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Short note on Cohen-Macaulay and Gorenstein amalgamated duplication

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Abstract Let A be a commutative ring and let I be an ideal of A. The amalgamated duplication of A along I is the subring of $A \times A$ given by $A \bowtie I = \{(a, a+i)/a \in A, i \in I\}$. In this paper, we are interested in understanding when $A \bowtie I$ is Cohen-Macaulay (resp. Gorenstein) in the general (not necessarily local) case. .

1 Introduction

Throughout, all rings considered are commutative with unity and all modules are unital. Let A be a ring and I and ideal of A, and $\pi : A \to A/I$ the canonical surjection. The amalgamated duplication of A along I, denoted by $A \bowtie I$, is the special pullback (or fiber product) of π and π; i.e., the subring of $A \times A$ given by

$$A \bowtie I := \pi \times_{A/I} \pi = \{(a, a+i) \mid a \in A,\, i \in I\}$$

This construction was introduced and its basic properties were studied by D'Anna and Fontana in [5, 6] and then it was investigated by D'Anna in [4] with the aim of applying it to curve singularities (over algebraic closed fields) where he proved that the amalgamated duplication of an algebroid curve along a regular canonical ideal yields a Gorenstein algebroid curve [4, Theorem 14 and Corollary 17]. Let A be a Noetherian local ring of Krull dimension d and I be an ideal of A. In [4], it is proved that $A \bowtie I$ is Cohen-Macaulay if and only if A is Cohen-Macaulay and I is a maximal Cohen-Macaulay A-module. Moreover, in [1], the authors showed that $A \bowtie I$ is Gorenstein if and only if A is Cohen-Macaulay and I is a canonical module for A, and then A/I is Cohen-Macaulay with $\dim(A/I) = d-1$ (if I is a non unit proper ideal). In this paper, we study when $A \bowtie I$ is Cohen-Macaulay (resp. Gorenstein) in the general (not necessarily local) case. As general reference for terminology and well-known results, we refer the reader to [2].

2 Results

The study of Cohen-Macaulay (resp. Gorenstein) rings is based on the localization of rings with their maximal ideals. Hence, we need the following lemma.

Lemma 2.1. *Let* $\mathfrak{m}$ *be a maximal ideal of* A *and set*

$$\mathfrak{m} \bowtie I := (\mathfrak{m} \times A) \cap (A \bowtie I) = \{(m, m+i) \mid m \in \mathfrak{m},\, i \in I\}$$

and

$$\overline{\mathfrak{m}} := (A \times \mathfrak{m}) \cap (A \bowtie I) = \{(a, a+i) \mid a \in A,\, i \in I,\, a+i \in \mathfrak{m}\}$$

Let M *be a maximal ideal of* $A \bowtie I$*. Then,*

(i) $I \times I \subseteq M \Leftrightarrow \exists \mathfrak{m} \in \mathrm{Max}(A)$ *such that* $I \subseteq \mathfrak{m}$ *and* $M = \overline{\mathfrak{m}} = \mathfrak{m} \bowtie I$.
In this case, we have

$$(A \bowtie I)_M \cong A_{\mathfrak{m}} \bowtie I_{\mathfrak{m}}$$

(ii) $I \times I \nsubseteq M \Leftrightarrow \exists \mathfrak{m} \in \mathrm{Max}(A)$ *such that* $I \nsubseteq \mathfrak{m}$ *and* $M = \overline{\mathfrak{m}}$ *or* $M = \mathfrak{m} \bowtie I$.
In this case,

$$(A \bowtie I)_M \cong A_{\mathfrak{m}}$$

Consequently, we have

$$\mathrm{Max}\,(A \bowtie I)) = \{\overline{\mathfrak{m}}, \mathfrak{m} \bowtie I \mid \mathfrak{m} \in \mathrm{Max}(A)\}$$

Proof. (1) ($\Rightarrow$) Assume that $I \times I \subseteq M$, and consider the ideal $\mathfrak{m}$ of A given by

$$\mathfrak{m} := \{m \in A \mid \exists i \in I \text{ such that } (m, m+i) \in M\}$$

Clearly, the fact that $I \times I \subseteq M$ forces $I \subseteq \mathfrak{m}$. So, we can see easily that $M = \mathfrak{m} \bowtie I = \overline{\mathfrak{m}}$. Moreover, by [4, Proposition 2.5], we have $\frac{A \bowtie I}{\mathfrak{m} \bowtie I} \cong \frac{A}{\mathfrak{m}}$. Hence, $\mathfrak{m}$ is a maximal ideal of A.
($\Leftarrow$) Follows from the isomorphism of rings $\frac{A \bowtie I}{\mathfrak{m} \bowtie I} \cong \frac{A}{\mathfrak{m}}$.
The last statement follows from [4, Proposition 2.7].
(2) ($\Rightarrow$) Assume $I \times I \nsubseteq M$. Applying [7, Lemma 1.1.4(3)], to the the following conductor square with conductor $\mathrm{Ker}(\mu_1) = I \times I$, where ι_2 is the natural embedding, μ_1 is the canonical surjection, and for each $a \in A$ and $i \in I$, $\mu_2(a, a+i) = \overline{a}$ and $\iota_1(\overline{a}) = (\overline{a}, \overline{a})$.

$$\begin{array}{ccc} A \bowtie I & \xrightarrow{\iota_2} & A \times A \\ \Big\downarrow{\scriptstyle \mu_2} & & \Big\downarrow{\scriptstyle \mu_1} \\ \dfrac{A}{I} & \xrightarrow{\iota_1} & \dfrac{A}{I} \times \dfrac{A}{I} \end{array}$$

there is a unique prime Q of $A \times A$ such that $I \times I \nsubseteq Q$ and

$$M = Q \cap A \bowtie I \text{ with } (A \times A)_Q = (A \bowtie I)_M.$$

Then either $Q = \mathfrak{m} \times A$ or $Q = A \times \mathfrak{m}$ for some prime ideal $\mathfrak{m}$ of A such that $I \nsubseteq \mathfrak{m}$. That is, $M = \overline{\mathfrak{m}}$ or $M = \mathfrak{m} \bowtie I$. Accordingly, we'll have

$$(A \bowtie I)_M \cong A_{\mathfrak{m}}$$

Moreover, by [4, Proposition 2.5], we have $\frac{A \bowtie I}{M} \cong \frac{A}{\mathfrak{m}}$. Hence, $\mathfrak{m}$ is a maximal ideal of A.
($\Leftarrow$) Follows from that last isomorphism of rings. □

The characterization of $A \bowtie I$ to be Cohen-Macaulay (resp. Gorenstein) is already done in the local case in [1, 4]. The results found are formed as follows.

Lemma 2.2 ([1, Theorem 1.8]). *Let A be a local ring and I a non-zero prpoer ideal of A. Then,*

(i) The ring $A \bowtie I$ is Cohen-Macaulay if and only if A is Cohen-Macaulay and I is a maximal Cohen-Macaulay A-module.

(ii) The ring $A \bowtie I$ is Gorenstein if and only if A is Cohen-Macaulay and I is a canonical A-module.

Remark 2.3. In our proofs, we encountered two trivial cases. The first one is when $I = A$. In this case, $A \bowtie A = A \times A$ (which is not local certainly) but it is well known that $A \times A$ is Cohen-Macaulay (resp. Gorenstein) if and only if A is Cohen-Macaulay (resp. Gorenstein), and certainly $I = A$ is a maximal Cohen-Macaulay (resp. canonical) A-module. The second trivial case is when $I = (0)$. In this case $A \bowtie (0) \cong A$ which is trivially Cohen-Macaulay (resp. Gorenstein) when A is Cohen-Macaulay (resp. Gorenstein).

The notations and the facts of the previous lemmas and remark will be used in the sequel without explicit reference.

The first result characterize when $A \bowtie I$ is Cohen-Macaulay (resp. Gorenstein) in the general case. For a given A-module M, let $\mathrm{Supp}(M)$ denote the support of M, that is;

$$\mathrm{Supp}(M) = \{\mathfrak{p} \in \mathrm{Spec}(A) \mid M_{\mathfrak{p}} \neq (0)\}$$

Proposition 2.4. *Let A be a ring and I a non zero ideal of A. Then,*

(i) the ring $A \bowtie I$ is Cohen-Macaulay if and only if A is Cohen-Macaulay and $I_{\mathfrak{m}}$ is a maximal Cohen-Macaulay $A_{\mathfrak{m}}$-module for each $\mathfrak{m} \in \mathrm{Supp}(I) \cap \mathrm{Max}(A)$.

(ii) the ring $A \bowtie I$ is Gorenstein if and only if A is Cohen-Macaulay, $I_{\mathfrak{m}}$ is a canonical $A_{\mathfrak{m}}$-module for each $\mathfrak{m} \in \mathrm{Supp}(I) \cap \mathrm{Max}(A)$ and $\mathrm{type}(A_{\mathfrak{m}}) = 1$ for each $m \in \mathrm{Max}(A) \backslash \mathrm{Supp}(I)$.

Proof. Assume that $A \bowtie I$ is a Cohen-Macaulay rings (resp. Gorenstein ring) and let $\mathfrak{m}$ be a maximal ideal of A. If $I \subseteq \mathfrak{m}$, then $\mathfrak{m} \bowtie I$ is a maximal ideal of $A \bowtie I$ and $(A \bowtie I)_{\mathfrak{m} \bowtie I} \cong A_{\mathfrak{m}} \bowtie I_{\mathfrak{m}}$ is a Cohen-Macaulay ring (resp. Gorenstein ring). Then, either $I_{\mathfrak{m}} = (0)$ and $A_{\mathfrak{m}} \bowtie I_{\mathfrak{m}} \cong A_{\mathfrak{m}}$ is a Cohen-Macaulay ring (resp. Gorenstein ring and so Cohen-Macaulay of type 1 by [2, Theorem 3.2.10]), or $I_{\mathfrak{m}} \neq (0)$, and so $A_{\mathfrak{m}}$ is Cohen-Macaulay and $I_{\mathfrak{m}}$ is a maximal Cohen-Macaulay (resp. canonical) $A_{\mathfrak{m}}$-module. Now, if $I \nsubseteq \mathfrak{m}$. There exists a maximal ideal M of $A \bowtie I$ such that $(A \bowtie I)_M \cong A_{\mathfrak{m}}$, and then $A_{\mathfrak{m}}$ is Cohen-Macaulay (resp. Gorenstein and so Cohen-Macaulay) and $I_{\mathfrak{m}} = A_{\mathfrak{m}}$ is a maximal Cohen-Macaulay (resp. canonical) $A_{\mathfrak{m}}$-module. Consequently, A is Cohen-Macaulay, $I_{\mathfrak{m}}$ is a maximal Cohen-Macaulay (resp. canonical) $A_{\mathfrak{m}}$-module for each $\mathfrak{m} \in \mathrm{Supp}(I) \cap \mathrm{Max}(A)$ (resp. and $A_{\mathfrak{m}}$ of type 1 for each $m \in \mathrm{Max}(A) \backslash \mathrm{Supp}(I)$).
Now, we will prove the converse implication in the assertion (1) (resp. (2)). Let M be a maximal ideal of $A \bowtie I$. If $I \times I \subseteq M$, there exists a maximal ideal $I \subseteq \mathfrak{m}$ of A such that $M = \mathfrak{m} \bowtie I$ and we have $(A \bowtie I)_M \cong A_{\mathfrak{m}} \bowtie I_{\mathfrak{m}}$. If $I_{\mathfrak{m}} = (0)$, then $(A \bowtie I)_M \cong A_{\mathfrak{m}}$ which is a Cohen-Macaulay ring (resp. Cohen-Macaulay ring of type 1, and so Gorenstein). Otherwise, $I_{\mathfrak{m}}$ is a maximal Cohen-Macaulay (resp. canonical) $A_{\mathfrak{m}}$-module and certainly $A_{\mathfrak{m}}$ is a Cohen-Macaulay ring. Thus, $(A \bowtie I)_M$ is a Cohen-Macaulay ring (resp. Gorenstein ring). Now, suppose that $I \times I \nsubseteq M$. There exist a maximal ideal $\mathfrak{m}$ of A such that $I \nsubseteq \mathfrak{m}$ and $(A \bowtie I)_M \cong A_{\mathfrak{m}}$ which is Cohen-Macaulay (resp. and $I_{\mathfrak{m}} = A_{\mathfrak{m}}$ is a canonical module, on so $A_{\mathfrak{m}}$ is Gorenstein by [2, Theorem 3.3.7]). Accordingly, $A \bowtie I$ is a Cohen-Macaulay ring (resp. Gorenstein ring). □

Corollary 2.5. *Let A be a ring and I a non zero ideal of A. Then,*

(i) If A is a Cohen-Macaulay ring and I is a maximal Cohen-Macaulay A-module, then $A \bowtie I$ is a Cohen-Macaulay ring.

(ii) If A is Cohen-Macaulay ring and I is a canonical A-module, then $A \bowtie I$ is a Gorenstein ring.

Proof. By definition, I is a maximal Cohen-Macaulay (resp. canonical) A-module if $I_{\mathfrak{m}}$ is a maximal Cohen-Macaulay (canonical) $A_{\mathfrak{m}}$-module for each $\mathfrak{m} \in \mathrm{Max}(A)$. Moreover, it is known that if I is a canonical A-module then $\mathrm{Supp}(I) = \mathrm{Spec}(A)$ and so $\mathrm{Max}(A) \backslash \mathrm{Supp}(I) = \emptyset$. Thus, our corollary follows directly from Proposition 2.4. □

Corollary 2.6. *Let I be a proper ideal of A such that* $\mathrm{ann}(I) \subseteq Jac(A)$. *Then,*

(i) the ring $A \bowtie I$ is Cohen-Macaulay ring if and only if A is Cohen-Macaulay and I is a maximal Cohen-Macaulay A-module.

(ii) the ring $A \bowtie I$ is Gorenstein ring if and only if A is Cohen-Macaulay and I is a canonical A-module.

Proof. Since $\mathrm{ann}_A(I) \subseteq Jac(A)$, we have $I \neq (0)$. Moreover, since A must be Noetherian in the context of our corollary (by [4, Remark 2.1]), we have $\mathrm{Supp}(I) = \mathrm{V}(\mathrm{ann}_A(I))$ (by [8, Theorem 3.3.22]). Hence, $\mathrm{Supp}(I) \cap \mathrm{Max}(A) = \mathrm{Max}(A)$ and $\mathrm{Max}(A) \backslash \mathrm{Supp}(I) = \emptyset$. Thus, the equivalences in (1) and (2) follow immediately from Proposition 2.4. □

In [4, Theorem 11], D'Anna proved that if A is a local Cohen-Macaulay ring and I is proper ideal, then $A \bowtie I$ is Gorenstein if and only of A has a canonical module ω_A and $I \cong \omega_A$. In D'Anna's proof, this is deduced from [4, Proposition 3]. But Shapiro (in [10]) pointed an error in [4, Proposition 3] and showed that it is true if and only if $\mathrm{ann}(I) = (0)$ ([10, Lemma 2.1]. Thus, we conclude that if A is a local Cohen-Macaulay ring and I is proper ideal containing a non-zerodivisor element such that $A \bowtie I$ is Gorenstein then I is a canonical module. The next corollary which a particular case of Corollary 2.7 recovers the D'Anna's result corrected by Shapiro.

Corollary 2.7. *Let I be a proper ideal of A such that* $\mathrm{ann}(I) = (0)$. *Then,*

(i) the ring $A \bowtie I$ is Cohen-Macaulay ring if and only if A is Cohen-Macaulay and I is a maximal Cohen-Macaulay A-module.

(ii) the ring $A \bowtie I$ is Gorenstein ring if and only if A is Cohen-Macaulay and I is a canonical A-module.

Recall that a ring R is called *quasi-Frobenius* [9] if it Noetherian and self injective. The quotient R/I where R is a principal ideal domain and I is any nonzero ideal of R is a classical example of quasi-Frobenius ring. Several characterizations of quasi-Frobenius rings were given in [9]. The characterization of $R \bowtie I$ to be quasi-Frobenius was done in [3]. However, we will find it again by using Proposition 2.4.

Corollary 2.8. *The ring $A \bowtie I$ is quasi-Frobenius if and only if A is quasi-Frobenius and I is generated by an idempotent.*

Proof. Following [4, Remark 2.1], $\dim(R \bowtie I) = \dim(R)$, and $R \bowtie I$ is Noetherian if and only if R is Noetherian. Thus, $A \bowtie I$ is Artinian if and only if A is Artinian. Moreover, recall that a ring is quasi-Frobenius if and only if it an Artinian Gorenstein ring.
($\Rightarrow$) Assume that $A \bowtie I$ is quasi-Frobenius. Then, $A \bowtie I$ is Artinian, and so is A. Then, $A_{\mathfrak{m}}$ is Artinian for each $\mathfrak{m} \in \mathrm{Max}(A)$. On the other hand, over local Artinian rings, the canonical module is the injective hull of the residue field. Thus, following Proposition 2.4, for each $\mathfrak{m} \in \mathrm{Max}(A)$, I_m is (0) or injective. Thus, I is an injective ideal since A is Noetherian and so it is generated by an idempotent element. Consequently, $I_{\mathfrak{m}} = (0)$ or $I_{\mathfrak{m}} = A_{\mathfrak{m}}$. If $I_{\mathfrak{m}} = (0)$, we have, by Proposition 2.4 again, $\mathrm{type}(A_{\mathfrak{m}}) = 1$. Thus, $A_{\mathfrak{m}}$ a Gorenstein Artinian ring, and so quasi-Frobenius. If $I_{\mathfrak{m}} = A_{\mathfrak{m}}$ then $A_{\mathfrak{m}}$ is self injective. Consequently, A is self injective, and so it is quasi-Frobenius.
($\Leftarrow$) Assume that A is quasi-Frobenius and I is generated by an idempotent. For each $\mathfrak{m} \in \mathrm{Max}(A)$, $A_{\mathfrak{m}}$ is Gorenstein, and so $\mathrm{type}(A_{\mathfrak{m}}) = 1$. Moreover, for each $\mathfrak{m} \in \mathrm{Supp}(I) \cap \mathrm{Max}(A)$, $I_{\mathfrak{m}} = A_{\mathfrak{m}}$, and so it is a canonical $A_{\mathfrak{m}}$-module. Thus, $A \bowtie I$ is Gorenstein. Hence, since $A \bowtie I$ is Artinian (because A is Artinian), we conclude that $A \bowtie I$ is quasi-Frobenius. □

Cohen-Macaulay (resp. canonical) modules have not necessary a finite projective dimension. However, when this is the case, we have the following result.

Proposition 2.9. *Let I be a proper ideal of A such that* $\mathrm{pd}_R(I) < \infty$. *Then,*

(i) the ring $A \bowtie I$ is a Cohen-Macaulay ring if and only if A is Cohen-Macaulay and I is projective.

(ii) the ring $A \bowtie I$ is a Gorenstein ring if and only if A is Gorenstein and I is projective.

Proof. (1) ($\Rightarrow$) Assume that $A \bowtie I$ is a Cohen-Macaulay ring. Following Proposition 2.4, it suffices to prove that I is projective. Since A is Noetherian, we have to prove that I_m is projective for each $\mathfrak{m} \in \mathrm{Max}(A)$ such that $I_{\mathfrak{m}} \neq (0)$. Let $\mathfrak{m}$ be such maximal ideal of A. Using Auslander-Buchsbaum formula (since $\mathrm{pd}_{A_{\mathfrak{m}}}(I_{\mathfrak{m}}) < \infty$), we have

$$\mathrm{pd}_{A_{\mathfrak{m}}}(I_{\mathfrak{m}}) + \mathrm{depth}(I_{\mathfrak{m}}) = \mathrm{depth}(A_{\mathfrak{m}})$$

On the other hand, from Proposition 2.4, $I_{\mathfrak{m}}$ is a maximal Cohen-Macaulay $A_{\mathfrak{m}}$. Thus, $\mathrm{depth}(I_{\mathfrak{m}}) = \mathrm{depth}(A_{\mathfrak{m}})$, and so $\mathrm{pd}_{A_{\mathfrak{m}}}(I_{\mathfrak{m}}) = 0$. Consequently, I is projective.
($\Leftarrow$) Assume that A is Cohen-Macaulay and I is projective. Let $\mathfrak{m}$ be a maximal ideal of A such that $I_{\mathfrak{m}} \neq (0)$. Then, I_m is a non zero free ideal of $A_{\mathfrak{m}}$. Thus, it is generated by a non-zerodivisor element, and so $\dim(I_{\mathfrak{m}}) = \dim(A_{\mathfrak{m}})$. On the other hand, by the Auslander-Buchsbaum formula, we have $\mathrm{depth}(I_{\mathfrak{m}}) = \mathrm{depth}(A_{\mathfrak{m}})$. Thus, since $\mathrm{depth}(A_{\mathfrak{m}}) = \dim(A_{\mathfrak{m}})$, it is clear that $I_{\mathfrak{m}}$ is a maximal Cohen-Macaulay $A_{\mathfrak{m}}$-module. Consequently, from Proposition 2.4, $A \bowtie I$ is a Cohen-Macaulay ring.
(2)($\Rightarrow$) Assume that $A \bowtie I$ is a Gorenstein ring. From (1), it suffices to prove that A is Gorenstein. Let $\mathfrak{m} \in \mathrm{Max}(A)$. If $I_{\mathfrak{m}} = (0)$, from Proposition 2.4, $A_{\mathfrak{m}}$ is a Cohen-Macaulay ring of type 1, and so it is a Gorenstein ring. Otherwise, $I_{\mathfrak{m}}$ is a canonical $A_{\mathfrak{m}}$-module. Moreover, since

I is projective, $I_{\mathfrak{m}}$ is non zero free ideal of $A_{\mathfrak{m}}$. Hence, $I_{\mathfrak{m}} \cong A_{\mathfrak{m}}$. Thus, from [2, Theorem 3.3.7], $A_{\mathfrak{m}}$ is Gorenstein. Consequently, A is Gorenstein.
($\Leftarrow$) Assume that A is Gorenstein and I is projective. Then, for each $\mathfrak{m} \in \mathrm{Max}(A)$, $A_{\mathfrak{m}}$ is Gorenstein, and so $\mathrm{type}(A_{\mathfrak{m}}) = 1$. Thus, following Proposition 2.4, it suffices to show that $I_{\mathfrak{m}}$ is a canonical $A_{\mathfrak{m}}$-module for each $\mathfrak{m} \in \mathrm{Supp}(I) \cap \mathrm{Max}(A)$. As in (1)($\Leftarrow$), we can prove that, for each $\mathfrak{m} \in \mathrm{Supp}(I) \cap \mathrm{Max}(A)$, I_m is a maximal Cohen-Macaulay $A_{\mathfrak{m}}$-module which is generated by a non-zerodivisor element. Thus, $I_{\mathfrak{m}} \cong A_{\mathfrak{m}}$. Hence, $\mathrm{id}_{A_{\mathfrak{m}}}(I_{\mathfrak{m}}) = \mathrm{id}_{A_{\mathfrak{m}}}(A_{\mathfrak{m}}) < \infty$, and $\dim_{A_{\mathfrak{m}}/\mathfrak{m}A_{\mathfrak{m}}} \mathrm{Ext}^t_{A_{\mathfrak{m}}}(A_{\mathfrak{m}}/\mathfrak{m}A_{\mathfrak{m}}, I_{\mathfrak{m}}) = \dim_{A_{\mathfrak{m}}/\mathfrak{m}A_{\mathfrak{m}}} \mathrm{Ext}^t_{A_{\mathfrak{m}}}(A_{\mathfrak{m}}/\mathfrak{m}A_{\mathfrak{m}}, A_{\mathfrak{m}}) = 1$ with $t = \mathrm{depth}(A_{\mathfrak{m}}) = \mathrm{depth}(I_{\mathfrak{m}})$. Thus, $I_{\mathfrak{m}}$ is a canonical $A_{\mathfrak{m}}$-module. Consequently, $A \bowtie I$ is a Gorenstein ring. □

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