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Additivity of Jordan higher Derivable maps on alternative rings

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Abstract: Let $\mathcal{R}$ be an alternative ring (not necessarily with the identity element) and let $\mathbb{N}$ be the set of non-negative integers. A family $D = \{d_n\}_{n\in\mathbb{N}}$ of mappings $d_n : \mathcal{R} \to \mathcal{R}$ (not necessarily additivity) such that $d_0 = I_{\mathcal{R}}$, the identity map on $\mathcal{R}$, is said to be a Jordan higher derivable map if $d_n(xy + yx) = \sum\limits_{i+j=n} d_i(x)d_j(y) + \sum\limits_{i+j=n} d_i(y)d_j(x)$ holds for all $x, y \in \mathcal{R}$ and for each $n \in \mathbb{N}$. In the present paper, it is shown that every Jordan higher derivable map d_n is additive for each $n \in \mathbb{N}$ under certain mild assumptions and hence $D = \{d_n\}_{n\in\mathbb{N}}$ is a Jordan higher derivation on $\mathcal{R}$.

1 Introduction

Throughout this paper, let $\mathcal{R}$ be an alternative ring unless otherwise mentioned. For any $x, y \in \mathcal{R}$, $x \circ y = xy + yx$ will denote the Jordan product on $\mathcal{R}$. A ring $\mathcal{R}$ (not necessarily associative or commutative) is called an alternative ring if $\mathcal{R}$ satisfies $x^2y = x(xy)$ and $yx^2 = (yx)x$ for all $x, y \in \mathcal{R}$ and flexible if $x(yx) = (xy)x$ holds for all $x, y \in \mathcal{R}$. It can be easily seen that all associative rings are alternative and every alternative ring is flexible. Hence throughout the product xyx will denote the product $x(yx)$ or $(xy)x$ for all $x, y \in \mathcal{R}$. The study of nonassociative rings has received fair amount of attention during the last few decades, and many authors studied nonassociative algebras (see [15] and references therein), in particular, alternative rings after the discovery of their connection with the theory of projective planes. A map (not necessarily additive) $d : \mathcal{R} \to \mathcal{R}$ is said to be a derivable (resp. Jordan derivable) map if $d(xy) = d(x)y + xd(y)\big(\text{resp. } d(x \circ y) = d(x) \circ y + x \circ d(y)\big)$ holds for all $x, y \in \mathcal{R}$. If we consider $d : \mathcal{R} \to \mathcal{R}$ as an additive maps in the above definitions, then a derivable map (resp. Jordan derivable) is called a derivation (resp. Jordan derivation). An alternative ring $\mathcal{R}$ is said to be 2-torsion free if $2x = 0$ implies that $x = 0$ for all $x \in \mathcal{R}$.

In the remaining part of the paper, we shall use the following notions : Let $\mathcal{R}$ be 2-torsion free alternative ring with a nontrivial idempotent e_1 and formally set $e_0 = 1 - e_1$ ($\mathcal{R}$ need not have an identity element). It can be easily seen that $(e_ix)e_j = e_i(xe_j)$, where $i, j = 0, 1$ for all $x \in \mathcal{R}$. By Peirce decomposition $\mathcal{R} = \mathcal{R}_{11} + \mathcal{R}_{10} + \mathcal{R}_{01} + \mathcal{R}_{00}$, where $\mathcal{R}_{ij} = e_i\mathcal{R}e_j$ for $i, j \in \{0, 1\}$. The notion x_{ij} denote an arbitrary element of $\mathcal{R}_{ij}$ and any element $x \in \mathcal{R}$ can be expressed as $x = x_{11} + x_{10} + x_{01} + x_{00}$. Peirce decomposition of alternative rings satisfy the following relations :

(i) $\mathcal{R}_{ij}\mathcal{R}_{jk} \subseteq \mathcal{R}_{ik}$, when $i, j, k \in \{0, 1\}$,

(ii) $\mathcal{R}_{ij}\mathcal{R}_{ij} \subseteq \mathcal{R}_{ji}$ with $x_{ij}^2 = x_{ij}y_{ij} + y_{ij}x_{ij} = 0$,

(iii) $\mathcal{R}_{ij}\mathcal{R}_{kl} = 0,\ (j \neq k),\ (i, j) \neq (k, l)$.

for all $x_{ij}, y_{ij} \in \mathcal{R}_{ij}$.

In recent years, the study of additivity of multiplicative maps on rings and algebras has attracted a vide circle of algebraists (see for reference [3, 12, 10, 11] where further references can be found). It was Martindale [12], who first studied this problem and raised the question : When is a multiplicative map additive? He answered this question for a multiplicative isomorphism of an associative ring with a family of idempotents under certain assumptions and proved that every bijective multiplicative map from a prime ring containing a nontrivial idempotent onto an arbitrary ring is necessarily additive. About two decades later, Daif [3] investigated this problem for multiplicative derivations on associative rings and obtained the following result:

Theorem 1.1. *Let $\mathcal{R}$ be an associative ring (not necessarily with identity element) containing a nontrivial idempotent e which satisfies:*

(i) $x\mathcal{R} = \{0\}$ *implies* $x = 0$,

(ii) $e\mathcal{R}x = \{0\}$ *implies* $x = 0$ *(hence* $\mathcal{R}x = \{0\}$ *implies* $x = 0$*),*

(iii) $exe\mathcal{R}(1-e) = \{0\}$ *implies* $exe = 0$.

If d is multiplicative derivation of $\mathcal{R}$, then d is additive.

In the year 2010, Lu [11] studied the Jordan derivable maps on associative prime rings containing a nontrivial idempotent and proved that if $\mathcal{R}$ is a 2-torsion free associative unital prime ring which contains a nontrivial idempotent, then under certain restrictions every Jordan derivable map on $\mathcal{R}$ is additive and hence a derivation. Further, Jing and Lu [10] generalized the result due to Lu [11] for a larger class of rings viz. arbitrary associative ring with a nontrivial idempotent. Very recently, Rodrigues et al.[13] initiated the study of this problem for alternative rings and obtained the result due to Daif [3] in this setting for derivable maps. Further this problem was studied by Ferreira and Ferreira [6] for Jordan derivable map in an alternative ring. More precisely, they obtained the following result :

Theorem 1.2. *Let $\mathcal{R}$ be an alternative ring containing a nontrivial idempotent and satisfying the following conditions for $i, j, k \in \{0, 1\}$:*

(i) *If* $a_{ij}x_{jk} = 0$ *for all* $x_{jk} \in \mathcal{R}_{jk}$, *then* $a_{ij} = 0$,

(ii) *If* $x_{ij}a_{jk} = 0$ *for all* $x_{ij} \in \mathcal{R}_{ij}$, *then* $a_{jk} = 0$,

(iii) *If* $a_{ii}x_{ii} + x_{ii}a_{ii}$ *for all* $x_{ii} \in \mathcal{R}_{ii}$, *then* $a_{ii} = 0$.

If d is a multiplicative Jordan derivation, then d is additive.

The concept of derivation was extended to higher derivation by Hasse and Schmidt [9]. Let $\mathbb{N}$ be the set of all non-negative integers and $D = \{d_n\}_{n\in\mathbb{N}}$ be a family of additive maps on a ring $\mathcal{R}$ such that $d_0 = I_{\mathcal{R}}$, the identity map on $\mathcal{R}$. Then D is said to be a higher derivation(resp. Jordan higher derivation) on a ring $\mathcal{R}$ if for each $n \in \mathbb{N}$ and for all $x, y \in \mathcal{R}$, $d_n(xy) = \sum_{i+j=n} d_i(x)d_j(y)$ (resp. $d_n(x^2) = \sum_{i+j=n} d_i(x)d_j(x)$). Further, many results were obtained for higher derivations by several authors (see [7, 8] and references therein.) Motivated by this, Ashraf and Praveen [1] introduced the notion of higher derivable (resp. Jordan higher derivable) map on a ring $\mathcal{R}$. A family $D = \{d_n\}_{n\in\mathbb{N}}$ of mappings $d_n : \mathcal{R} \to \mathcal{R}$ (without assumption of additivity) such that $d_0 = I_{\mathcal{R}}$ is said to be higher derivable map (resp. Jordan higher derivable map) if $d_n(xy) = \sum_{i+j=n} d_i(x)d_j(y)$ (resp. $d_n(xy + yx) = \sum_{i+j=n} d_i(x)d_j(y) + \sum_{i+j=n} d_i(y)d_j(x)$) holds for all $x, y \in \mathcal{R}$ and for each $n \in \mathbb{N}$. Furthermore, they provided a sufficient condition on a ring $\mathcal{R}$ under which a Jordan higher derivable map becomes a higher derivation and generalized the result due to Jing and Lu [10] in the setting of higher derivable maps. Motivated by these observations, in this article, we discuss the question raised by Martindale [12] for alternative rings with Jordan higher derivable maps and prove that every Jordan higher derivable map is additive under certain assumptions.

2 Jordan derivable maps

Motivated by Lu [11], Jing and Lu [10] studied the additivity of Jordan derivable maps on an associative ring and prove the following result:

Theorem 2.1. *Let $\mathcal{R}$ be a 2-torsion free associative ring containing a nontrivial idempotent and satisfying the following conditions for $i, j, k \in \{0, 1\}$:*

(i) *If* $a_{ij}x_{jk} = 0$ *for all* $x_{jk} \in \mathcal{R}_{jk}$, *then* $a_{ij} = 0$,

(ii) *If* $x_{ij}a_{jk} = 0$ *for all* $x_{ij} \in \mathcal{R}_{ij}$, *then* $a_{jk} = 0$,

(iii) *If* $a_{ii}x_{ii} + x_{ii}a_{ii}$ *for all* $x_{ii} \in \mathcal{R}_{ii}$, *then* $a_{ii} = 0$.

If a mapping $d : \mathcal{R} \to \mathcal{R}$ *satisfies* $d(xy + yx) = d(x)y + xd(y) + d(y)x + yd(x)$ *holds for all* $x, y \in \mathcal{R}$, *then* d *is additive and hence* d *is a Jordan derivation.*

In this section, we generalize the above result for alternative rings and prove the following result:

Theorem 2.2. *Let* $\mathcal{R}$ *be a* 2*-torsion free alternative ring containing a nontrivial idempotent satisfying* $a_{ij}t_{jk} = 0$ *or* $t_{ki}a_{ij} = 0$ *implies that* $a_{ij} = 0$ *for all* $t_{jk} \in \mathcal{R}_{jk}, t_{ki} \in \mathcal{R}_{ki}$ *and* $i, j, k \in \{0, 1\}$. *If a map* $d : \mathcal{R} \to \mathcal{R}$ *satisfies* $d(xy + yx) = d(x)y + xd(y) + d(y)x + yd(x)$ *for all* $x, y \in \mathcal{R}$, *then* d *is additive and hence* d *is a Jordan derivation on* $\mathcal{R}$.

In order to prove the above theorem, we need to prove the following facts :

Fact 2.1. $d(0) = 0$.

Proof.

$$\begin{aligned} d(0) &= d(0 \circ 0) \\ &= d(0)0 + 0d(0) + d(0)0 + 0d(0) \\ &= 0. \end{aligned}$$

□

Fact 2.2. For $x_{ii} \in \mathcal{R}_{ii}, y_{jk} \in \mathcal{R}_{jk}$, where $i, j, k \in \{0, 1\}$ and $j \neq k$ such that

$$d(x_{ii} + y_{jk}) = d(x_{ii}) + d(y_{jk}).$$

Proof. First we prove that $d(x_{00}+y_{10}) = d(x_{00})+d(y_{10})$. Now, for any $x_{00} \in \mathcal{R}_{00}$ and $y_{10} \in \mathcal{R}_{10}$, we have

$$\begin{aligned} &[d(x_{00} + y_{10}) - d(x_{00}) - d(y_{10})] \circ e_1 \\ &= d\big((x_{00} + y_{10})e_1 + e_1(x_{00} + y_{10})\big) - (x_{00} + y_{10})d(e_1) - d(e_1)(x_{00} + y_{10}) \\ &\quad -d\big(x_{00}e_1 + e_1x_{00}\big) + x_{00}d(e_1) + d(e_1)x_{00} - d\big(y_{10}e_1 + e_1y_{10}\big) + y_{10}d(e_1) + d(e_1)y_{10} \\ &= 0. \end{aligned}$$

This implies that

$$\begin{aligned} e_1[d(x_{00} + y_{10}) - d(x_{00}) - d(y_{10})]e_1 &= 0, \\ e_1[d(x_{00} + y_{10}) - d(x_{00}) - d(y_{10})]e_0 &= 0, \\ e_0[d(x_{00} + y_{10}) - d(x_{00}) - d(y_{10})]e_1 &= 0. \end{aligned}$$

Also, using Fact 2.1, we get

$$\begin{aligned} &[d(x_{00} + y_{10}) - d(x_{00}) - d(y_{10})] \circ t_{10} \\ &= d\big((x_{00} + y_{10})t_{10} + t_{10}(x_{00} + y_{10})\big) - (x_{00} + y_{10})d(t_{10}) \\ &\quad -d(t_{10})(x_{00} + y_{10}) - d\big(x_{00}t_{10} + t_{10}x_{00}\big) + x_{00}d(t_{10}) \\ &\quad +d(t_{10})x_{00} - d\big(y_{10}t_{10} + t_{10}y_{10}\big) + y_{10}d(t_{10}) + d(t_{10})y_{10} \\ &= 0. \end{aligned}$$

By assumption, we can find that $e_0[d(x_{00} + y_{10}) - d(x_{00}) - d(y_{10})]e_0 = 0$ and hence $d(x_{00} + y_{10}) = d(x_{00}) + d(y_{10})$.

In the similar way, we can prove the rest of cases. □

Fact 2.3. For any $x_{10}, y_{10} \in \mathcal{R}_{10}; x_{01}, z_{01} \in \mathcal{R}_{01}$ and $y_{00}, z_{00} \in \mathcal{R}_{00}$,

(i) $d(x_{10} + y_{10}z_{00}) = d(x_{10}) + d(y_{10}z_{00})$,

(ii) $d(x_{01} + y_{00}z_{01}) = d(x_{01}) + d(y_{00}z_{01})$.

Proof. (i) For any $x_{10}, y_{10} \in \mathcal{R}_{10}, z_{00} \in \mathcal{R}_{00}$ and using Facts 2.1, 2.2, we have

$$\begin{aligned} d(x_{10} + y_{10}z_{00}) &= d[(e_1 + y_{10})(x_{10} + z_{00}) + (x_{10} + z_{00})(e_1 + y_{10})] \\ &= d(e_1 + y_{10})(x_{10} + z_{00}) + d(x_{10} + z_{00})(e_1 + y_{10}) \\ &\quad +(e_1 + y_{10})d(x_{10} + z_{00}) + (x_{10} + z_{00})d(e_1 + y_{10}) \\ &= [d(e_1) + d(y_{10})](x_{10} + z_{00}) + [d(x_{10}) + d(z_{00})](e_1 + y_{10}) \\ &\quad +(e_1 + y_{10})[d(x_{10}) + d(z_{00})] + (x_{10} + z_{00})[d(e_1) + d(y_{10})] \\ &= d(e_1x_{10} + x_{10}e_1) + d(x_{10}y_{10} + y_{10}x_{10}) + d(e_1z_{00} + z_{00}e_1) \\ &\quad +d(y_{10}z_{00} + z_{00}y_{10}) \\ &= d(x_{10}) + d(y_{10}z_{00}). \end{aligned}$$

(ii) Similar to (i). □

Fact 2.4. For any $x_{10}, y_{10} \in \mathcal{R}_{10}$ and $x_{01}, y_{01} \in \mathcal{R}_{01}$,

(i) $d(x_{10} + y_{10}) = d(x_{10}) + d(y_{10})$,

(ii) $d(x_{01} + y_{01}) = d(x_{01}) + d(y_{01})$.

Proof. (i) For any $x_{10}, y_{10} \in \mathcal{R}_{10}$, we obtain

$$\begin{aligned} &[d(x_{10} + y_{10}) - d(x_{10}) - d(y_{10})] \circ e_0 \\ &= d\big((x_{10} + y_{10})e_0 + e_0(x_{10} + y_{10})\big) - (x_{10} + y_{10})d(e_0) - d(e_0)(x_{10} + y_{10}) \\ &\quad -d\big(x_{10}e_0 + e_0x_{10}\big) + x_{10}d(e_0) + d(e_0)x_{10} - d\big(y_{10}e_0 + e_0y_{10}\big) + y_{10}d(e_0) + d(e_0)y_{10} \\ &= d(x_{10} + y_{10}) - d(x_{10}) - d(y_{10}). \end{aligned}$$

It follows that

$$\begin{aligned} e_0[d(x_{10} + y_{10}) - d(x_{10}) - d(y_{10})]e_0 &= 0, \\ e_1[d(x_{10} + y_{10}) - d(x_{10}) - d(y_{10})]e_1 &= 0. \end{aligned}$$

Now, for any $x_{10}, y_{10}, t_{10} \in \mathcal{R}_{10}$ and using Fact 2.1, we obtain that

$$\begin{aligned} &[d(x_{10} + y_{10}) - d(x_{10}) - d(y_{10})] \circ t_{10} \\ &= d\big((x_{10} + y_{10})t_{10} + t_{10}(x_{10} + y_{10})\big) - (x_{10} + y_{10})d(t_{10}) - d(t_{10})(x_{10} + y_{10}) \\ &\quad -d\big(x_{10}t_{10} + t_{10}x_{10}\big) + x_{10}d(t_{10}) + d(t_{10})x_{10} - d\big(y_{10}t_{10} + t_{10}y_{10}\big) \\ &\quad +y_{10}d(t_{10}) + d(t_{10})y_{10} \\ &= 0. \end{aligned}$$

This implies that $t_{10}[d(x_{10}+y_{10})-d(x_{10})-d(y_{10})]+[d(x_{10}+y_{10})-d(x_{10})-d(y_{10})]t_{10}$ which gives

$$e_0[d(x_{10} + y_{10}) - d(x_{10}) - d(y_{10})]e_1 = 0.$$

Similarly, we find that

$$e_1[d(x_{10} + y_{10}) - d(x_{10}) - d(y_{10})]e_0 = 0.$$

Hence,

$$d(x_{10} + y_{10}) = d(x_{10}) + d(y_{10}).$$

(ii) Similar to (i). □

Fact 2.5. For any $x_{00}, y_{00} \in \mathcal{R}_{00}$ and $x_{11}, y_{11} \in \mathcal{R}_{11}$,

(i) $d(x_{00}+y_{00}) = d(x_{00}) + d(y_{00})$,

(ii) $d(x_{11}+y_{11}) = d(x_{11}) + d(y_{11})$.

Proof. For any $x_{00}, y_{00} \in \mathcal{R}_{00}$

$$\begin{aligned}
&[d(x_{00}+y_{00}) - d(x_{00}) - d(y_{00})] \circ e_1 \\
= \;& d\big((x_{00}+y_{00})e_1 + e_1(x_{00}+y_{00})\big) - (x_{00}+y_{00})d(e_1) - d(e_1)(x_{00}+y_{00}) \\
& -d\big(x_{00}e_1 + e_1x_{00}\big) + x_{00}d(e_1) + d(e_1)x_{00} - d\big(y_{00}e_1 + e_1y_{00}\big) + y_{00}d(e_1) + d(e_1)y_{00} \\
= \;& 0.
\end{aligned}$$

This implies that

$$\begin{aligned}
e_1[d(x_{00}+y_{00}) - d(x_{00}) - d(y_{00})]e_1 &= 0, \\
e_1[d(x_{00}+y_{00}) - d(x_{00}) - d(y_{00})]e_0 &= 0, \\
e_0[d(x_{00}+y_{00}) - d(x_{00}) - d(y_{00})]e_1 &= 0.
\end{aligned}$$

Using Facts 2.1, 2.4, we get

$$\begin{aligned}
&[d(x_{00}+y_{00}) - d(x_{00}) - d(y_{00})] \circ t_{10} \\
= \;& d\big((x_{00}+y_{00})t_{10} + t_{10}(x_{00}+y_{00})\big) - (x_{00}+y_{00})d(t_{10}) - d(t_{10})(x_{00}+y_{00}) \\
& -d\big(x_{00}t_{10} + t_{10}x_{00}\big) + x_{00}d(t_{10}) + d(t_{10})x_{00} - d\big(y_{00}t_{10} + t_{10}y_{00}\big) \\
& +y_{00}d(t_{10}) + d(t_{10})y_{00} \\
= \;& 0.
\end{aligned}$$

This shows that $e_0[d(x_{00}+y_{00}) - d(x_{00}) - d(y_{00})]e_0 = 0$ and by our assumption $d(x_{00}+y_{00}) = d(x_{00}) + d(y_{00})$. In the similar way, we can obtain (ii). □

Fact 2.6. For any $x_{10}, \in \mathcal{R}_{10}$ and $y_{01} \in \mathcal{R}_{01}$, $d(x_{10}+y_{01}) = d(x_{10}) + d(y_{01})$.

Proof. For any $x_{10} \in \mathcal{R}_{10}$ and $y_{01} \in \mathcal{R}_{01}$, we find that

$$\begin{aligned}
&[d(x_{10}+y_{01}) - d(x_{10}) - d(y_{01})] \circ e_0 \\
= \;& d\big((x_{10}+y_{01})e_0 + e_0(x_{10}+y_{01})\big) - (x_{10}+y_{01})d(e_0) - d(e_0)(x_{10}+y_{01}) \\
& -d\big(x_{10}e_0 + e_0x_{10}\big) + x_{10}d(e_0) + d(e_0)x_{10} - d\big(y_{01}e_0 + e_0y_{01}\big) + y_{01}d(e_0) + d(e_0)y_{01} \\
= \;& d(x_{10}+y_{01}) - d(x_{10}) - d(y_{01}).
\end{aligned}$$

This implies that

$$\begin{aligned}
e_1[d(x_{10}+y_{01}) - d(x_{10}) - d(y_{01})]e_1 &= 0, \\
e_0[d(x_{10}+y_{01}) - d(x_{10}) - d(y_{01})]e_0 &= 0.
\end{aligned}$$

Also, for any $x_{10} \in \mathcal{R}_{10}$ and $y_{01} \in \mathcal{R}_{01}$, we get

$$\begin{aligned}
&[d(x_{10}+y_{01}) - d(x_{10}) - d(y_{01})] \circ t_{10} \\
= \;& d\big((x_{10}+y_{01})t_{10} + t_{10}(x_{10}+y_{01})\big) - (x_{10}+y_{01})d(t_{10}) - d(t_{10})(x_{10}+y_{01}) \\
& -d\big(x_{10}t_{10} + t_{10}x_{10}\big) + x_{10}d(t_{10}) + d(t_{10})x_{10} - d\big(y_{01}t_{10} + t_{10}y_{01}\big) \\
& +y_{01}d(t_{10}) + d(t_{10})y_{01} \\
= \;& 0.
\end{aligned}$$

By assumption it follows that $t_{10}[d(x_{10}+y_{01}) - d(x_{10}) - d(y_{01})] + [d(x_{10}+y_{01}) - d(x_{10}) - d(y_{01})]t_{10} = 0$. This yields that

$$e_0[d(x_{10}+y_{01}) - d(x_{10}) - d(y_{01})]e_1 = 0.$$

Similarly, we can find that $e_1[d(x_{10}+y_{01}) - d(x_{10}) - d(y_{01})]e_0 = 0$ and hence $d(x_{10}+y_{01}) = d(x_{10}) + d(y_{01})$. □

Fact 2.7. For any $x_{00} \in \mathcal{R}_{00}$ and $y_{11} \in \mathcal{R}_{11}$, $d(x_{00} + y_{11}) = d(x_{00}) + d(y_{11})$.

Proof. For any $x_{00} \in \mathcal{R}_{00}$ and $y_{11} \in \mathcal{R}_{11}$, we find that

$$\begin{aligned} &[d(x_{00} + y_{11}) - d(x_{00}) - d(y_{11})] \circ e_0 \\ &= d\big((x_{00} + y_{11})e_0 + e_0(x_{00} + y_{11})\big) - (x_{00} + y_{11})d(e_0) - d(e_0)(x_{00} + y_{11}) \\ &\quad -d\big(x_{00}e_0 + e_0x_{00}\big) + x_{00}d(e_0) + d(e_0)x_{00} - d\big(y_{11}e_0 + e_0y_{11}\big) + y_{11}d(e_0) + d(e_0)y_{11} \\ &= 0. \end{aligned}$$

This implies that

$$\begin{aligned} e_0[d(x_{00} + y_{11}) - d(x_{00}) - d(y_{11})]e_0 &= 0, \\ e_0[d(x_{00} + y_{11}) - d(x_{00}) - d(y_{11})]e_1 &= 0, \\ e_1[d(x_{00} + y_{11}) - d(x_{00}) - d(y_{11})]e_0 &= 0. \end{aligned}$$

Now, using Facts 2.4, we arrive at

$$\begin{aligned} &[d(x_{00} + y_{11}) - d(x_{00}) - d(y_{11})] \circ t_{10} \\ &= d\big((x_{00} + y_{11})t_{10} + t_{10}(x_{00} + y_{11})\big) - (x_{00} + y_{11})d(t_{10}) - d(t_{10})(x_{00} + y_{11}) \\ &\quad -d\big(x_{00}t_{10} + t_{10}x_{00}\big) + x_{00}d(t_{10}) + d(t_{10})x_{00} - d\big(y_{11}t_{10} + t_{10}y_{11}\big) \\ &\quad +y_{11}d(t_{10}) + d(t_{10})y_{11} \\ &= 0. \end{aligned}$$

This yields that $e_1[d(x_{00} + y_{11}) - d(x_{00}) - d(y_{11})]e_1 = 0$ and hence $d(x_{00} + y_{11}) = d(x_{00}) + d(y_{11})$. □

Fact 2.8. For any $x_{00} \in \mathcal{R}_{00}; z_{01} \in \mathcal{R}_{01}; z_{10} \in \mathcal{R}_{10}$ and $y_{11} \in \mathcal{R}_{11}$,

(i) $d(x_{00} + y_{11} + z_{10}) = d(x_{00}) + d(y_{11}) + d(z_{10})$,

(ii) $d(x_{00} + y_{11} + z_{01}) = d(x_{00}) + d(y_{11}) + d(z_{01})$.

Proof. Now, using Fact 2.2, we get

$$\begin{aligned} &[d(x_{00} + y_{11} + z_{10}) - d(x_{00}) - d(y_{11}) - d(z_{10})] \circ e_0 \\ &= d\big((x_{00} + y_{11} + z_{10})e_0 + e_0(x_{00} + y_{11} + z_{10})\big) - (x_{00} + y_{11} + z_{10})d(e_0) \\ &\quad -d(e_0)(x_{00} + y_{11} + z_{10}) - d\big(x_{00}e_0 + e_0x_{00}\big) + x_{00}d(e_0) + d(e_0)x_{00} \\ &\quad -d\big(y_{11}e_0 + e_0y_{11}\big) + y_{11}d(e_0) + d(e_0)y_{11} - d\big(z_{10}e_0 + e_0z_{10}\big) + z_{10}d(e_0) + d(e_0)z_{10} \\ &= 0. \end{aligned}$$

Therefore, we obtain that

$$\begin{aligned} e_0[d(x_{00} + y_{11} + z_{10}) - d(x_{00}) - d(y_{11}) - d(z_{10})]e_0 &= 0, \\ e_0[d(x_{00} + y_{11} + z_{10}) - d(x_{00}) - d(y_{11}) - d(z_{10})]e_1 &= 0, \\ e_1[d(x_{00} + y_{11} + z_{10}) - d(x_{00}) - d(y_{11}) - d(z_{10})]e_0 &= 0. \end{aligned}$$

Also, using Fact 2.4, we get

$$\begin{aligned} &[d(x_{00} + y_{11} + z_{10}) - d(x_{00}) - d(y_{11}) - d(z_{10})] \circ t_{10} \\ &= d\big((x_{00} + y_{11} + z_{10})t_{10} + t_{10}(x_{00} + y_{11} + z_{10})\big) - (x_{00} + y_{11} + z_{10})d(t_{10}) \\ &\quad -d(t_{10})(x_{00} + y_{11} + z_{10}) - d\big(x_{00}t_{10} + t_{10}x_{00}\big) + x_{00}d(t_{10}) + d(t_{10})x_{00} \\ &\quad -d\big(y_{11}t_{10} + t_{10}y_{11}\big) + y_{11}d(t_{10}) + d(t_{10})y_{11} - d\big(z_{10}t_{10} + t_{10}z_{10}\big) \\ &\quad +z_{10}d(t_{10}) + d(t_{10})z_{10} \\ &= 0. \end{aligned}$$

From here we can find that

$$e_1[d(x_{00}+y_{11}+z_{10})-d(x_{00})-d(y_{11})-d(z_{10})]e_1=0.$$

Hence,

$$d(x_{00}+y_{11}+z_{10})=d(x_{00})+d(y_{11})+d(z_{10}).$$

In the similar way, we can obtain (ii). □

Fact 2.9. For any $x_{00}\in\mathcal{R}_{00}; y_{01}\in\mathcal{R}_{01}; z_{10}\in\mathcal{R}_{10}$ and $x_{11}\in\mathcal{R}_{11}$,

(i) $d(x_{00}+y_{01}+z_{10})=d(x_{00})+d(y_{01})+d(z_{10})$,

(ii) $d(x_{11}+y_{01}+z_{10})=d(x_{11})+d(y_{01})+d(z_{10})$.

Proof. (i) Using Fact 2.6, we have

$$\begin{aligned}
&[d(x_{00}+y_{01}+z_{10})-d(x_{00})-d(y_{01})-d(z_{10})]\circ e_1\\
&= d\big((x_{00}+y_{01}+z_{10})e_1+e_1(x_{00}+y_{01}+z_{10})\big)-(x_{00}+y_{01}+z_{10})d(e_1)\\
&\quad -d(e_1)(x_{00}+y_{01}+z_{10})-d\big(x_{00}e_1+e_1x_{00}\big)+x_{00}d(e_1)+d(e_1)x_{00}\\
&\quad -d\big(y_{01}e_1+e_1y_{01}\big)+y_{01}d(e_1)+d(e_1)y_{01}-d\big(z_{10}e_1+e_1z_{10}\big)+z_{10}d(e_1)+d(e_1)z_{10}\\
&= 0.
\end{aligned}$$

This implies that

$$\begin{aligned}
e_1[d(x_{00}+y_{01}+z_{10})-d(x_{00})-d(y_{01})-d(z_{10})]e_1 &= 0,\\
e_0[d(x_{00}+y_{01}+z_{10})-d(x_{00})-d(y_{01})-d(z_{10})]e_1 &= 0,\\
e_1[d(x_{00}+y_{01}+z_{10})-d(x_{00})-d(y_{01})-d(z_{10})]e_0 &= 0.
\end{aligned}$$

Also, using Facts 2.7, 2.8, we obtain that

$$\begin{aligned}
&[d(x_{00}+y_{01}+z_{10})-d(x_{00})-d(y_{01})-d(z_{10})]\circ t_{10}\\
&= d\big((x_{00}+y_{01}+z_{10})t_{10}+t_{10}(x_{00}+y_{01}+z_{10})\big)-(x_{00}+y_{01}+z_{10})d(t_{10})\\
&\quad -d(t_{10})(x_{00}+y_{01}+z_{10})-d\big(x_{00}t_{10}+t_{10}x_{00}\big)+x_{00}d(t_{10})+d(t_{10})x_{00}\\
&\quad -d\big(y_{01}t_{10}+t_{10}y_{01}\big)+y_{01}d(t_{10})+d(t_{10})y_{01}-d\big(z_{10}t_{10}+t_{10}z_{10}\big)\\
&\quad +z_{10}d(t_{10})+d(t_{10})z_{10}\\
&= 0.
\end{aligned}$$

This yields that $e_0[d(x_{00}+y_{01}+z_{10})-d(x_{00})-d(y_{01})-d(z_{10})]e_0=0$ and hence we arrive at $d(x_{00}+y_{01}+z_{10})=d(x_{00})+d(y_{01})+d(z_{10})$. In the similar way, we can obtain (ii). □

Fact 2.10. For any $x_{00}\in\mathcal{R}_{00}, y_{01}\in\mathcal{R}_{01}, z_{10}\in\mathcal{R}_{10}$ and $w_{11}\in\mathcal{R}_{11}$,

$$d(x_{00}+y_{01}+z_{10}+w_{11})=d(x_{00})+d(y_{01})+d(z_{10})+d(w_{11}).$$

Proof. Using Fact 2.9, we find that

$$\begin{aligned}
&[d(x_{00}+y_{01}+z_{10}+w_{11})-d(x_{00})-d(y_{01})-d(z_{10})-d(w_{11})]\circ e_1\\
&= d\big((x_{00}+y_{01}+z_{10}+w_{11})e_1+e_1(x_{00}+y_{01}+z_{10}+w_{11})\big)\\
&\quad -(x_{00}+y_{01}+z_{10}+w_{11})d(e_1)-d(e_1)(x_{00}+y_{01}+z_{10}+w_{11})-d\big(x_{00}e_1+e_1x_{00}\big)\\
&\quad +x_{00}d(e_1)+d(e_1)x_{00}-d\big(y_{01}e_1+e_1y_{01}\big)+y_{01}d(e_1)+d(e_1)y_{01}-d\big(z_{10}e_1+e_1z_{10}\big)\\
&\quad +z_{10}d(e_1)+d(e_1)z_{10}-d\big(w_{11}e_1+e_1w_{11}\big)+w_{11}d(e_1)+d(e_1)w_{11}\\
&= 0.
\end{aligned}$$

This implies that

$$\begin{aligned} e_1[d(x_{00}+y_{01}+z_{10}+w_{11})-d(x_{00})-d(y_{01})-d(z_{10})-d(w_{11})]e_1 &= 0,\\ e_0[d(x_{00}+y_{01}+z_{10}+w_{11})-d(x_{00})-d(y_{01})-d(z_{10})-d(w_{11})]e_1 &= 0,\\ e_1[d(x_{00}+y_{01}+z_{10}+w_{11})-d(x_{00})-d(y_{01})-d(z_{10})-d(w_{11})]e_0 &= 0. \end{aligned}$$

Now, using Fact 2.8, we get

$$\begin{aligned} &[d(x_{00}+y_{01}+z_{10}+w_{11})-d(x_{00})-d(y_{01})-d(z_{10})-d(w_{11})]\circ t_{10}\\ &= d\big((x_{00}+y_{01}+z_{10}+w_{11})t_{10}+t_{10}(x_{00}+y_{01}+z_{10}+w_{11})\big)\\ &\quad -(x_{00}+y_{01}+z_{10}+w_{11})d(t_{10})-d(t_{10})(x_{00}+y_{01}+z_{10}+w_{11})-d\big(x_{00}t_{10}+t_{10}x_{00}\big)\\ &\quad +x_{00}d(t_{10})+d(t_{10})x_{00}-d\big(y_{01}t_{10}+t_{10}y_{01}\big)+y_{01}d(t_{10})+d(t_{10})y_{01}+z_{10}d(t_{10})\\ &\quad -d\big(z_{10}t_{10}+t_{10}z_{10}\big)+d(t_{10})z_{10}-d\big(w_{11}t_{10}+t_{10}w_{11}\big)+w_{11}d(t_{10})+d(t_{10})w_{11}\\ &= 0. \end{aligned}$$

By assumption, we have $e_0[d(x_{00}+y_{01}+z_{10}+w_{11})-d(x_{00})-d(y_{01})-d(z_{10})-d(w_{11})]e_0=0$, which yields that $d(x_{00}+y_{01}+z_{10}+w_{11})=d(x_{00})+d(y_{01})+d(z_{10})+d(w_{11})$. □

Proof of Theorem 2.2. For any $x,y\in\mathcal{R}$, let $x=x_{11}+x_{10}+x_{01}+x_{00}$ and $y=y_{11}+y_{10}+y_{01}+y_{00}$ for all $x_{ij},y_{ij}\in\mathcal{R}_{ij}$ where $i,j\in\{0,1\}$. On using Facts 2.4, 2.5 and 2.10, we have

$$\begin{aligned} d(x+y) &= d(x_{11}+x_{10}+x_{01}+x_{00}+y_{11}+y_{10}+y_{01}+y_{00})\\ &= d(x_{11}+y_{11})+d(x_{01}+y_{01})+d(x_{01}+y_{01})+d(x_{00}+y_{00})\\ &= d(x_{11})+d(y_{11})+d(x_{01})+d(y_{01})+d(x_{01})+d(y_{01})+d(x_{00})+d(y_{00})\\ &= d(x_{11}+x_{10}+x_{01}+x_{00})+d(y_{11}+y_{10}+y_{01}+y_{00})\\ &= d(x)+d(y). \end{aligned}$$

This implies that d is additive on $\mathcal{R}$ and hence d is a Jordan derivation on $\mathcal{R}$. □

3 Jordan higher derivable maps

Motivated by Ferreira and Ferreira [6], in this section we study the additivity of Jordan higher derivable maps on alternative rings under certain assumptions as follows:

Theorem 3.1. *Let $\mathcal{R}$ be an alternative ring with a nontrivial idempotent satisfying the following conditions for $i,j,k\in\{1,0\}$:*

(i) *If $a_{ij}x_{jk}=0$ for all $x_{jk}\in\mathcal{R}_{jk}$, then $a_{ij}=0$,*

(ii) *If $x_{ij}a_{jk}=0$ for all $x_{ij}\in\mathcal{R}_{ij}$, then $a_{jk}=0$,*

(iii) *If $a_{ii}x_{ii}+x_{ii}a_{ii}$ for all $x_{ii}\in\mathcal{R}_{ii}$, then $a_{ii}=0$.*

If the family $D=\{d_n\}_{n\in\mathbb{N}}$ of mappings $d_n:\mathcal{R}\to\mathcal{R}$ such that $d_0=I_{\mathcal{R}}$ satisfies $d_n(xy+yx)=\sum_{i+j=n}d_i(x)d_j(y)+\sum_{i+j=n}d_i(y)d_j(x)$ for all $x,y\in\mathcal{R}$, then d_n is additive for each $n\in\mathbb{N}$ and hence $D=\{d_n\}_{n\in\mathbb{N}}$ is additive and hence D is a Jordan higher derivation on $\mathcal{R}$.

Now we prove this theorem in following sequence of lemmas:

Lemma 3.2. $d_n(0)=0$.

Proof.

$$\begin{aligned} d_n(0) &= d_n(0\circ 0)\\ &= d_n(0)0+0d_n(0)+d_n(0)0+0d_n(0)+2\sum_{\substack{i+j=n\\0<i,j<n}}d_i(0)d_j(0)\\ &= 0. \end{aligned}$$

□

Lemma 3.3. *For $x_{ii} \in \mathcal{R}_{ii}, y_{jk} \in \mathcal{R}_{jk}$, where $i, j, k \in \{0, 1\}$ and $j \neq k$ such that*

$$d_n(x_{ii} + y_{jk}) = d_n(x_{ii}) + d_n(y_{jk}).$$

Proof. Let us take $i, j = 1$ and $k = 0$. Then we have to prove that $d_n(x_{11} + y_{10}) = d_n(x_{11}) + d_n(y_{10})$.

$$\begin{aligned} & d_n[(x_{11} + y_{10})t_{00} + t_{00}(x_{11} + y_{10})] \\ = \; & d_n(x_{11} + y_{10})t_{00} + t_{00}d_n(x_{11} + y_{10}) + (x_{11} + y_{10})d_n(t_{00}) \\ & + d_n(t_{00})(x_{11} + y_{10}) + \sum_{\substack{i+j=n \\ 0<i,j<n}} d_i(x_{11} + y_{10})d_j(t_{00}) + \sum_{\substack{i+j=n \\ 0<i,j<n}} d_i(t_{00})d_j(x_{11} + y_{10}). \end{aligned}$$

On the other hand, we obtain that

$$\begin{aligned} & d_n[(x_{11} + y_{10})t_{00} + t_{00}(x_{11} + y_{10})] \\ = \; & d_n(y_{10}t_{00} + t_{00}y_{10}) + d_n(x_{11}t_{00} + t_{00}x_{11}) \\ = \; & d_n(x_{11})t_{00} + t_{00}d_n(x_{11}) + x_{11}d_n(t_{00}) + d_n(t_{00})x_{11} + \sum_{\substack{i+j=n \\ 0<i,j<n}} d_i(x_{11})d_j(t_{00}) \\ & + \sum_{\substack{i+j=n \\ 0<i,j<n}} d_i(t_{00})d_j(x_{11}) + d_n(y_{10})t_{00} + t_{00}d_n(y_{10}) + y_{10}d_n(t_{00}) + d_n(t_{00})y_{10} \\ & + \sum_{\substack{i+j=n \\ 0<i,j<n}} d_i(y_{10})d_j(t_{00}) + \sum_{\substack{i+j=n \\ 0<i,j<n}} d_i(t_{00})d_j(y_{10}). \end{aligned}$$

From last two expressions we conclude that

$$[d_n(x_{11} + y_{10}) - d_n(x_{11}) - d_n(y_{10})]t_{00} + t_{00}[d_n(x_{11} + y_{10}) - d_n(x_{11}) - d_n(y_{10})] = 0.$$

On applying assumptions, we find that

$$\begin{aligned} e_1[d_n(x_{11} + y_{10}) - d_n(x_{11}) - d_n(y_{10})]e_0 &= 0, \\ e_0[d_n(x_{11} + y_{10}) - d_n(x_{11}) - d_n(y_{10})]e_0 &= 0, \\ e_0[d_n(x_{11} + y_{10}) - d_n(x_{11}) - d_n(y_{10})]e_1 &= 0. \end{aligned}$$

To show that $e_1[d_n(x_{11} + y_{10}) - d_n(x_{11}) - d_n(y_{10})]e_1 = 0$, we consider that

$$\begin{aligned} & d_n[(x_{11} + y_{10})t_{10} + t_{10}(x_{11} + y_{10})] \\ = \; & d_n(y_{10}t_{10} + t_{10}y_{10}) + d_n(x_{11}t_{10} + t_{10}x_{11}) \\ = \; & d_n(x_{11})t_{10} + t_{10}d_n(x_{11}) + x_{11}d_n(t_{10}) + d_n(t_{10})x_{11} + \sum_{\substack{i+j=n \\ 0<i,j<n}} d_i(x_{11})d_j(t_{10}) \\ & + \sum_{\substack{i+j=n \\ 0<i,j<n}} d_i(t_{10})d_j(x_{11}) + d_n(y_{10})t_{10} + t_{10}d_n(y_{10}) + y_{10}d_n(t_{10}) + d_n(t_{10})y_{10} \\ & + \sum_{\substack{i+j=n \\ 0<i,j<n}} d_i(y_{10})d_j(t_{10}) + \sum_{\substack{i+j=n \\ 0<i,j<n}} d_i(t_{10})d_j(y_{10}). \end{aligned}$$

On the other way

$$\begin{aligned} & d_n[(x_{11} + y_{10})t_{10} + t_{10}(x_{11} + y_{10})] \\ = \; & d_n(x_{11} + y_{10})t_{10} + t_{10}d_n(x_{11} + y_{10}) + (x_{11} + y_{10})d_n(t_{10}) \\ & + d_n(t_{10})(x_{11} + y_{10}) + \sum_{\substack{i+j=n \\ 0<i,j<n}} d_i(x_{11} + y_{10})d_j(t_{10}) + \sum_{\substack{i+j=n \\ 0<i,j<n}} d_i(t_{10})d_j(x_{11} + y_{10}). \end{aligned}$$

From last two expression we arrive at $e_1[d_n(x_{11}+y_{10})-d_n(x_{11})-d_n(y_{10})]e_1=0$, this proves the result. In the similar way we can prove the rest of cases. □

Lemma 3.4. *For* $i \neq j$ *and* $i,j \in \{0,1\}$

(i) $d_n(x_{ij}+y_{ij}z_{jj}) = d_n(x_{ij}) + d_n(y_{ij}z_{jj})$,

(ii) $d_n(x_{ji}+y_{jj}z_{ji}) = d_n(x_{ji}) + d_n(y_{jj}z_{ji})$.

Proof. Consider the case for $i=1$ and $j=0$. For any $x_{10}, y_{10} \in \mathcal{R}_{10}, z_{00} \in \mathcal{R}_{00}$ and using Lemmas 3.2 and 3.3, we have

$$
\begin{aligned}
& d_n(x_{10}+y_{10}z_{00}) \\
&= d_n[(e_1+y_{10})(x_{10}+z_{00})+(x_{10}+z_{00})(e_1+y_{10})] \\
&= d_n(e_1+y_{10})(x_{10}+z_{00})+d_n(x_{10}+z_{00})(e_1+y_{10}) \\
&\quad +(e_1+y_{10})d_n(x_{10}+z_{00})+(x_{10}+z_{00})d_n(e_1+y_{10}) \\
&\quad +\sum_{\substack{i+j=n\\0<i,j<n}} d_i(x_{10}+z_{00})d_j(e_1+y_{10})+\sum_{\substack{i+j=n\\0<i,j<n}} d_i(e_1+y_{10})d_j(x_{10}+z_{00}) \\
&= [d_n(e_1)+d(y_{10})](x_{10}+z_{00})+[d_n(x_{10})+d_n(z_{00})](e_1+y_{10}) \\
&\quad +(e_1+y_{10})[d_n(x_{10})+d_n(z_{00})]+(x_{10}+z_{00})[d_n(e_1)+d_n(y_{10})] \\
&\quad +\sum_{\substack{i+j=n\\0<i,j<n}} d_i(x_{10})d_j(e_1)+\sum_{\substack{i+j=n\\0<i,j<n}} d_i(e_1)d_j(x_{10}) \\
&\quad +\sum_{\substack{i+j=n\\0<i,j<n}} d_i(x_{10})d_j(y_{10})+\sum_{\substack{i+j=n\\0<i,j<n}} d_i(y_{10})d_j(x_{10})+\sum_{\substack{i+j=n\\0<i,j<n}} d_i(z_{00})d_j(e_1) \\
&\quad +\sum_{\substack{i+j=n\\0<i,j<n}} d_i(e_1)d_j(z_{00})+\sum_{\substack{i+j=n\\0<i,j<n}} d_i(z_{00})d_j(y_{10})+\sum_{\substack{i+j=n\\0<i,j<n}} d_i(y_{10})d_j(z_{00}) \\
&= d_n(e_1x_{10}+x_{10}e_1)+d_n(x_{10}y_{10}+y_{10}x_{10})+d_n(e_1z_{00}+z_{00}e_1)+d_n(y_{10}z_{00}+z_{00}y_{10}) \\
&= d_n(x_{10})+d_n(y_{10}z_{00}).
\end{aligned}
$$

In the similar way we can prove the rest of the cases. □

Lemma 3.5. $d_n(x_{ij}+y_{ij}) = d_n(x_{ij})+d_n(y_{ij})$, *for* $i \neq j$ *and* $i,j \in \{0,1\}$.

Proof. Consider the case of $i=1$ and $j=0$. Then we have to prove that $d_n(x_{10}+y_{10}) = d_n(x_{10})+d_n(y_{10})$.

$$
\begin{aligned}
& d_n[(x_{10}+y_{10})t_{00}+t_{00}(x_{10}+y_{10})] \\
&= d_n(x_{10}+y_{10})t_{00}+t_{00}d_n(x_{10}+y_{10})+(x_{10}+y_{10})d_n(t_{00}) \\
&\quad +d_n(t_{00})(x_{10}+y_{10})+\sum_{\substack{i+j=n\\0<i,j<n}} d_i(x_{10}+y_{10})d_j(t_{00})+\sum_{\substack{i+j=n\\0<i,j<n}} d_i(t_{00})d_j(x_{10}+y_{10}).
\end{aligned}
$$

On the other way, we have

$$\begin{aligned}
& d_n[(x_{10}+y_{10})t_{00}+t_{00}(x_{10}+y_{10})] \\
&= d_n(y_{10}t_{00}+x_{10}y_{00}) \\
&= d_n(y_{10}t_{00}+t_{00}y_{10})+d_n(x_{10}t_{00}+t_{00}x_{10}) \\
&= d_n(x_{10})t_{00}+t_{00}d_n(x_{10})+x_{10}d_n(t_{00})+d_n(t_{00})x_{10}+\sum_{\substack{i+j=n\\0<i,j<n}} d_i(x_{10})d_j(t_{00}) \\
&\quad +\sum_{\substack{i+j=n\\0<i,j<n}} d_i(t_{00})d_j(x_{10})+d_n(y_{10})t_{00}+t_{00}d_n(y_{10})+y_{10}d_n(t_{00})+d_n(t_{00})y_{10} \\
&\quad +\sum_{\substack{i+j=n\\0<i,j<n}} d_i(y_{10})d_j(t_{00})+\sum_{\substack{i+j=n\\0<i,j<n}} d_i(t_{00})d_j(y_{10}).
\end{aligned}$$

Above two expressions implies that

$$[d_n(x_{10}+y_{10})-d_n(x_{10})-d_n(y_{10})]t_{00}+t_{00}[d_n(x_{10}+y_{10})-d_n(x_{10})-d_n(y_{10})]=0.$$

On applying assumptions, we find that

$$\begin{aligned}
e_1[d_n(x_{10}+y_{10})-d_n(x_{10})-d_n(y_{10})]e_0 &= 0, \\
e_0[d_n(x_{10}+y_{10})-d_n(x_{10})-d_n(y_{10})]e_0 &= 0, \\
e_0[d_n(x_{10}+y_{10})-d_n(x_{10})-d_n(y_{10})]e_1 &= 0.
\end{aligned}$$

Now we have to show that $e_1[d_n(x_{10}+y_{10})-d_n(x_{10})-d_n(y_{10})]e_1=0$, on considering

$$\begin{aligned}
& d_n[(x_{10}+y_{10})t_{10}+t_{10}(x_{10}+y_{10})] \\
&= d_n(y_{10}t_{10}+t_{10}y_{10})+d_n(x_{10}t_{10}+t_{10}x_{10}) \\
&= d_n(x_{10})t_{10}+t_{10}d_n(x_{10})+x_{10}d_n(t_{10})+d_n(t_{10})x_{10}+\sum_{\substack{i+j=n\\0<i,j<n}} d_i(x_{10})d_j(t_{10}) \\
&\quad +\sum_{\substack{i+j=n\\0<i,j<n}} d_i(t_{10})d_j(x_{10})+d_n(y_{10})t_{10}+t_{10}d_n(y_{10})+y_{10}d_n(t_{10})+d_n(t_{10})y_{10} \\
&\quad +\sum_{\substack{i+j=n\\0<i,j<n}} d_i(y_{10})d_j(t_{10})+\sum_{\substack{i+j=n\\0<i,j<n}} d_i(t_{10})d_j(y_{10}).
\end{aligned}$$

On the other hand

$$\begin{aligned}
& d_n[(x_{10}+y_{10})t_{10}+t_{10}(x_{10}+y_{10})] \\
&= d_n(x_{10}+y_{10})t_{10}+t_{10}d_n(x_{10}+y_{10})+(x_{10}+y_{10})d_n(t_{10}) \\
&\quad +d_n(t_{10})(x_{10}+y_{10})+\sum_{\substack{i+j=n\\0<i,j<n}} d_i(x_{10}+y_{10})d_j(t_{10})+\sum_{\substack{i+j=n\\0<i,j<n}} d_i(t_{10})d_j(x_{10}+y_{10}).
\end{aligned}$$

From above two expressions we arrive at $e_1[d_n(x_{10}+y_{10})-d_n(x_{10})-d_n(y_{10})]e_1=0$, this proves the result. In the similar way we can prove the rest of cases. □

Lemma 3.6. *$d_n(x_{ii}+y_{ii})=d_n(x_{ii})+d_n(y_{ii})$ for $i\in\{0,1\}$.*

Proof. Let us consider the case $i=1$

$$\begin{aligned}
0 &= d_n[(x_{11}+y_{11})t_{00}+t_{00}(x_{11}+y_{11})] \\
&= d_n(x_{11}+y_{11})t_{00}+t_{00}d_n(x_{11}+y_{11})+(x_{11}+y_{11})d_n(t_{00})+d_n(t_{00})(x_{11}+y_{11}) \\
&\quad +\sum_{\substack{i+j=n\\0<i,j<n}} d_i(x_{11}+y_{11})d_j(t_{00})+\sum_{\substack{i+j=n\\0<i,j<n}} d_i(t_{00})d_j(x_{11}+y_{11}).
\end{aligned}$$

Again consider that

$$\begin{aligned}
0 &= d_n[(x_{11}+y_{11})t_{00}+t_{00}(x_{11}+y_{11})] \\
&= d_n(x_{11}t_{00}+t_{00}x_{11})+d_n(y_{11}t_{00}+t_{00}y_{11}) \\
&= d_n(x_{11})t_{00}+t_{00}d_n(x_{11})+x_{11}d_n(t_{00})+d_n(t_{00})x_{11}+\sum_{\substack{i+j=n\\0<i,j<n}} d_i(x_{11})d_j(t_{00}) \\
&\quad +\sum_{\substack{i+j=n\\0<i,j<n}} d_i(t_{00})d_j(x_{11})+d_n(y_{11})t_{00}+t_{00}d_n(y_{11})+y_{11}d_n(t_{00})+d_n(t_{00})y_{11} \\
&\quad +\sum_{\substack{i+j=n\\0<i,j<n}} d_i(y_{11})d_j(t_{00})+\sum_{\substack{i+j=n\\0<i,j<n}} d_i(t_{00})d_j(y_{11}).
\end{aligned}$$

Combining above two expressions and applying assumptions, we find that

$$\begin{aligned}
e_1[d_n(x_{11}+y_{11})-d_n(x_{11})-d_n(y_{11})]e_0 &= 0, \\
e_0[d_n(x_{11}+y_{11})-d_n(x_{11})-d_n(y_{11})]e_0 &= 0, \\
e_0[d_n(x_{11}+y_{11})-d_n(x_{11})-d_n(y_{11})]e_1 &= 0.
\end{aligned}$$

On using similar steps as used in last lemma we can find $e_1[d_n(x_{11}+y_{11})-d_n(x_{11})-d_n(y_{11})]e_1 = 0$, this proves the result. In the similar way we can prove the rest of cases. □

Lemma 3.7. $d_n(x_{10}+y_{01}) = d_n(x_{10})+d_n(y_{01})$.

Proof. For any $x_{10}\in\mathcal{R}_{10}$ and $y_{01}\in\mathcal{R}_{01}$, we find that

$$\begin{aligned}
&[d_n(x_{10}+y_{01})-d_n(x_{10})-d_n(y_{01})]\circ e_0 \\
&= d_n\big((x_{10}+y_{01})e_0+e_0(x_{10}+y_{01})\big)-(x_{10}+y_{01})d_n(e_0)-d_n(e_0)(x_{10}+y_{01}) \\
&\quad -\sum_{\substack{i+j=n\\0<i,j<n}} d_i(x_{10}+y_{01})d_j(e_0)-\sum_{\substack{i+j=n\\0<i,j<n}} d_i(e_0)d_j(x_{10}+y_{01})-d_n\big(x_{10}e_0+e_0x_{10}\big) \\
&\quad +x_{10}d_n(e_0)+d_n(e_0)x_{10}+\sum_{\substack{i+j=n\\0<i,j<n}} d_i(x_{10})d_j(e_0)+\sum_{\substack{i+j=n\\0<i,j<n}} d_i(e_0)d_j(x_{10})+y_{01}d_n(e_0) \\
&\quad -d_n\big(y_{01}e_0+e_0y_{01}\big)+d_n(e_0)y_{01}+\sum_{\substack{i+j=n\\0<i,j<n}} d_i(y_{01})d_j(e_0)+\sum_{\substack{i+j=n\\0<i,j<n}} d_i(e_0)d_j(y_{01}) \\
&= d_n(x_{10}+y_{01})-d_n(x_{10})-d_n(y_{01}).
\end{aligned}$$

This implies that

$$\begin{aligned}
e_1[d_n(x_{10}+y_{01})-d_n(x_{10})-d_n(y_{01})]e_1 &= 0, \\
e_0[d_n(x_{10}+y_{01})-d_n(x_{10})-d_n(y_{01})]e_0 &= 0.
\end{aligned}$$

Also, for any $x_{10} \in \mathcal{R}_{10}$ and $y_{01} \in \mathcal{R}_{01}$, we get

$$
\begin{aligned}
&[d_n(x_{10}+y_{01}) - d_n(x_{10}) - d_n(y_{01})] \circ t_{10} \\
= \;& d_n\big((x_{10}+y_{01})t_{10} + t_{10}(x_{10}+y_{01})\big) - (x_{10}+y_{01})d_n(t_{10}) - d_n(t_{10})(x_{10}+y_{01}) \\
& - \sum_{\substack{i+j=n \\ 0<i,j<n}} d_i(x_{10}+y_{01})d_j(t_{10}) - \sum_{\substack{i+j=n \\ 0<i,j<n}} d_i(t_{10})d_j(x_{10}+y_{01}) - d_n\big(x_{10}t_{10} + t_{10}x_{10}\big) \\
& + x_{10}d_n(t_{10}) + d_n(t_{10})x_{10} + \sum_{\substack{i+j=n \\ 0<i,j<n}} d_i(x_{10})d_j(t_{10}) + \sum_{\substack{i+j=n \\ 0<i,j<n}} d_i(t_{10})d_j(x_{10}) + y_{01}d_n(t_{10}) \\
& - d_n\big(y_{01}t_{10} + t_{10}y_{01}\big) + d_n(t_{10})y_{01} + \sum_{\substack{i+j=n \\ 0<i,j<n}} d_i(y_{01})d_j(t_{10}) + \sum_{\substack{i+j=n \\ 0<i,j<n}} d_i(t_{10})d_j(y_{01}) \\
= \;& 0.
\end{aligned}
$$

By assumption it follows that $t_{10}[d_n(x_{10}+y_{01}) - d_n(x_{10}) - d_n(y_{01})] + [d_n(x_{10}+y_{01}) - d_n(x_{10}) - d_n(y_{01})]t_{10} = 0$. This yields that

$$e_0[d_n(x_{10}+y_{01}) - d_n(x_{10}) - d_n(y_{01})]e_1 = 0.$$

Similarly, we can find that $e_1[d_n(x_{10}+y_{01}) - d_n(x_{10}) - d_n(y_{01})]e_0 = 0$ and hence $d_n(x_{10}+y_{01}) = d_n(x_{10}) + d_n(y_{01})$. □

Lemma 3.8. *$d_n(x_{ii}+y_{ji}+z_{ij}) = d_n(x_{ii}) + d_n(y_{ji}) + d_n(z_{ij})$ for $i \in \{0,1\}$ and $i \neq j$.*

Proof. Let us consider the possibility of $i = 1$ and $j = 0$, we have

$$
\begin{aligned}
& d_n[(x_{11}+y_{01}+z_{10})t_{00} + t_{00}(x_{11}+y_{01}+z_{10})] \\
& \quad = (x_{11}+y_{01}+z_{10})d_n(t_{00}) + d_n(t_{00})(x_{11}+y_{01}+z_{10}) \\
& \qquad + \sum_{\substack{i+j=n \\ 0<i,j<n}} d_i(x_{11}+y_{01}+z_{10})d_j(t_{00}) + \sum_{\substack{i+j=n \\ 0<i,j<n}} d_i(t_{00})d_j(x_{11}+y_{01}+z_{10})
\end{aligned}
$$

On the other way

$$
\begin{aligned}
& d_n[(x_{11}+y_{01}+z_{10})t_{00} + t_{00}(x_{11}+y_{01}+z_{10})] \\
= \;& d_n(z_{10}t_{00} + t_{00}y_{01}) \\
= \;& d_n(z_{10}t_{00} + t_{00}z_{10}) + d_n(x_{11}t_{00} + t_{00}x_{11}) + d_n(y_{01}t_{00} + t_{00}y_{01}) \\
= \;& z_{10}d_n(t_{00}) + d_n(t_{00})z_{10} + t_{00}d_n(z_{10}) + d_n(z_{10})t_{00} + \sum_{\substack{i+j=n \\ 0<i,j<n}} d_i(z_{10})d_j(t_{00}) \\
& + \sum_{\substack{i+j=n \\ 0<i,j<n}} d_i(t_{00})d_j(z_{10}) + x_{11}d_n(t_{00}) + d_n(t_{00})x_{11} + t_{00}d_n(x_{11}) + d_n(x_{11})t_{00} \\
& + \sum_{\substack{i+j=n \\ 0<i,j<n}} d_i(x_{11})d_j(t_{00}) + \sum_{\substack{i+j=n \\ 0<i,j<n}} d_i(t_{00})d_j(x_{11}) + y_{01}d_n(t_{00}) + d_n(t_{00})y_{01} \\
& + t_{00}d_n(y_{01}) + d_n(y_{01})t_{00} + \sum_{\substack{i+j=n \\ 0<i,j<n}} d_i(y_{01})d_j(t_{00}) + \sum_{\substack{i+j=n \\ 0<i,j<n}} d_i(t_{00})d_j(y_{01}).
\end{aligned}
$$

It follows that

$$
\begin{aligned}
e_1[d_n(x_{11}+y_{01}+z_{10}) - d_n(x_{11}) - d_n(y_{01}) - d_n(z_{10})]e_0 &= 0, \\
e_0[d_n(x_{11}+y_{01}+z_{10}) - d_n(x_{11}) - d_n(y_{01}) - d_n(z_{10})]e_0 &= 0, \\
e_0[d_n(x_{11}+y_{01}+z_{10}) - d_n(x_{11}) - d_n(y_{01}) - d_n(z_{10})]e_1 &= 0.
\end{aligned}
$$

Now applying similar steps as used in previous lemmas we can obtain that $e_1[d_n(x_{11}+y_{01}+z_{10})-d_n(x_{11})-d_n(y_{01})-d_n(z_{10})]e_1=0$, this leads to the result. Similarly, we can prove the rest of cases. □

Lemma 3.9. *For any* $x_{00}\in\mathcal{R}_{00}, y_{01}\in\mathcal{R}_{01}, z_{10}\in\mathcal{R}_{10}$ *and* $w_{11}\in\mathcal{R}_{11}$,

$$d_n(x_{00}+y_{01}+z_{10}+w_{11})=d_n(x_{00})+d_n(y_{01})+d_n(z_{10})+d_n(w_{11}).$$

Proof. Consider that

$$\begin{aligned}
&d_n[(x_{00}+y_{01}+z_{10}+w_{11})t_{00}+t_{00}(x_{00}+y_{01}+z_{10}+w_{11})]\\
&= d_n(x_{00}+y_{01}+z_{10}+w_{11})t_{00}+t_{00}d_n(x_{00}+y_{01}+z_{10}+w_{11})\\
&\quad +(x_{00}+y_{01}+z_{10}+w_{11})d_n(t_{00})+d_n(t_{00})(x_{00}+y_{01}+z_{10}+w_{11})\\
&\quad +\sum_{\substack{i+j=n\\0<i,j<n}} d_i(x_{00}+y_{01}+z_{10}+w_{11})d_j(t_{00})+\sum_{\substack{i+j=n\\0<i,j<n}} d_i(t_{00})d_j(x_{00}+y_{01}+z_{10}+w_{11}).
\end{aligned}$$

On the other way

$$\begin{aligned}
&d_n[(x_{00}+y_{01}+z_{10}+w_{11})t_{00}+t_{00}(x_{00}+y_{01}+z_{10}+w_{11})]\\
&= d_n(x_{00}t_{00}+t_{00}x_{00})+d_n(y_{01}t_{00}+t_{00}y_{01})+d_n(w_{11}t_{00}+t_{00}w_{11})+d_n(z_{10}t_{00}+t_{00}z_{10})\\
&= d_n(x_{00})t_{00}+t_{00}d_n(x_{00})+x_{00}d_n(t_{00})+d_n(t_{00})x_{00}+\sum_{\substack{i+j=n\\0<i,j<n}} d_i(x_{00})d_j(t_{00})\\
&\quad +\sum_{\substack{i+j=n\\0<i,j<n}} d_i(t_{00})d_j(x_{00})+d_n(y_{01})t_{00}+t_{00}d_n(y_{01})+y_{01}d_n(t_{00})+d_n(t_{00})y_{01}\\
&\quad +\sum_{\substack{i+j=n\\0<i,j<n}} d_i(y_{01})d_j(t_{00})+\sum_{\substack{i+j=n\\0<i,j<n}} d_i(t_{00})d_j(y_{01})+d_n(z_{10})t_{00}+t_{00}d_n(z_{10})+z_{10}d_n(t_{00})\\
&\quad +d_n(t_{00})z_{10}+\sum_{\substack{i+j=n\\0<i,j<n}} d_i(z_{10})d_j(t_{00})+\sum_{\substack{i+j=n\\0<i,j<n}} d_i(t_{00})d_j(z_{10})+d_n(w_{11})t_{00}+t_{00}d_n(w_{11})\\
&\quad +w_{11}d_n(t_{00})+d_n(t_{00})w_{11}+\sum_{\substack{i+j=n\\0<i,j<n}} d_i(w_{11})d_j(t_{00})+\sum_{\substack{i+j=n\\0<i,j<n}} d_i(t_{00})d_j(w_{11}).
\end{aligned}$$

On combining above two expressions, we find that

$$\begin{aligned}
e_1[d_n(x_{00}+y_{01}+z_{10}+w_{11})-d_n(x_{00})-d_n(y_{01})-d_n(z_{10})-d_n(w_{11})]e_0 &= 0,\\
e_0[d_n(x_{00}+y_{01}+z_{10}+w_{11})-d_n(x_{00})-d_n(y_{01})-d_n(z_{10})-d_n(w_{11})]e_0 &= 0,\\
e_0[d_n(x_{00}+y_{01}+z_{10}+w_{11})-d_n(x_{00})-d_n(y_{01})-d_n(z_{10})-d_n(w_{11})]e_1 &= 0.
\end{aligned}$$

On applying similar method as used in previous lemmas, we get that

$$e_1[d_n(x_{00}+y_{01}+z_{10}+w_{11})-d_n(x_{00})-d_n(y_{01})-d_n(z_{10})-d_n(w_{11})]e_1=0.$$

This proves our result. □

Proof of Theorem 3.1. For any $x,y\in\mathcal{R}$, let $x=x_{11}+x_{10}+x_{01}+x_{00}$ and $y=y_{11}+y_{10}+y_{01}+y_{00}$ for all $x_{ij},y_{ij}\in\mathcal{R}_{ij}$ where $i,j\in\{0,1\}$. On using Lemmas 3.5, 3.6 and 3.9, we have

$$\begin{aligned}
d_n(x+y) &= d_n(x_{11}+x_{10}+x_{01}+x_{00}+y_{11}+y_{10}+y_{01}+y_{00})\\
&= d_n(x_{11}+y_{11})+d_n(x_{01}+y_{01})+d_n(x_{01}+y_{01})+d_n(x_{00}+y_{00})\\
&= d_n(x_{11})+d_n(y_{11})+d_n(x_{01})+d_n(y_{01})+d_n(x_{01})+d_n(y_{01})\\
&\quad +d_n(x_{00})+d_n(y_{00})\\
&= d_n(x_{11}+x_{10}+x_{01}+x_{00})+d_n(y_{11}+y_{10}+y_{01}+y_{00})\\
&= d_n(x)+d_n(y).
\end{aligned}$$

This implies that $D = \{d_n\}_{n\in\mathbb{N}}$ is additive on $\mathcal{R}$ and hence $D = \{d_n\}_{n\in\mathbb{N}}$ is a Jordan higher derivation on $\mathcal{R}$. □

In case of 2-torsion free alternative rings we can drop the condition (iii) of above theorem and we get the similar result as follows:

Theorem 3.10. *Let $\mathcal{R}$ be a 2-torsion free alternative ring containing a nontrivial idempotent satisfying $a_{ij}t_{jk} = 0$ or $t_{ki}a_{ij} = 0$ implies that $a_{ij} = 0$ for all $t_{jk} \in \mathcal{R}_{jk}, t_{ki} \in \mathcal{R}_{ki}$ and $i, j, k \in \{0, 1\}$. If the family $D = \{d_n\}_{n\in\mathbb{N}}$ of mappings $d_n : \mathcal{R} \to \mathcal{R}$ such that $d_0 = I_{\mathcal{R}}$ satisfies $d_n(xy+yx) = \sum_{i+j=n} d_i(x)d_j(y) + \sum_{i+j=n} d_i(y)d_j(x)$ for all $x, y \in \mathcal{R}$, then d_n is additive for each $n \in \mathbb{N}$ and hence $D = \{d_n\}_{n\in\mathbb{N}}$ is additive and hence D is a Jordan higher derivation on $\mathcal{R}$.*

In view of Theorem 2.2, it is clear that d_1 is additive. We will use this result throughout this section whenever needed without specific mention.

We facilitate our discussion with the following facts :

Fact 3.1. $d_n(0) = 0$.

Proof.

$$\begin{aligned} d_n(0) &= d_n(0 \circ 0) \\ &= d_n(0)0 + 0d_n(0) + d_n(0)0 + 0d_n(0) + 2\sum_{\substack{i+j=n \\ 0<i,j<n}} d_i(0)d_j(0) \\ &= d_n(0)0 + 0d_n(0) + d_n(0)0 + 0d_n(0) \\ &= 0. \end{aligned}$$

□

Fact 3.2. For $x_{ii} \in \mathcal{R}_{ii}, y_{jk} \in \mathcal{R}_{jk}$, where $i, j, k \in \{0, 1\}$ and $j \neq k$ such that

$$d_n(x_{ii} + y_{jk}) = d_n(x_{ii}) + d_n(y_{jk}).$$

Proof. First we prove that $d_n(x_{00} + y_{10}) = d_n(x_{00}) + d_n(y_{10})$. Now, for any $x_{00} \in \mathcal{R}_{00}$ and $y_{10} \in \mathcal{R}_{10}$, we have

$$\begin{aligned} & [d_n(x_{00} + y_{10}) - d_n(x_{00}) - d_n(y_{10})] \circ e_1 \\ &= d_n\big((x_{00} + y_{10})e_1 + e_1(x_{00} + y_{10})\big) - (x_{00} + y_{10})d_n(e_1) - d_n(e_1)(x_{00} + y_{10}) \\ &\quad - \sum_{\substack{i+j=n \\ 0<i,j<n}} d_i(x_{00} + y_{10})d_j(e_1) - \sum_{\substack{i+j=n \\ 0<i,j<n}} d_i(e_1)d_j(x_{00} + y_{10}) - d_n\big(x_{00}e_1 + e_1x_{00}\big) \\ &\quad + x_{00}d_n(e_1) + d_n(e_1)x_{00} + \sum_{\substack{i+j=n \\ 0<i,j<n}} d_i(x_{00})d_j(e_1) - \sum_{\substack{i+j=n \\ 0<i,j<n}} d_i(e_1)d_j(x_{00}) + y_{10}d_n(e_1) \\ &\quad - d_n\big(y_{10}e_1 + e_1y_{10}\big) + d_n(e_1)y_{10} + \sum_{\substack{i+j=n \\ 0<i,j<n}} d_i(y_{10})d_j(e_1) + \sum_{\substack{i+j=n \\ 0<i,j<n}} d_i(e_1)d_j(y_{10}) \\ &= 0. \end{aligned}$$

This implies that

$$\begin{aligned} e_1[d_n(x_{00} + y_{10}) - d_n(x_{00}) - d_n(y_{10})]e_1 &= 0, \\ e_1[d_n(x_{00} + y_{10}) - d_n(x_{00}) - d_n(y_{10})]e_0 &= 0, \\ e_0[d_n(x_{00} + y_{10}) - d_n(x_{00}) - d_n(y_{10})]e_1 &= 0. \end{aligned}$$

Also, using Fact 3.1, we get

$$\begin{aligned}
&[d_n(x_{00}+y_{10})-d_n(x_{00})-d_n(y_{10})]\circ t_{10}\\
=\;& d_n\big((x_{00}+y_{10})t_{10}+t_{10}(x_{00}+y_{10})\big)-(x_{00}+y_{10})d_n(t_{10})-d(t_{10})(x_{00}+y_{10})\\
&-\sum_{\substack{i+j=n\\0<i,j<n}} d_i(x_{00}+y_{10})d_j(t_{10})-\sum_{\substack{i+j=n\\0<i,j<n}} d_i(t_{10})d_j(x_{00}+y_{10})-d_n\big(x_{00}t_{10}+t_{10}x_{00}\big)\\
&+x_{00}d_n(t_{10})+d_n(t_{10})x_{00}+\sum_{\substack{i+j=n\\0<i,j<n}} d_i(x_{00})d_j(t_{10})+\sum_{\substack{i+j=n\\0<i,j<n}} d_i(t_{10})d_j(x_{00})+y_{10}d(t_{10})\\
&-d_n\big(y_{10}t_{10}+t_{10}y_{10}\big)+d_n(t_{10})y_{10}+\sum_{\substack{i+j=n\\0<i,j<n}} d_i(y_{10})d_j(t_{10})+\sum_{\substack{i+j=n\\0<i,j<n}} d_i(t_{10})d_j(y_{10})\\
=\;& 0.
\end{aligned}$$

By assumption, we can find that $e_0[d_n(x_{00}+y_{10})-d_n(x_{00})-d_n(y_{10})]e_0=0$ and hence $d_n(x_{00}+y_{10})=d_n(x_{00})+d_n(y_{10})$.

In the similar way, we can prove rest of the cases. □

Fact 3.3. For any $x_{10},y_{10}\in\mathcal{R}_{10};x_{01},z_{01}\in\mathcal{R}_{01}$ and $y_{00},z_{00}\in\mathcal{R}_{00}$,

(i) $d_n(x_{10}+y_{10}z_{00})=d_n(x_{10})+d_n(y_{10}z_{00})$,

(ii) $d_n(x_{01}+y_{00}z_{01})=d_n(x_{01})+d_n(y_{00}z_{01})$.

Proof. Same as the proof of Lemma 3.4. □

Fact 3.4. For any $x_{10},y_{10}\in\mathcal{R}_{10}$ and $x_{01},y_{01}\in\mathcal{R}_{01}$,

(i) $d_n(x_{10}+y_{10})=d_n(x_{10})+d_n(y_{10})$,

(ii) $d_n(x_{01}+y_{01})=d_n(x_{01})+d_n(y_{01})$.

Proof. (i) For any $x_{10},y_{10}\in\mathcal{R}_{10}$, we obtain

$$\begin{aligned}
&[d_n(x_{10}+y_{10})-d_n(x_{10})-d_n(y_{10})]\circ e_0\\
=\;& d_n\big((x_{10}+y_{10})e_0+e_0(x_{10}+y_{10})\big)-(x_{10}+y_{10})d_n(e_0)-d_n(e_0)(x_{10}+y_{10})\\
&-\sum_{\substack{i+j=n\\0<i,j<n}} d_i(x_{10}+y_{10})d_j(e_0)-\sum_{\substack{i+j=n\\0<i,j<n}} d_i(e_0)d_j(x_{10}+y_{10})-d_n\big(x_{10}e_0+e_0x_{10}\big)\\
&+x_{10}d_n(e_0)+d_n(e_0)x_{10}+\sum_{\substack{i+j=n\\0<i,j<n}} d_i(x_{10})d_j(e_0)+\sum_{\substack{i+j=n\\0<i,j<n}} d_i(e_0)d_j(x_{10})+y_{10}d_n(e_0)\\
&-d_n\big(y_{10}e_0+e_0y_{10}\big)+d_n(e_0)y_{10}+\sum_{\substack{i+j=n\\0<i,j<n}} d_i(y_{10})d_j(e_0)+\sum_{\substack{i+j=n\\0<i,j<n}} d_i(e_0)d_j(y_{10})\\
=\;& d_n(x_{10}+y_{10})-d_n(x_{10})-d_n(y_{10}).
\end{aligned}$$

It follows that

$$\begin{aligned}
e_0[d_n(x_{10}+y_{10})-d_n(x_{10})-d_n(y_{10})]e_0 &= 0,\\
e_1[d_n(x_{10}+y_{10})-d_n(x_{10})-d_n(y_{10})]e_1 &= 0.
\end{aligned}$$

Now, for any $x_{10}, y_{10}, t_{10} \in \mathcal{R}_{10}$ and using Fact 3.1, we obtain that

$$\begin{aligned}
&[d_n(x_{10}+y_{10}) - d_n(x_{10}) - d_n(y_{10})] \circ t_{10} \\
&= d_n\big((x_{10}+y_{10})t_{10} + t_{10}(x_{10}+y_{10})\big) - (x_{10}+y_{10})d_n(t_{10}) - d_n(t_{10})(x_{10}+y_{10}) \\
&\quad - \sum_{\substack{i+j=n \\ 0<i,j<n}} d_i(x_{10}+y_{10})d_j(t_{10}) - \sum_{\substack{i+j=n \\ 0<i,j<n}} d_i(t_{10})d_j(x_{10}+y_{10}) - d_n\big(x_{10}t_{10} + t_{10}x_{10}\big) \\
&\quad + x_{10}d_n(t_{10}) + d_n(t_{10})x_{10} + \sum_{\substack{i+j=n \\ 0<i,j<n}} d_i(x_{10})d_j(t_{10}) + \sum_{\substack{i+j=n \\ 0<i,j<n}} d_i(t_{10})d_j(x_{10}) + y_{10}d_n(t_{10}) \\
&\quad - d_n\big(y_{10}t_{10} + t_{10}y_{10}\big) + d_n(t_{10})y_{10} + \sum_{\substack{i+j=n \\ 0<i,j<n}} d_i(y_{10})d_j(t_{10}) + \sum_{\substack{i+j=n \\ 0<i,j<n}} d_i(t_{10})d_j(y_{10}) \\
&= 0.
\end{aligned}$$

This implies that

$$t_{10}[d_n(x_{10}+y_{10}) - d_n(x_{10}) - d_n(y_{10})] + [d_n(x_{10}+y_{10}) - d_n(x_{10}) - d_n(y_{10})]t_{10} = 0.$$

This gives us $e_0[d_n(x_{10}+y_{10}) - d_n(x_{10}) - d_n(y_{10})]e_1 = 0$. Similarly, we find that $e_1[d_n(x_{10}+y_{10}) - d_n(x_{10}) - d_n(y_{10})]e_0 = 0$ and hence $d_n(x_{10}+y_{10}) = d_n(x_{10}) + d_n(y_{10})$.
(ii) Similar to (i). □

Fact 3.5. For any $x_{00}, y_{00} \in \mathcal{R}_{00}$ and $x_{11}, y_{11} \in \mathcal{R}_{11}$,

(i) $d_n(x_{00}+y_{00}) = d_n(x_{00}) + d_n(y_{00})$,

(ii) $d_n(x_{11}+y_{11}) = d_n(x_{11}) + d_n(y_{11})$.

Proof. For any $x_{00}, y_{00} \in \mathcal{R}_{00}$

$$\begin{aligned}
&[d_n(x_{00}+y_{00}) - d_n(x_{00}) - d_n(y_{00})] \circ e_1 \\
&= d_n\big((x_{00}+y_{00})e_1 + e_1(x_{00}+y_{00})\big) - (x_{00}+y_{00})d_n(e_1) - d_n(e_1)(x_{00}+y_{00}) \\
&\quad - \sum_{\substack{i+j=n \\ 0<i,j<n}} d_i(x_{00}+y_{00})d_j(e_1) - \sum_{\substack{i+j=n \\ 0<i,j<n}} d_i(e_1)d_j(x_{00}+y_{00}) - d_n\big(x_{00}e_1 + e_1x_{00}\big) \\
&\quad + x_{00}d_n(e_1) + d_n(e_1)x_{00} + \sum_{\substack{i+j=n \\ 0<i,j<n}} d_i(x_{00})d_j(e_1) + \sum_{\substack{i+j=n \\ 0<i,j<n}} d_i(e_1)d_j(x_{00}) + y_{00}d_n(e_1) \\
&\quad - d_n\big(y_{00}e_1 + e_1y_{00}\big) + d_n(e_1)y_{00} + \sum_{\substack{i+j=n \\ 0<i,j<n}} d_i(y_{00})d_j(e_1) + \sum_{\substack{i+j=n \\ 0<i,j<n}} d_i(e_1)d_j(y_{00}) \\
&= 0.
\end{aligned}$$

This implies that

$$\begin{aligned}
e_1[d_n(x_{00}+y_{00}) - d_n(x_{00}) - d_n(y_{00})]e_1 &= 0, \\
e_1[d_n(x_{00}+y_{00}) - d_n(x_{00}) - d_n(y_{00})]e_0 &= 0, \\
e_0[d_n(x_{00}+y_{00}) - d_n(x_{00}) - d_n(y_{00})]e_1 &= 0.
\end{aligned}$$

Using Facts 3.1 and 3.4, we get

$$\begin{aligned}
&[d_n(x_{00}+y_{00})-d_n(x_{00})-d_n(y_{00})]\circ t_{10}\\
=\ & d_n\big((x_{00}+y_{00})t_{10}+t_{10}(x_{00}+y_{00})\big)-(x_{00}+y_{00})d_n(t_{10})-d_n(t_{10})(x_{00}+y_{00})\\
&-\sum_{\substack{i+j=n\\0<i,j<n}} d_i(x_{00}+y_{00})d_j(t_{10})-\sum_{\substack{i+j=n\\0<i,j<n}} d_i(t_{10})d_j(x_{00}+y_{00})-d_n\big(x_{00}t_{10}+t_{10}x_{00}\big)\\
&+x_{00}d_n(t_{10})+d_n(t_{10})x_{00}+\sum_{\substack{i+j=n\\0<i,j<n}} d_i(x_{00})d_j(t_{10})+\sum_{\substack{i+j=n\\0<i,j<n}} d_i(t_{10})d_j(x_{00})+y_{00}d_n(t_{10})\\
&-d_n\big(y_{00}t_{10}+t_{10}y_{00}\big)+d_n(t_{10})y_{00}+\sum_{\substack{i+j=n\\0<i,j<n}} d_i(y_{00})d_j(t_{10})+\sum_{\substack{i+j=n\\0<i,j<n}} d_i(t_{10})d_j(y_{00})\\
=\ & 0.
\end{aligned}$$

This shows that $e_0[d_n(x_{00}+y_{00})-d_n(x_{00})-d_n(y_{00})]e_0 = 0$ and by our assumption $d_n(x_{00}+y_{00})=d_n(x_{00})+d_n(y_{00})$. In the similar way, we can obtain (ii). □

Fact 3.6. For any $x_{10},\in\mathcal{R}_{10}$ and $y_{01}\in\mathcal{R}_{01}$, $d_n(x_{10}+y_{01})=d_n(x_{10})+d_n(y_{01})$.

Proof. For any $x_{10}\in\mathcal{R}_{10}$ and $y_{01}\in\mathcal{R}_{01}$, we find that

$$\begin{aligned}
&[d_n(x_{10}+y_{01})-d_n(x_{10})-d_n(y_{01})]\circ e_0\\
=\ & d_n\big((x_{10}+y_{01})e_0+e_0(x_{10}+y_{01})\big)-(x_{10}+y_{01})d_n(e_0)-d_n(e_0)(x_{10}+y_{01})\\
&-\sum_{\substack{i+j=n\\0<i,j<n}} d_i(x_{10}+y_{01})d_j(e_0)-\sum_{\substack{i+j=n\\0<i,j<n}} d_i(e_0)d_j(x_{10}+y_{01})-d_n\big(x_{10}e_0+e_0x_{10}\big)\\
&+x_{10}d_n(e_0)+d_n(e_0)x_{10}+\sum_{\substack{i+j=n\\0<i,j<n}} d_i(x_{10})d_j(e_0)+\sum_{\substack{i+j=n\\0<i,j<n}} d_i(e_0)d_j(x_{10})+y_{01}d_n(e_0)\\
&-d_n\big(y_{01}e_0+e_0y_{01}\big)+d_n(e_0)y_{01}+\sum_{\substack{i+j=n\\0<i,j<n}} d_i(y_{01})d_j(e_0)+\sum_{\substack{i+j=n\\0<i,j<n}} d_i(e_0)d_j(y_{01})\\
=\ & d_n(x_{10}+y_{01})-d_n(x_{10})-d_n(y_{01}).
\end{aligned}$$

This implies that

$$\begin{aligned}
e_1[d_n(x_{10}+y_{01})-d_n(x_{10})-d_n(y_{01})]e_1 &= 0,\\
e_0[d_n(x_{10}+y_{01})-d_n(x_{10})-d_n(y_{01})]e_0 &= 0.
\end{aligned}$$

Also, for any $x_{10}\in\mathcal{R}_{10}$ and $y_{01}\in\mathcal{R}_{01}$, we get

$$\begin{aligned}
&[d_n(x_{10}+y_{01})-d_n(x_{10})-d_n(y_{01})]\circ t_{10}\\
=\ & d_n\big((x_{10}+y_{01})t_{10}+t_{10}(x_{10}+y_{01})\big)-(x_{10}+y_{01})d_n(t_{10})-d_n(t_{10})(x_{10}+y_{01})\\
&-\sum_{\substack{i+j=n\\0<i,j<n}} d_i(x_{10}+y_{01})d_j(t_{10})-\sum_{\substack{i+j=n\\0<i,j<n}} d_i(t_{10})d_j(x_{10}+y_{01})-d_n\big(x_{10}t_{10}+t_{10}x_{10}\big)\\
&+x_{10}d_n(t_{10})+d_n(t_{10})x_{10}+\sum_{\substack{i+j=n\\0<i,j<n}} d_i(x_{10})d_j(t_{10})+\sum_{\substack{i+j=n\\0<i,j<n}} d_i(t_{10})d_j(x_{10})+y_{01}d_n(t_{10})\\
&-d_n\big(y_{01}t_{10}+t_{10}y_{01}\big)+d_n(t_{10})y_{01}+\sum_{\substack{i+j=n\\0<i,j<n}} d_i(y_{01})d_j(t_{10})+\sum_{\substack{i+j=n\\0<i,j<n}} d_i(t_{10})d_j(y_{01})\\
=\ & 0.
\end{aligned}$$

By assumption it follows that $t_{10}[d_n(x_{10}+y_{01})-d_n(x_{10})-d_n(y_{01})]+[d_n(x_{10}+y_{01})-d_n(x_{10})-d_n(y_{01})]t_{10}=0$. This yields that

$$e_0[d_n(x_{10}+y_{01})-d_n(x_{10})-d_n(y_{01})]e_1=0.$$

Similarly, we can find that $e_1[d_n(x_{10}+y_{01})-d_n(x_{10})-d_n(y_{01})]e_0=0$ and hence $d_n(x_{10}+y_{01})=d_n(x_{10})+d_n(y_{01})$. □

Fact 3.7. For any $x_{00}\in\mathcal{R}_{00}$ and $y_{11}\in\mathcal{R}_{11}$, $d_n(x_{00}+y_{11})=d_n(x_{00})+d_n(y_{11})$.

Proof. For any $x_{00}\in\mathcal{R}_{00}$ and $y_{11}\in\mathcal{R}_{11}$, we find that

$$\begin{aligned}
&[d_n(x_{00}+y_{11})-d_n(x_{00})-d_n(y_{11})]\circ e_0\\
&= d_n\big((x_{00}+y_{11})e_0+e_0(x_{00}+y_{11})\big)-(x_{00}+y_{11})d_n(e_0)-d_n(e_0)(x_{00}+y_{11})\\
&\quad-\sum_{\substack{i+j=n\\0<i,j<n}}d_i(x_{00}+y_{11})d_j(e_0)\sum_{\substack{i+j=n\\0<i,j<n}}d_i(e_0)d_j(x_{00}+y_{11})-d_n\big(x_{00}e_0+e_0x_{00}\big)\\
&\quad+x_{00}d_n(e_0)+d_n(e_0)x_{00}+\sum_{\substack{i+j=n\\0<i,j<n}}d_i(x_{00})d_j(e_0)+\sum_{\substack{i+j=n\\0<i,j<n}}d_i(e_0)d_j(x_{00})+y_{11}d_n(e_0)\\
&\quad-d_n\big(y_{11}e_0+e_0y_{11}\big)+d_n(e_0)y_{11}+\sum_{\substack{i+j=n\\0<i,j<n}}d_i(y_{11})d_j(e_0)+\sum_{\substack{i+j=n\\0<i,j<n}}d_i(e_0)d_j(y_{11})\\
&= 0.
\end{aligned}$$

This implies that

$$\begin{aligned}
e_0[d_n(x_{00}+y_{11})-d_n(x_{00})-d_n(y_{11})]e_0 &= 0,\\
e_0[d_n(x_{00}+y_{11})-d_n(x_{00})-d_n(y_{11})]e_1 &= 0,\\
e_1[d_n(x_{00}+y_{11})-d_n(x_{00})-d_n(y_{11})]e_0 &= 0.
\end{aligned}$$

Now, using Fact 3.4, we arrive at

$$\begin{aligned}
&[d_n(x_{00}+y_{11})-d_n(x_{00})-d_n(y_{11})]\circ t_{10}\\
&= d_n\big((x_{00}+y_{11})t_{10}+t_{10}(x_{00}+y_{11})\big)-(x_{00}+y_{11})d_n(t_{10})-d_n(t_{10})(x_{00}+y_{11})\\
&\quad-\sum_{\substack{i+j=n\\0<i,j<n}}d_i(x_{00}+y_{11})d_j(t_{10})\sum_{\substack{i+j=n\\0<i,j<n}}d_i(t_{10})d_j(x_{00}+y_{11})-d_n\big(x_{00}t_{10}+t_{10}x_{00}\big)\\
&\quad+x_{00}d_n(t_{10})+d_n(t_{10})x_{00}+\sum_{\substack{i+j=n\\0<i,j<n}}d_i(x_{00})d_j(t_{10})+\sum_{\substack{i+j=n\\0<i,j<n}}d_i(t_{10})d_j(x_{00})+y_{11}d_n(t_{10})\\
&\quad-d_n\big(y_{11}t_{10}+t_{10}y_{11}\big)+d_n(t_{10})y_{11}+\sum_{\substack{i+j=n\\0<i,j<n}}d_i(y_{11})d_j(t_{10})+\sum_{\substack{i+j=n\\0<i,j<n}}d_i(t_{10})d_j(y_{11})\\
&= 0.
\end{aligned}$$

This yields that $e_1[d_n(x_{00}+y_{11})-d_n(x_{00})-d_n(y_{11})]e_1=0$ and hence $d_n(x_{00}+y_{11})=d_n(x_{00})+d_n(y_{11})$. □

Fact 3.8. For any $x_{00}\in\mathcal{R}_{00}$; $z_{01}\in\mathcal{R}_{01}$; $z_{10}\in\mathcal{R}_{10}$ and $y_{11}\in\mathcal{R}_{11}$,

(i) $d_n(x_{00}+y_{11}+z_{10})=d_n(x_{00})+d(y_{11})+d_n(z_{10})$,

(ii) $d_n(x_{00}+y_{11}+z_{01})=d_n(x_{00})+d_n(y_{11})+d_n(z_{01})$.

Proof. Now, using Fact 3.2, we get

$$
\begin{aligned}
&[d_n(x_{00}+y_{11}+z_{10})-d_n(x_{00})-d_n(y_{11})-d_n(z_{10})]\circ e_0\\
&= d_n\big((x_{00}+y_{11}+z_{10})e_0+e_0(x_{00}+y_{11}+z_{10})\big)-(x_{00}+y_{11}+z_{10})d_n(e_0)\\
&\quad -d_n(e_0)(x_{00}+y_{11}+z_{10})-\sum_{\substack{i+j=n\\0<i,j<n}} d_i(x_{00}+y_{11}+z_{10})d_j(e_0)\\
&\quad -\sum_{\substack{i+j=n\\0<i,j<n}} d_i(e_0)d_j(x_{00}+y_{11}+z_{10})-d_n\big(x_{00}e_0+e_0x_{00}\big)+x_{00}d_n(e_0)+d_n(e_0)x_{00}\\
&\quad +\sum_{\substack{i+j=n\\0<i,j<n}} d_i(x_{00})d_j(e_0)+\sum_{\substack{i+j=n\\0<i,j<n}} d_i(e_0)d_j(x_{00})-d_n\big(y_{11}e_0+e_0y_{11}\big)+y_{11}d_n(e_0)\\
&\quad +d_n(e_0)y_{11}-d_n\big(z_{10}e_0+e_0z_{10}\big)+z_{10}d_n(e_0)+d_n(e_0)z_{10}+\sum_{\substack{i+j=n\\0<i,j<n}} d_i(y_{11})d_j(e_0)\\
&\quad +\sum_{\substack{i+j=n\\0<i,j<n}} d_i(e_0)d_j(y_{11})+\sum_{\substack{i+j=n\\0<i,j<n}} d_i(z_{10})d_j(e_0)+\sum_{\substack{i+j=n\\0<i,j<n}} d_i(e_0)d_j(z_{10})\\
&= 0.
\end{aligned}
$$

Therefore, we obtain that

$$
\begin{aligned}
e_0[d_n(x_{00}+y_{11}+z_{10})-d_n(x_{00})-d_n(y_{11})-d_n(z_{10})]e_0 &= 0,\\
e_0[d_n(x_{00}+y_{11}+z_{10})-d_n(x_{00})-d_n(y_{11})-d_n(z_{10})]e_1 &= 0,\\
e_1[d_n(x_{00}+y_{11}+z_{10})-d_n(x_{00})-d_n(y_{11})-d_n(z_{10})]e_0 &= 0.
\end{aligned}
$$

Also, using Fact 3.4, we get

$$
\begin{aligned}
&[d_n(x_{00}+y_{11}+z_{10})-d_n(x_{00})-d_n(y_{11})-d_n(z_{10})]\circ t_{10}\\
&= d_n\big((x_{00}+y_{11}+z_{10})t_{10}+t_{10}(x_{00}+y_{11}+z_{10})\big)-(x_{00}+y_{11}+z_{10})d_n(t_{10})\\
&\quad -d_n(t_{10})(x_{00}+y_{11}+z_{10})-\sum_{\substack{i+j=n\\0<i,j<n}} d_i(x_{00}+y_{11}+z_{10})d_j(t_{10})\\
&\quad -\sum_{\substack{i+j=n\\0<i,j<n}} d_i(t_{10})d_j(x_{00}+y_{11}+z_{10})-d_n\big(x_{00}t_{10}+t_{10}x_{00}\big)+x_{00}d_n(t_{10})+d_n(t_{10})x_{00}\\
&\quad +\sum_{\substack{i+j=n\\0<i,j<n}} d_i(x_{00})d_j(t_{10})+\sum_{\substack{i+j=n\\0<i,j<n}} d_i(t_{10})d_j(x_{00})-d_n\big(y_{11}t_{10}+t_{10}y_{11}\big)+y_{11}d_n(t_{10})\\
&\quad +d(t_{10})y_{11}-d_n\big(z_{10}t_{10}+t_{10}z_{10}\big)+z_{10}d_n(t_{10})+d_n(t_{10})z_{10}+\sum_{\substack{i+j=n\\0<i,j<n}} d_i(y_{11})d_j(t_{10})\\
&\quad +\sum_{\substack{i+j=n\\0<i,j<n}} d_i(t_{10})d_j(y_{11})+\sum_{\substack{i+j=n\\0<i,j<n}} d_i(z_{10})d_j(t_{10})+\sum_{\substack{i+j=n\\0<i,j<n}} d_i(t_{10})d_j(z_{10})\\
&= 0.
\end{aligned}
$$

From here we can find that $e_1[d(x_{00}+y_{11}+z_{10})-d(x_{00})-d(y_{11})-d(z_{10})]e_1=0$, and hence $d_n(x_{00}+y_{11}+z_{10})=d_n(x_{00})+d_n(y_{11})+d_n(z_{10})$. In the similar way, we can obtain (ii). □

Fact 3.9. For any $x_{00}\in\mathcal{R}_{00}; y_{01}\in\mathcal{R}_{01}; z_{10}\in\mathcal{R}_{10}$ and $x_{11}\in\mathcal{R}_{11}$,

(i) $d_n(x_{00}+y_{01}+z_{10})=d_n(x_{00})+d_n(y_{01})+d_n(z_{10})$,

(ii) $d_n(x_{11}+y_{01}+z_{10})=d_n(x_{11})+d_n(y_{01})+d_n(z_{10})$.

Proof. (i) Using Fact 3.6, we have

$$\begin{aligned}
&[d_n(x_{00}+y_{01}+z_{10})-d_n(x_{00})-d_n(y_{01})-d_n(z_{10})]\circ e_1\\
&= d_n\big((x_{00}+y_{01}+z_{10})e_1+e_1(x_{00}+y_{01}+z_{10})\big)-(x_{00}+y_{01}+z_{10})d_n(e_1)\\
&\quad -\sum_{\substack{i+j=n\\0<i,j<n}} d_i(x_{00}+y_{01}+z_{10})d_j(e_1)-\sum_{\substack{i+j=n\\0<i,j<n}} d_i(e_1)d_j(x_{00}+y_{01}+z_{10})\\
&\quad -d_n(e_1)(x_{00}+y_{01}+z_{10})-d_n\big(x_{00}e_1+e_1x_{00}\big)+x_{00}d_n(e_1)+d_n(e_1)x_{00}\\
&\quad +\sum_{\substack{i+j=n\\0<i,j<n}} d_i(x_{00})d_j(e_1)+\sum_{\substack{i+j=n\\0<i,j<n}} d_i(e_1)d_j(x_{00})-d_n\big(y_{01}e_1+e_1y_{01}\big)+y_{01}d_n(e_1)\\
&\quad +d_n(e_1)y_{01}+\sum_{\substack{i+j=n\\0<i,j<n}} d_i(y_{01})d_j(e_1)+\sum_{\substack{i+j=n\\0<i,j<n}} d_i(e_1)d_j(y_{01})-d_n\big(z_{10}e_1+e_1z_{10}\big)\\
&\quad +z_{10}d_n(e_1)+d_n(e_1)z_{10}+\sum_{\substack{i+j=n\\0<i,j<n}} d_i(z_{10})d_j(e_1)+\sum_{\substack{i+j=n\\0<i,j<n}} d_i(e_1)d_j(z_{10})\\
&= 0.
\end{aligned}$$

This implies that

$$\begin{aligned}
e_1[d_n(x_{00}+y_{01}+z_{10})-d_n(x_{00})-d_n(y_{01})-d_n(z_{10})]e_1 &= 0,\\
e_0[d_n(x_{00}+y_{01}+z_{10})-d_n(x_{00})-d_n(y_{01})-d_n(z_{10})]e_1 &= 0,\\
e_1[d_n(x_{00}+y_{01}+z_{10})-d_n(x_{00})-d_n(y_{01})-d_n(z_{10})]e_0 &= 0.
\end{aligned}$$

Also, using Facts 3.7 and 3.8, we obtain that

$$\begin{aligned}
&[d_n(x_{00}+y_{01}+z_{10})-d_n(x_{00})-d_n(y_{01})-d_n(z_{10})]\circ t_{10}\\
&= d_n\big((x_{00}+y_{01}+z_{10})t_{10}+t_{10}(x_{00}+y_{01}+z_{10})\big)-(x_{00}+y_{01}+z_{10})d_n(t_{10})\\
&\quad -\sum_{\substack{i+j=n\\0<i,j<n}} d_i(x_{00}+y_{01}+z_{10})d_j(t_{10})-\sum_{\substack{i+j=n\\0<i,j<n}} d_i(t_{10})d_j(x_{00}+y_{01}+z_{10})\\
&\quad -d_n(t_{10})(x_{00}+y_{01}+z_{10})-d_n\big(x_{00}t_{10}+t_{10}x_{00}\big)+x_{00}d_n(t_{10})+d_n(t_{10})x_{00}\\
&\quad +\sum_{\substack{i+j=n\\0<i,j<n}} d_i(x_{00})d_j(t_{10})+\sum_{\substack{i+j=n\\0<i,j<n}} d_i(t_{10})d_j(x_{00})-d_n\big(y_{01}t_{10}+t_{10}y_{01}\big)+y_{01}d_n(t_{10})\\
&\quad +d_n(t_{10})y_{01}+\sum_{\substack{i+j=n\\0<i,j<n}} d_i(y_{01})d_j(t_{10})+\sum_{\substack{i+j=n\\0<i,j<n}} d_i(t_{10})d_j(y_{01})-d_n\big(z_{10}t_{10}+t_{10}z_{10}\big)\\
&\quad +z_{10}d_n(t_{10})+d_n(t_{10})z_{10}+\sum_{\substack{i+j=n\\0<i,j<n}} d_i(z_{10})d_j(t_{10})+\sum_{\substack{i+j=n\\0<i,j<n}} d_i(t_{10})d_j(z_{10})\\
&= 0.
\end{aligned}$$

This yields that $e_0[d_n(x_{00}+y_{01}+z_{10})-d_n(x_{00})-d_n(y_{01})-d_n(z_{10})]e_0=0$ and hence we arrive at $d_n(x_{00}+y_{01}+z_{10})=d_n(x_{00})+d_n(y_{01})+d_n(z_{10})$. In the similar way, we can obtain (ii). □

Fact 3.10. For any $x_{00}\in\mathcal{R}_{00}, y_{01}\in\mathcal{R}_{01}, z_{10}\in\mathcal{R}_{10}$ and $w_{11}\in\mathcal{R}_{11}$,

$$d_n(x_{00}+y_{01}+z_{10}+w_{11})=d_n(x_{00})+d_n(y_{01})+d_n(z_{10})+d_n(w_{11}).$$

Proof. Using Fact 3.9, we find that

$$\begin{aligned}
&[d_n(x_{00}+y_{01}+z_{10}+w_{11})-d_n(x_{00})-d_n(y_{01})-d_n(z_{10})-d_n(w_{11})]\circ e_1\\
&= d_n\big((x_{00}+y_{01}+z_{10}+w_{11})e_1+e_1(x_{00}+y_{01}+z_{10}+w_{11})\big)\\
&\quad -(x_{00}+y_{01}+z_{10}+w_{11})d(e_1)-d_n(e_1)(x_{00}+y_{01}+z_{10}+w_{11})\\
&\quad -\sum_{\substack{i+j=n\\0<i,j<n}} d_i(x_{00}+y_{01}+z_{10}+w_{11})d_j(e_1)-\sum_{\substack{i+j=n\\0<i,j<n}} d_i(e_1)d_j(x_{00}+y_{01}+z_{10}+w_{11})\\
&\quad -d_n\big(x_{00}e_1+e_1x_{00}\big)+x_{00}d_n(e_1)+d_n(e_1)x_{00}+\sum_{\substack{i+j=n\\0<i,j<n}} d_i(x_{00})d_j(e_1)\\
&\quad +\sum_{\substack{i+j=n\\0<i,j<n}} d_i(e_1)d_j(x_{00})-d_n\big(y_{01}e_1+e_1y_{01}\big)+y_{01}d_n(e_1)+d_n(e_1)y_{01}\\
&\quad +\sum_{\substack{i+j=n\\0<i,j<n}} d_i(y_{01})d_j(e_1)+\sum_{\substack{i+j=n\\0<i,j<n}} d_i(e_1)d_j(y_{01})-d_n\big(z_{10}e_1+e_1z_{10}\big)+z_{10}d_n(e_1)\\
&\quad +d_n(e_1)z_{10}+\sum_{\substack{i+j=n\\0<i,j<n}} d_i(z_{10})d_j(e_1)+\sum_{\substack{i+j=n\\0<i,j<n}} d_i(e_1)d_j(z_{10})-d_n\big(w_{11}e_1+e_1w_{11}\big)\\
&\quad +w_{11}d_n(e_1)+d_n(e_1)w_{11}+\sum_{\substack{i+j=n\\0<i,j<n}} d_i(w_{11})d_j(e_1)+\sum_{\substack{i+j=n\\0<i,j<n}} d_i(e_1)d_j(w_{11})\\
&= 0.
\end{aligned}$$

This implies that

$$\begin{aligned}
e_1[d_n(x_{00}+y_{01}+z_{10}+w_{11})-d_n(x_{00})-d_n(y_{01})-d_n(z_{10})-d_n(w_{11})]e_1 &= 0,\\
e_0[d_n(x_{00}+y_{01}+z_{10}+w_{11})-d_n(x_{00})-d_n(y_{01})-d_n(z_{10})-d_n(w_{11})]e_1 &= 0,\\
e_1[d_n(x_{00}+y_{01}+z_{10}+w_{11})-d_n(x_{00})-d_n(y_{01})-d_n(z_{10})-d_n(w_{11})]e_0 &= 0.
\end{aligned}$$

Now, using Fact 3.8, we get

$$\begin{aligned}
&[d_n(x_{00}+y_{01}+z_{10}+w_{11})-d_n(x_{00})-d_n(y_{01})-d_n(z_{10})-d_n(w_{11})]\circ t_{10}\\
&= d_n\big((x_{00}+y_{01}+z_{10}+w_{11})t_{10}+t_{10}(x_{00}+y_{01}+z_{10}+w_{11})\big)\\
&\quad -(x_{00}+y_{01}+z_{10}+w_{11})d_n(t_{10})-d_n(t_{10})(x_{00}+y_{01}+z_{10}+w_{11})\\
&\quad -\sum_{\substack{i+j=n\\0<i,j<n}} d_i(x_{00}+y_{01}+z_{10}+w_{11})d_j(t_{10})-\sum_{\substack{i+j=n\\0<i,j<n}} d_i(t_{10})d_j(x_{00}+y_{01}+z_{10}+w_{11})\\
&\quad -d_n\big(x_{00}t_{10}+t_{10}x_{00}\big)+x_{00}d_n(t_{10})+d_n(t_{10})x_{00}+\sum_{\substack{i+j=n\\0<i,j<n}} d_i(x_{00})d_j(t_{10})\\
&\quad +\sum_{\substack{i+j=n\\0<i,j<n}} d_i(t_{10})d_j(x_{00})-d_n\big(y_{01}t_{10}+t_{10}y_{01}\big)+y_{01}d_n(t_{10})+d_n(t_{10})y_{01}\\
&\quad +\sum_{\substack{i+j=n\\0<i,j<n}} d_i(y_{01})d_j(t_{10})+\sum_{\substack{i+j=n\\0<i,j<n}} d_i(t_{10})d_j(y_{01})-d_n\big(z_{10}t_{10}+t_{10}z_{10}\big)+d_n(t_{10})z_{10}\\
&\quad +z_{10}d_n(t_{10})+\sum_{\substack{i+j=n\\0<i,j<n}} d_i(z_{10})d_j(t_{10})+\sum_{\substack{i+j=n\\0<i,j<n}} d_i(t_{10})d_j(z_{10})-d_n\big(w_{11}t_{10}+t_{10}w_{11}\big)\\
&\quad +w_{11}d_n(t_{10})+d_n(t_{10})w_{11}+\sum_{\substack{i+j=n\\0<i,j<n}} d_i(w_{11})d_j(t_{10})+\sum_{\substack{i+j=n\\0<i,j<n}} d_i(t_{10})d_j(w_{11})\\
&= 0.
\end{aligned}$$

By assumption, we have $e_0[d_n(x_{00}+y_{01}+z_{10}+w_{11})-d_n(x_{00})-d_n(y_{01})-d_n(z_{10})-d_n(w_{11})]e_0 = 0$, which yields that $d_n(x_{00}+y_{01}+z_{10}+w_{11}) = d_n(x_{00})+d_n(y_{01})+d_n(z_{10})+d_n(w_{11})$. □

Proof of Theorem 3.10. Same as the proof Theorem 3.1. □

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