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Generalized Kato spectrum of operator matrices on the Banach space

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Abstract In this note, we prove equality up to $S(T) \cup S(T^*)$ between the generalized Drazin spectrum and the generalized Kato spectrum, $S(T)$ is the set where T fails to have the SVEP. As applications, we investigate some classes of operators as the supercyclic and multiplier operators, also we give sufficient conditions which assure that the generalized Kato decomposition spectrum of an upper triangular operator matrices is the union of its diagonal entries spectra.

1 Introduction and Preliminaries

Throughout, X denotes a complex Banach space and $\mathcal{B}(X)$ denotes the Banach algebra of all bounded linear operators on X, we denote by T^*, $N(T)$, $R(T)$, $R^\infty(T) = \bigcap_{n\geq 0} R(T^n)$, $K(T)$, $H_0(T)$, $\rho(T)$, $\sigma(T)$, respectively the adjoint, the null space, the range, the hyper-range, the analytic core, the quasinilpotent part, the resolvent set, the spectrum of T.

Recall that $T \in \mathcal{B}(X)$ is said to be Kato operator or semi-regular if $R(T)$ is closed and $N(T) \subseteq R^\infty(T)$. Denote by $\rho_K(T)$:
$\rho_K(T) = \{\lambda \in \mathbb{C} : T - \lambda I$ is Kato $\}$ the Kato resolvent and $\sigma_K(T) = \mathbb{C}\backslash\rho_K(T)$ the Kato spectrum of T. It is well known that $\rho_K(T)$ is an open subset of $\mathbb{C}$.
According to [1, Definition 1.40], we say that $T \in \mathcal{B}(X)$ admits a generalized Kato decomposition, abbreviated GKD or pesudo-Fredholm operator if there exists a pair of T-invariant closed subspaces (M, N) such that $X = M \oplus N$, the restriction $T_{|M}$ is semi-regular, and $T_{|N}$ is quasinilpotent. Obviously, every Kato operator admits a GKD because in this case $M = X$ and $N = \{0\}$, again the quasi-nilpotent operator admits a GKD: Take $M = \{0\}$ and $N = X$. If we suppose that $T_{|N}$ is nilpotent of order $d \in \mathbb{N}$ then T is said to be of Kato type of order d. Finally T is said essentially semi-regular if it admits a GKD (M, N) such that N is finite-dimensional. Evidently every essentially semi-regular operator is of Kato type. The generalized Kato spectrum of T is defined by

$$\sigma_{gK}(T) := \{\lambda \in \mathbb{C} : T - \lambda I \text{ does not admit a generalized Kato decomposition}\}.$$

Evidently $\sigma_{gK}(T) \subseteq \sigma_K(T)$. We refer to [1] for more information about the topics of GKD.

Next, let $T \in \mathcal{B}(X)$, T is said to have the single valued extension property at $\lambda_0 \in \mathbb{C}$ (SVEP) if for every open neighbourhood $U \subseteq \mathbb{C}$ of λ_0, the only analytic function $f : U \longrightarrow X$ which satisfies the equation $(T - zI)f(z) = 0$ for all $z \in U$ is the function $f \equiv 0$. T is said to have the SVEP if T has the SVEP for every $\lambda \in \mathbb{C}$. Denote by $A(T) = \{\lambda \in \mathbb{C} : T \text{ has the SVEP at } \lambda\}$ and $S(T) = \mathbb{C}\backslash A(T)$, by [13, proposition 1.2.16] $A(T) = \mathbb{C}$ if and only if $X_T(\emptyset) = \{0\}$, if and only if $X_T(\emptyset)$ is closed where $X_T(\Omega)$ is the local spectral subspace of T associated with the open set Ω.
Obviously, every operator $T \in \mathcal{B}(X)$ has the SVEP at every $\lambda \in \rho(T)$, then T and T^* have the SVEP at every point of the boundary $\partial(\sigma(T))$ of the spectrum.
An operator $T \in \mathcal{B}(X)$ is said to be decomposable if, for any open covering U_1, U_2 of the complex plane $\mathbb{C}$, there are two closed T-invariant subspaces X_1 and X_2 of X such that $X_1 + X_2 = X$ and $\sigma(T|X_k) \subset U_k$, $k = 1, 2$.

Note that T is decomposable implies that T and T^* have the SVEP.

Let $T \in \mathcal{B}(X)$, the ascent of T is defined by $a(T) = min\{p \in \mathbb{N} : N(T^p) = N(T^{p+1})\}$, if such p does not exist we let $a(T) = \infty$. Analogously the descent of T is $d(T) = min\{q \in \mathbb{N} : R(T^q) = R(T^{q+1})\}$, if such q does not exist we let $d(T) = \infty$ [14]. It is well known that if both $a(T)$ and $d(T)$ are finite then $a(T) = d(T)$ and we have the decomposition $X = R(T^p) \oplus N(T^p)$ where $p = a(T) = d(T)$.
The descent and ascent spectra of $T \in \mathcal{B}(X)$ are defined by :

$$\sigma_{des}(T) = \{\lambda \in \mathbb{C},\ T - \lambda I \text{ has not finite descent}\}$$

$$\sigma_{ac}(T) = \{\lambda \in \mathbb{C},\ T - \lambda I \text{ has not finite ascent }\}$$

In [9], Drazin.M.P introduced the concept of Drazin inverse for semigroups. $T \in \mathcal{B}(X)$ is said to be a Drazin invertible if there exists a positive integer k and an operator $S \in \mathcal{B}(X)$ such that

$$ST = TS,\ \ T^{k+1}S = T^k \ \ and\ S^2T = S.$$

which is also equivalent to the fact that $T = T_1 \oplus T_2$; where T_1 is invertible and T_2 is nilpotent. Recall that an operator T is Drazin invertible if it has a finite ascent and descent. The concept of Drazin invertible has been generalized by Koliha [12]. In fact $T \in \mathcal{B}(X)$ is generalized Drazin invertible if and only if $0 \notin acc\sigma(T)$, the set of accumulation points of $\sigma(T)$, which is also equivalent to the fact that $T = T_1 \oplus T_2$ where T_1 is invertible and T_2 is quasinilpotent. The following statements are equivalent:

(i) T is generalized Drazin invertible,

(ii) 0 is an isolated point in the spectrum $\sigma(T)$ of T ;

(iii) $K(T)$ is closed and $X = K(T) \oplus H_0(T)$,

The Drazin and generalized Drazin spectra of $T \in \mathcal{B}(X)$ are defined by :

$$\sigma_{gD}(T) = \{\lambda \in \mathbb{C},\ T - \lambda I \text{ is not generalized Drazin}\}$$

$$\sigma_D(T) = \{\lambda \in \mathbb{C},\ T - \lambda I \text{ is not Drazin invertible }\}$$

Let E be a subset of X. E is said T-invariant if $T(E) \subseteq E$. If E and F are two closed T-invariant subspaces of X such that $X = E \oplus F$, we say that T is completely reduced by the pair (E, F) and it is denoted by $(E, F) \in Red(T)$. In this case we write $T = T_{|E} \oplus T_{|F}$ and say that T is the direct sum of $T_{|E}$ and $T_{|F}$.
In [7], M D. Cvetković and SČ. Živković-Zlatanović introduced and studied a new concept of generalized Drazin invertibility of bounded operators as a generalization of generalized Drazin invertible operators. In fact, an operator $T \in \mathcal{B}(X)$ is said to be generalized Drazin bounded below if $H_0(T)$ is closed and complemented with a subspace M in X such that $(M, H_0(T)) \in Red(T)$ and $T(M)$ is closed which is equivalent to there exists $(M, N) \in Red(T)$ such that $T_{|M}$ is bounded below and $T_{|N}$ is quasi-nilpotent, see [7, Theorem 3.6]. An operator $T \in \mathcal{B}(X)$ is said to be generalized Drazin surjective if $K(T)$ is closed and complemented with a subspace N in X such that $N \subseteq H_0(T)$ and $(K(T), N) \in Red(T)$ which is equivalent to there exists $(M, N) \in Red(T)$ such that $T_{|M}$ is surjective and $T_{|N}$ is quasi-nilpotent, see [7, Theorem 3.7].
The generalized Drazin bounded below and surjective spectra of $T \in \mathcal{B}(X)$ are defined respectively by:

$$\sigma_{gD\mathcal{M}}(T) = \{\lambda \in \mathbb{C},\ T - \lambda I \text{ is not generalized Drazin bounded below}\};$$

$$\sigma_{gD\mathcal{Q}}(T) = \{\lambda \in \mathbb{C},\ T - \lambda I \text{ is not generalized Drazin surjective}\}.$$

From [7], we have:

$$\sigma_{gD}(T) = \sigma_{gD\mathcal{M}}(T) \cup \sigma_{gD\mathcal{Q}}(T).$$

The aim of this paper is to present the relationship between $\sigma_{gK}(.)$, $\sigma_{gD\mathcal{M}}(.)$ and $\sigma_{gD\mathcal{Q}}(.)$ and we apply this results to same classes of operator as multipliers and supercyclic operators. Finally, we prove that if A and B are decomposable, then for every $C \in \mathcal{B}(Y, X)$ we have :

$$\sigma_{gK}(M_C) = \sigma_{gK}(A) \cup \sigma_{gK}(B) \text{ where } M_C = \begin{pmatrix} A & C \\ 0 & B \end{pmatrix}$$

2 SVEP and generalized Kato spectrum

Lemma 2.1. *Let $T \in \mathcal{B}(X)$. Suppose that T has the SVEP at $\lambda \in \mathbb{C}$. Then :*

$$T - \lambda \textit{ is bounded below if and only if } T - \lambda \textit{ is semi regular}$$

Proof. We have $T - \lambda$ bounded below implies that $T - \lambda$ is semi regular.
Conversely, if $T - \lambda$ is semi regular then $R(T - \lambda)$ is closed. Suppose that $T - \lambda$ is not injectif then $N(T - \lambda) \neq \{0\}$, since $T - \lambda$ is semi regular, then $\{0\} \neq N(T - \lambda) \subseteq R^\infty(T - \lambda) = K(T - \lambda)$, hence $N(T - \lambda) \cap K(T - \lambda) \neq \{0\}$, this contradict that T has the SVEP at λ (see [1, Theorem 2.22]. Therefore $T - \lambda$ is bonded below. □

By duality we have

Lemma 2.2. *Let $T \in \mathcal{B}(X)$. Suppose that T^* has the SVEP at $\lambda \in \mathbb{C}$. Then :*

$$T - \lambda \textit{ is surjectif if and only if } T - \lambda \textit{ is semi regular}$$

Lemma 2.3. *Let $T \in \mathcal{B}(X)$. Then:*

$$S(T) \subset \sigma_{gD\mathcal{M}}(T) \textit{ and } S(T^*) \subset \sigma_{gD\mathcal{Q}}(T).$$

Proof. Let $\lambda \notin \sigma_{gD\mathcal{M}}(T)$, then $T - \lambda$ is generalized Drazin bounded below, hence $H_0(T - \lambda)$ is closed. By [2, Theorem 1.7], T has the SVEP at λ
Let $\lambda \notin \sigma_{gD\mathcal{Q}}(T)$, then $T - \lambda$ is generalized Drazin surjective, hence $K(T - \lambda)$ is closed and $K(T - \lambda) \oplus N = X$ where $N \subseteq H_0(T - \lambda)$, then $K(T - \lambda) + H_0(T - \lambda) = X$. By [2, Theorem 1.7], T^* has the SVEP at λ

□

Proposition 2.4. *Let $T \in \mathcal{B}(X)$. Then :*

$$\sigma_{gD\mathcal{M}}(T) = \sigma_{gK}(T) \cup S(T)$$

Proof. Since $\sigma_{gK}(T) \subset \sigma_{gD\mathcal{M}}(T)$, and by lemma 2.3 we have $\sigma_{gK}(T) \cup S(T) \subset \sigma_{gD\mathcal{M}}(T)$.
Now, let $\lambda \notin \sigma_{gK}(T) \cup S(T)$, then $T - \lambda$ is a pseudo Fredholm operator, hence there exists two T-invariant closed subspaces of X, M and N such that $(T - \lambda)_{|M}$ is semi-regular and $(T - \lambda)_{|N}$ is quasinilpotent. Since T has the SVEP at λ then $(T - \lambda)_{|M}$ and $(T - \lambda)_{|N}$ have the SVEP at λ (see [1, Theorem 2.9]). By Lemma 2.1 $(T - \lambda)_{|M}$ is bounded below which implies that $T - \lambda$ is generalized Drazin bounded below. This complete the proof. □

Corollary 2.5. *Let $T \in \mathcal{B}(X)$ and suppose that T has SVEP, then :*

$$\sigma_{gD\mathcal{M}}(T) = \sigma_{gK}(T)$$

Proposition 2.6. *Let $T \in \mathcal{B}(X)$. Then :*

$$\sigma_{gD\mathcal{Q}}(T) = \sigma_{gK}(T) \cup S(T^*)$$

Proof. Since $\sigma_{gK}(T) \subset \sigma_{gD\mathcal{Q}}(T)$ and by lemma 2.3 we have $\sigma_{gK}(T) \cup S(T^*) \subset \sigma_{gD\mathcal{Q}}(T)$.
Let $\lambda \notin \sigma_{gK}(T)$, then $T - \lambda$ is a pseudo Fredholm operator, hence there exists two T- invariant closed subspaces of X, M and N such that $(T - \lambda)_{|M}$ is semi-regular and $(T - \lambda)_{|N}$ is quasinilpotent. Since T^* has the SVEP at λ this implies that $(T - \lambda)^*_{|N^\perp}$ and $(T - \lambda)^*_{|M^\perp}$ have the SVEP at λ (see [1, Theorem 2.9]). By Lemma 2.2 $(T - \lambda)_{|M}$ is surjective which implies that $T - \lambda$ is generalized Drazin surjective. This complete the proof. □

Corollary 2.7. *Let $T \in \mathcal{B}(X)$ and suppose that T^* has SVEP, then:*

$$\sigma_{gD\mathcal{Q}}(T) = \sigma_{gK}(T)$$

Example 2.8. Let C_p the Cesaro operator on the classical Hardy space $H^p(\mathcal{D})$, where $\mathcal{D}$ the open unit disc of $\mathbb{C}$ and $1 \leq p < \infty$, is given by:

$$C_p f(\lambda) := \frac{1}{\lambda} \int_0^{\lambda} \frac{f(\zeta)}{1-\zeta} d\zeta$$

C_p has the SVEP whenever $1 < p < \infty$ and $\sigma_{gK}(C_p) = \partial \Gamma_p$, Γ_p is the closed disc centered at $\frac{p}{2}$ with radius $\frac{p}{2}$. Then $\sigma_{gD\mathcal{M}}(C_p) = \partial \Gamma_p$.

Example 2.9. Let T be defined on $l^2(\mathbb{N})$ by :

$$T(x_1, x_2, ...) = (0, x_1, x_2, x_3, ...)$$

We have $\sigma_{gD}(T) = \{\lambda \in \mathbb{C}, \ \ |\lambda| \leq 1\}$. Since T has the SVEP then

$$\sigma_{gK}(T) = \sigma_{gD\mathcal{M}}(T)$$

Theorem 2.10. *Let $T \in \mathcal{B}(X)$. Then:*

$$\sigma_{gD}(T) = \sigma_{gK}(T) \cup (S(T) \cup S(T^*))$$

Proof. We have $\sigma_{gD}(T) \supseteq \sigma_{gK}(T) \cup (S(T) \cup S(T^*))$. Conversely, let $\lambda \notin \sigma_{gK}(T) \cup (S(T) \cup S(T^*))$, then $\lambda \notin \sigma_{gK}(T)$ and $\lambda \notin (S(T) \cup S(T^*))$. By proposition 2.4 and proposition 2.6 we have $\lambda \notin \sigma_{gD\mathcal{Q}}(T) \cup \sigma_{gD\mathcal{M}}(T)$. Since $\sigma_{gD}(T) = \sigma_{gD\mathcal{Q}}(T) \cup \sigma_{gD\mathcal{M}}(T)$, then $\lambda \notin \sigma_{gD}(T)$. □

Corollary 2.11. *Let $T \in \mathcal{B}(X)$ and suppose that T and T^* have the SVEP. Then :*

$$\sigma_{gD}(T) = \sigma_{gK}(T)$$

Corollary 2.12. *Let $T \in \mathcal{B}(X)$, be decomposable. Then :*

$$\sigma_{gD}(T) = \sigma_{gK}(T)$$

Example 2.13. Let T be the unilateral weighted shift on $l^2(\mathbb{N})$ defined by:

$$Te_n = \begin{cases} 0, & \text{if } n = p! \ for \ some \ p \in \mathbb{N} \\ e_{n+1} & \text{otherwise.} \end{cases}$$

The adjoint operator of T is :

$$T^* e_n = \begin{cases} 0 & \text{if } n = 0 \ or \ n = p! + 1 \ for \ some \ p \in \mathbb{N} \\ e_{n-1} & \text{otherwise.} \end{cases}$$

We have $\sigma(T) = \overline{D(0,1)}$ the unit closed disc. The point spectrum of T and T^* are : $\sigma_p(T) = \sigma_p(T^*) = \{0\}$, hence T and T^* have the SVEP. Then $\sigma_{ap}(T) = \sigma_{su}(T) = \sigma(T)$, hence $\sigma_{ap}(T)$ cluster at every point where $\sigma_{su}(T)$ and $\sigma_{ap}(T)$ respectively the surjective and approximative spectrum. From [11, Theorem 3.5], $\sigma_{gK}(T) = \sigma(T) = \overline{D(0,1)}$
According to corollaries, $\sigma_{gD}(T) = \sigma_{gD\mathcal{Q}}(T) = \sigma_{gD\mathcal{M}}(T) = \sigma_{gK}(T) = \overline{D(0,1)}$.

In the next proposition, we prove equality up to $\sigma_{des}(T)$ between the Drazin spectrum and the generalized Drazin spectrum.

Proposition 2.14. *Let $T \in \mathcal{B}(X)$. Then :*

$$\sigma_D(T) = \sigma_{gD}(T) \cup \sigma_{des}(T)$$

Proof. Let $\lambda \notin \sigma_{gD}(T) \cup \sigma_{des}(T)$, without loss of generality we can assume that $\lambda = 0$, then $T = T_1 \oplus T_2$ with T_1 is invertible operator and T_2 is quasinilpotent. Since T has finite descent, then T_1 and T_2 have finite descent. As, T_2 is quasinilpotent with finite descent, then it is a nilpotent operator (see [14]). Thus T is a Drazin invertible operator. □

3 Applications

A bounded linear operator T is called supercyclic provided there is some $x \in X$ such that the set $\{\lambda T^n,\ \lambda \in \mathbb{C}\ , n = 0, 1, 2, ..\}$ is dense in X. It is well now that if T is supercyclic then $\sigma_p(T^*) = \{0\}$ or $\sigma_p(T^*) = \{\alpha\}$ for some nonzero $\alpha \in \mathbb{C}$. Since an operator with countable point spectrum has SVEP, then we have the following:

Proposition 3.1. *Let $T \in \mathcal{B}(X)$, a supercyclic operator, then :*

$$\sigma_{gD\mathcal{Q}}(T) = \sigma_{gK}(T)$$

Since, Every hyponormal operator T on a Hilbert space has the single valued extension property, we have

Proposition 3.2. *Let T a hyponormal operator on a Hilbert space, then:*

$$\sigma_{gD\mathcal{M}}(T) = \sigma_{gK}(T)$$

In particular, If T is auto-adjoint, we have : $\sigma_{gD\mathcal{M}}(T) = \sigma_{gK}(T)$

Let $\mathcal{A}$ be a semi-simple commutative Banach algebra.
The mapping $T : \mathcal{A} \longrightarrow \mathcal{A}$ is said to be a multiplier of $\mathcal{A}$ if $T(x)y = xT(y)$ for all $x, y \in \mathcal{A}$. It is well known each multiplier on $\mathcal{A}$ is a continuous linear operator and that the set of all multiplier on $\mathcal{A}$ is a unital closed commutative subalgebra of $\mathcal{B}(\mathcal{A})$ [13, Proposition 4.1.1]. Also the semi-simplicity of $\mathcal{A}$ implies that every multiplier has the SVEP (see [13, Proposition 2.2.1]). According to proposition 2.4, we have :

Proposition 3.3. *Let T be a multiplier on a semi-simple commutative Banach algebra $\mathcal{A}$, then the following assertions are equivalent*

(i) T is pseudo-Fredholm .

(ii) T is generalized Drazin bounded below.

Now if we assume in additional that $\mathcal{A}$ is regular and Tauberian (see [13, Definition 4.9.7]), then every multiplier T^* has SVEP. Hence we have the following result,

Proposition 3.4. *Let T be a multiplier on a semi-simple regular and Tauberian commutative Banach algebra $\mathcal{A}$, then the following assertions are equivalent:*

(i) T is pseudo-Fredholm .

(ii) T is generalized Drazin invertible.

Let G a locally compact abelian group, with group operation + and Haar measure μ, let $L^1(G)$ consist of all $\mathbb{C}$-valued functions on G integrable with respect to Haar measure and $M(G)$ the Banach algebra of regular complex Borel measures on G. We recall that $L^1(G)$ is a regular semi-simple Tauberian commutative Banach algebra. Then we have the following:

Corollary 3.5. *Let G be a locally compact abelian group, $\mu \in M(G)$. Then every convolution operator $T : L^1(G) \longrightarrow L^1(G)$, $T(k) = \mu \star k$ is pseudo Fredholm if and only if is generalized Drazin invertible.*

4 Generalized Kato Decomposition for Operator Matrices

Let X and Y be Banach spaces and $\mathcal{B}(X, Y)$ denote the space of all bounded linear operator from X to Y. For $A \in \mathcal{B}(X)$, $B \in \mathcal{B}(Y)$, we denote by $M_C \in \mathcal{B}(X \oplus Y)$ the operator defined on $X \oplus Y$ by

$$\begin{pmatrix} A & C \\ 0 & B \end{pmatrix}$$

It is well known that, in the case of infinite dimensional, the inclusion $\sigma(M_C) \subset \sigma(A) \cup \sigma(B)$, may be strict. This motivates serval authors to study the defect $(\sigma_*(A) \cup \sigma_*(B)) \setminus \sigma_*(M_C)$ where

σ_* runs different type spectra.
In [8], they proved that : $\sigma_{gD}(M_C) \subset \sigma_{gD}(A) \cup \sigma_{gD}(B)$, this inclusion may be strict (see [18, Example 3.4]. In this section we interested and motivated by the relationship between $\sigma_{gK}(M_C)$ and $\sigma_{gK}(A) \cup \sigma_{gK}(B)$. We start by the following :

Proposition 4.1. *Let $A \in \mathcal{B}(X)$, $B \in \mathcal{B}(Y)$ and $C \in \mathcal{B}(Y, X)$. Then:*

$$\sigma_{gK}(M_C) = \sigma_{gK}(A) \cup \sigma_{gK}(B) \Longrightarrow \sigma_{gD}(M_C) = \sigma_{gD}(A) \cup \sigma_{gD}(B)$$

Proof. Let $\lambda \notin \sigma_{gD}(M_C)$, then $\lambda \notin \sigma_{gK}(M_C) = \sigma_{gK}(A) \cup \sigma_{gK}(B)$ this implies that $\lambda \notin \sigma_{gK}(A)$ and $\lambda \notin \sigma_{gK}(B)$. Suppose that $\lambda \in \sigma_{gD}(A)$, from Theorem 2.10 $\lambda \in S(A) \cup S(A^*)$. Then $S(A) \cup S(A^*) \subset S(M_C) \cup S(M_C^*) \subset \sigma_{gD}(M_C)$. This contradict that $\lambda \notin \sigma_{gD}(M_C)$. Hence $\lambda \notin \sigma_{gD}(A)$. According to [19, Lemmma 2.4], $\lambda \notin \sigma_{gD}(B)$. Thus $\lambda \notin \sigma_{gD}(A) \cup \sigma_{gD}(B)$. We conclude that $\sigma_{gD}(A) \cup \sigma_{gD}(B) \subset \sigma_{gD}(M_C)$. Since $\sigma_{gD}(M_C) \subset \sigma_{gD}(A) \cup \sigma_{gD}(B)$. This complete the proof. □

Proposition 4.1 and [18, Proposition 3.12] give the following:

Corollary 4.2. *Let $A \in \mathcal{B}(X)$, $B \in \mathcal{B}(Y)$ and $C \in \mathcal{B}(Y, X)$. Then :*

$$\sigma_{gK}(M_C) = \sigma_{gK}(A) \cup \sigma_{gK}(B) \Longrightarrow \sigma(M_C) = \sigma(A) \cup \sigma(B)$$

Remark 4.3. Let $A, B, C \in \mathcal{B}(l^2(\mathbb{N}))$ defined by:

$$A(x_1, x_2, x_3, ...) = (0, x_1, x_2, x_3, ...);$$

$$B = A^*;$$

$$C = I - AB.$$

We have $\sigma_{gK}(A) \cup \sigma_{gK}(B) = \{\lambda \in \mathbb{C}, |\lambda| \leq 1\}$. M_C is unitary, then $\sigma_{gK}(M_C) \subseteq \{\lambda \in \mathbb{C}; |\lambda| = 1\}$. So $\sigma_{gK}(M_C) \neq \sigma_{gK}(A) \cup \sigma_{gK}(B)$. Note that A^* and B have not the SVEP. This result will lead to us a sufficient condition that ensures the equality.

In the following theorem, we give sufficient condition for $\sigma_{gK}(M_C) = \sigma_{gK}(A) \cup \sigma_{gK}(B)$ holds for every $C \in \mathcal{B}(Y, X)$.

Theorem 4.4. *Let $A \in \mathcal{B}(X)$, $B \in \mathcal{B}(Y)$. If A, A^*, B and B^* have the SVEP, then for every $C \in \mathcal{B}(Y, X)$ we have:*

$$\sigma_{gK}(M_C) = \sigma_{gK}(A) \cup \sigma_{gK}(B)$$

Proof. A, A^*, B and B^* have the SVEP according to [10, Proposition 3.1], M_C and M_C^* have the SVEP. Hence by corollary 2.11

$$\sigma_{gK}(M_C) = \sigma_{gD}(M_C)$$

$$\sigma_{gK}(A) = \sigma_{gD}(A) \ \ and \ \ \sigma_{gK}(B) = \sigma_{gD}(B)$$

By [18, Corollary 3.6] and Corollary 2.11, we have

$$\sigma_{gD}(M_C) = \sigma_{gD}(A) \cup \sigma_{gD}(B) = \sigma_{gK}(A) \cup \sigma_{gK}(B).$$

Therefore :

$$\sigma_{gK}(M_C) = \sigma_{gK}(A) \cup \sigma_{gK}(B)$$

□

Corollary 4.5. *Let $A \in \mathcal{B}(X)$, $B \in \mathcal{B}(Y)$. If A and B are decomposable , then for every $C \in \mathcal{B}(Y, X)$ we have:*

$$\sigma_{gK}(M_C) = \sigma_{gK}(A) \cup \sigma_{gK}(B)$$

In particular, If A and B are algebraic or compact.

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