomposable if $Idem(R) = \{0, 1\}$.

Proposition 3.6. *Let R be an indecomposable ring and $n \geq 2$ be a integer. Then R is n-almost clean if and only if for each $x \in R \setminus Reg(R)$, $x - \alpha$ is regular for some $\alpha \in \sqrt[n-1]{1}$.*

Proof. Let $a \in R \setminus Reg(R)$. Write $a = r + x$ where $r \in Reg(R)$ and $x^n = x$. We have $(x^{n-1})^2 = x^{2n-2} = x^n x^{n-2} = xx^{n-2} = x^{n-1}$. Thus, x^{n-1} is an idempotent element of R. Moreover, $x^{n-1} \neq 0$. Otherwise, $x = x^n = x^{n-1}x = 0$, and so $a = r$ is regular, a contradiction. Thus, $x^{n-1} = 1$. Then, $x \in \sqrt[n-1]{1}$ and $ax = r$ is regular. Conversely, let $a \in R$. If a is not regular then there exists $\alpha \in \sqrt[n-1]{1}$ such that $a - \alpha = r$ is regular. Therefore, $a = r + \alpha$ and $\alpha^n = \alpha^{n-1}\alpha = \alpha$. Thus, R is n-almost clean. □

Let us give the following definition:

Definition 3.7. Let $n \geq 2$ be an integer. A ring R is called n-indecomposable if for each $x \in R$, $x^n = x$ implies that $x = 0$ or 1.

Indecomposable rings are just the 2-indecomposable rings. It is clear also that, for each $n \geq 2$, every n-indecomposable ring is indecomposable. The converse implication is not true. For example, $\mathbb{Z}_{15}$ is an indecomposable ring which is not $(2p + 1)$-indecomposable for each $p \geq 1$ (since $(\overline{4^{2p+1}} = \overline{4}$ but $\neq \overline{0}, \overline{1}$).

Proposition 3.8. *Let $n \geq 2$ be a positive integer. If R is an n-indecomposable ring then every n-almost clean ring is almost clean.*

Proof. Clear since for each n-indecomposable n-almost clean ring, every element $x \in R$ can be written as $x = e + r$ where $e = 0$ or 1 and $r \in Reg(R)$. Thus, R is almost clean ring. □

Theorem 3.9. *Consider $n \geq 2$ a positive integer. Let R be a ring and E an R-module. Then $R \propto E$ is n-almost clean if and only if each $x \in R$ can be written in the form $x = r + e$ where $r \in R - (Z(R) \cup Z(E))$ and $e^n = e$.*

Proof. We first observe that, if $(r,0) \in Reg(R \propto E)$, then $r \in R - (Z(R) \cup Z(E))$. For if $r \in Z(R)$, then $rs = 0$ where $s \neq 0$ and then $(r,0)(s,0) = (0,0)$, while if $r \in Z(E)$ then $rm = 0$ where $m \neq 0$ and then $(r,0)(0,m) = (0,0)$. Conversely, if $r \in R - (Z(R) \cup Z(E))$, then (r,m) is regular for each $m \in E$. For $(r,m)(s,n) = (0,0)$ gives $rs = 0$ and hence $s = 0$ and then $rn = 0$ and hence $n = 0$.
Suppose that $R \propto E$ is n-almost clean. Thus, for each $x \in R$ and from above, $(x,0) = (r,0) + (e,0)$ where $(r,0) \in Reg(R \propto E)$ and $(e,0)^n = (e,0)$. Thus, $r \in R - (Z(R) \cup Z(E))$ and $e^n = e$, and $x = r + e$. Conversely, let $x \in R$ and $m \in E$. Write $x = r + e$ where $r \in R - (Z(R) \cup Z(M))$ and $e^n = e$. Then, $(x,m) = (r,m) + (e,0)$ and we have just prove that $(r,m) \in Reg(R \propto E)$ and $(e,0)^n = (e,0)$. Hence, $R \propto E$ is n-almost clean. □

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