Palestine Journal of Mathematics

Vol. 7(Special Issue: I, 2018) , 79–87

On power serieswise Armendariz rings

Mounir El Ouarrachi and Najib Mahdou

Communicated by Ayman Badawi

MSC 2010 Classifications: 13F25, 16S70, 13P05, 15B33.

Keywords and phrases: Power serieswise Armendariz ring, Armendariz ring, trivial ring extension, direct product of rings.

Abstract In this paper, we investigate the transfer property of power serieswise Armendariz to trivial ring extensions, direct product of rings and the homomorphic image. The article includes a brief discussion of the scope and precision of our results.

1 Introduction

Throughout this paper, all rings are associative with identity elements. It is suitable to use "local" to refer to (not necessarily Noetherian) ring with a unique maximal ideal. A subring of a ring need not have the same unit. The polynomial ring and the formal power series ring with an indeterminate X over a ring R are denoted by $R[X]$ and $R[[X]]$ respectively, and $nil(R)$ denotes the set of nilpotent element (the nilradical) of R.

In [4], Armendariz proved that $a_ib_j = 0$ for all i, j whenever polynomials $f = \sum_{i=0}^{i=n} a_ix^i$ and $g = \sum_{i=0}^{i=m} b_ix^i$ over a reduced ring satisfy $fg = 0$. In [25], Rege and Chhawchharia (1997) called such a ring (not necessarily reduced) Armendariz. Armendariz rings are thus a generalization of reduced rings. It is easy to see that subring of Armendariz rings are also Armendariz. Also, D. D. Anderson and V. Camillo [1], show that a ring R is Gaussian if and only if every homomorphic image of R is Armendariz. See for instance [1, 4, 21, 25].

In [2], Ramon Antoine (2008) called nil-Armendariz rings if whenever the product of two polynomials $f(x) = \sum_{i=0}^{i=n} a_ix^i$ and $g(x) = \sum_{j=0}^{j=m} b_jx^j$ in $R[x]$ satisfies $f(x)g(x) \in nil(R)[x]$ we have $a_ib_j \in nil(R)$ for each i, j. Armendariz ring is nil-Armendariz [2, Proposition 2.7]. It is easy to see that a subring of nil-Armendariz ring is nil-Armendariz. In [23], Liu and Zhao (2006), introduced weakArmendariz as generalization of Armendariz. A ring R is called a weakArmendariz ring if whenever the product of two polynomials $f(x) = \sum_{i=0}^{i=n} a_ix^i$ and $g(x) = \sum_{j=0}^{j=m} b_jx^j$ in $R[x]$ satisfies $f(x)g(x) = 0$ we have $a_ib_j \in nil(R)$ for each i, j. It is clear that subring of weakArmendariz ring is also weakArmendariz. Obviously, nil-Armendariz rings are weakArmendariz rings. The following diagram of implications summarizes the relation between them (see for instance [2, 23]) :

$$Reduced \Longrightarrow Armendariz \Longrightarrow nil - Armendariz \Longrightarrow weakArmendariz.$$

A ring R is semicommutative if for all $a, b \in R$, $ab = 0$ implies $aRb = 0$. This is equivalent to the usual definition by Shin [11, Lemma 1.2] or Huh et al. [6, Lemma 1]. By Huh et al., reduced rings are semicommutative. Semicommutative ring is nil-Armendariz [23, Proposition 3.3] . Thus weakArmendariz rings and nil-Armendariz rings are a common generalization of semicommutative rings and Armendariz rings. Also, a ring R is called abelian if every idempotent in R is central. Armendariz rings are abelian by the proof of [1, Theorem 6]).

In [20], Kim et al. define power serieswise Armendariz rings as ring such that for every $f(x) = \sum_{i=0}^{\infty} a_ix^i$ and $g(x) = \sum_{i=0}^{\infty} b_ix^i \in R[[x]]$ such that $fg = 0$, then $a_ib_j = 0$ for every i and j. Power serieswise Armendariz rings are clearly Armendariz rings, but the converse is false by [3, Example 2]. Recall that a reduced ring is power serieswise Armendariz. It is easy to see that subring of power serieswise Armendariz is also power serieswise Armendariz. See for instance [2, 3, 11, 20].

Let A be a ring and a bi-module ${}_AE_A$. $A \propto E$ is the set of pairs (a, e) with pairwise addition and multiplication given by $(a, e)(b, f) = (ab, af + eb)$. $A \propto E$ is called the trivial ring extension of A by E (also called the idealization of E over A).

In this paper, we investigate the transfer property of power serieswise Armendariz to trivial ring extensions, direct product of rings and the homomorphic image. Our results generate new and original examples which enrich the current literature with new families of power serieswise Armendariz rings.

2 Main Results

Now we study the transfer property of power serieswise Armendariz to the trivial ring extensions.

Let A be a ring. We claim that n-by-n upper triangular matrix rings over A are not power serieswise Armendariz, where $n \geq 2$. It is enough to show that the 2-by-2 upper triangular matrix ring over A is not power serieswise Armendariz because each subring of a power serieswise Armendariz ring is also power serieswise Armendariz. Let S be the 2-by-2 upper triangular matrix ring over A, and let $f(x) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix} x$, and $g(x) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} x$ be polynomials in $S[[x]]$. Then $f(x)g(x) = 0$, but $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} \neq 0$. So S is not power serieswise Armendariz and consequently every n-by-n upper triangular matrix rings over A is not power serieswise Armendariz. But we may find subrings of the 3-by-3 upper triangular matrix rings which may be power serieswise Armendariz, as shown by the next result.

Proposition 2.1. *Let A be a reduced ring. Then*
S={ $\begin{pmatrix} a & b & c \\ 0 & a & d \\ 0 & 0 & a \end{pmatrix}$ */$a, b, c, d \in A$} is a power serieswise Armendariz ring.*

Proof. We use the method in the proof of [27, Proposition 2.5]. First notice that for $\begin{pmatrix} a_1 & b_1 & c_1 \\ 0 & a_1 & d_1 \\ 0 & 0 & a_1 \end{pmatrix}$, $\begin{pmatrix} a_2 & b_2 & c_2 \\ 0 & a_2 & d_2 \\ 0 & 0 & a_2 \end{pmatrix} \in S$, we can denote their addition and multiplication by

$$(a_1, b_1, c_1, d_1) + (a_2, b_2, c_2, d_2) = (a_1 + a_2, b_1 + b_2, c_1 + c_2, d_1 + d_2)$$

and

$$(a_1, b_1, c_1, d_1)(a_2, b_2, c_2, d_2) = (a_1a_2, a_1b_2 + b_1a_2, a_1c_2 + b_1d_2 + c_1a_2, a_1d_2 + d_1a_2)$$

respectively. So every polynomials in $S[[x]]$ can be expressed in the form $(p_0(x), p_1(x), p_2(x), p_3(x))$ for some $p_i(x)$ in $A[[x]]$.

Let $f(x) = (f_0(x), f_1(x), f_2(x), f_3(x))$ and $g(x) = (g_0(x), g_1(x), g_2(x), g_3(x))$ be elements of $S[[x]]$. Assume that $f(x)g(x) = 0$. Then $f(x)g(x) = (f_0(x)g_0(x), f_0(x)g_1(x)+f_1(x)g_0(x), f_0(x)g_2(x)+ f_1(x)g_3(x) + f_2(x)g_0(x), f_0(x)g_3(x) + f_3(x)g_0(x)) = 0$. So we have the following system of equations:

(i) $f_0(x)g_0(x) = 0$;

(ii) $f_0(x)g_1(x) + f_1(x)g_0(x) = 0$;

(iii) $f_0(x)g_2(x) + f_1(x)g_3(x) + f_2(x)g_0(x) = 0$;

(iv) $f_0(x)g_3(x) + f_3(x)g_0(x) = 0$.

From Equation (1), we see that $g_0(x)f_0(x) = 0$ since $A[[x]]$ is reduced. If we multiply equation (2) on the right side by $f_0(x)$, then $f_0(x)g_1(x)f_0(x)+f_1(x)g_0(x)f_0(x) = 0$. So $f_0(x)g_1(x)f_0(x) = 0$, if we multiply by $g_1(x)$ on the right side and use the fact that $R[[x]]$ is reduced, we have $f_0(x)g_1(x) = 0$ and hence $f_1(x)g_0(x) = 0$. Also if we multiply equation (4) on the right side by $f_0(x)$, then $f_0(x)g_3(x)f_0(x)+f_3(x)g_0(x)f_0(x) = 0$. So $f_0(x)g_3(x) = 0$ and hence $f_3(x)g_0(x) = 0$. Now if we multiply equation (3) on the right side by $f_0(x)$, then $f_0(x)g_2(x)f_0(x)+f_1(x)g_3(x)f_0(x)+f_2(x)g_0(x)f_0(x) = 0$. So $f_0(x)g_2(x) = 0$ and hence equation (3) becomes $f_1(x)g_3(x) + f_2(x)g_0(x) = 0$. If we multiply the last equation on the right side by $f_1(x)$, then we have $f_1(x)g_3(x) = 0$ and so $f_2(x)g_0(x) = 0$. Now let $f(x) = \sum_{i=0}^{\infty} \begin{pmatrix} a_i & b_i & c_i \\ 0 & a_i & d_i \\ 0 & 0 & a_i \end{pmatrix} x^i$ and $g(x) = \sum_{j=0}^{\infty} \begin{pmatrix} a'_j & b'_j & c'_j \\ 0 & a'_j & d'_j \\ 0 & 0 & a'_j \end{pmatrix} x^j$ where $f_0 = \sum_{i=0}^{\infty} a_i x^i$, $f_1 = \sum_{i=0}^{\infty} b_i x^i$, $f_2 = \sum_{i=0}^{\infty} c_i x^i$, $f_3 = \sum_{i=0}^{\infty} d_i x^i$, $g_0 = \sum_{j=0}^{\infty} a'_j x^j$, $g_1 = \sum_{j=0}^{\infty} b'_j x^j$, $g_2 = \sum_{j=0}^{\infty} c'_j x^j$, $g_3 = \sum_{j=0}^{\infty} d'_j x^j$. Then we obtain that $a_i a'_j = 0$, $a_i b'_j = 0$, $b_i a'_j = 0$, $a_i c'_j = 0$, $b_i d'_j = 0$,$c_i a'_j = 0$, $a_i d'_j = 0$ and $d_i a'_j = 0$ for all i, j by the preceding results, and condition that A is reduced. Consequently $\begin{pmatrix} a_i & b_i & c_i \\ 0 & a_i & d_i \\ 0 & 0 & a_i \end{pmatrix} \begin{pmatrix} a'_i & b'_i & c'_i \\ 0 & a'_i & d'_i \\ 0 & 0 & a'_i \end{pmatrix}$ for all i, j, therefore S is a power serieswise Armendariz ring. □

Let S be a reduced ring and let

$A_n = \{ \begin{pmatrix} a & a_{12} & a_{13} & . & . & . & a_{1n} \\ 0 & a & a_{23} & . & . & . & a_{2n} \\ 0 & 0 & a & . & . & . & a_{3n} \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ 0 & 0 & 0 & . & . & . & a \end{pmatrix} / a, a_{ij} \in S\}$. Based on Proposition 2.1, one may suspect that A_n may be also a power serieswise Armendariz ring for $n \geq 4$. But the following example erases this possibility.

Example 2.2. Let S be a ring. Then

$A_4 = \{ \begin{pmatrix} a & a_{12} & a_{13} & a_{14} \\ 0 & a & a_{23} & a_{24} \\ 0 & 0 & a & a_{34} \\ 0 & 0 & 0 & a \end{pmatrix} / a, a_{ij} \in S\}$ is not power serieswise Armendariz.

Proof. Let $f(x) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} x$

and $g(x) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} x$ be polynomials in $A_4[[x]]$.

Then $f(x)g(x) = 0$, but $\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \neq 0$, as desired. □

Given a ring A and a bimodule ${}_AE_A$, we have $A \propto E$ is isomorphic to the ring of all matrices $\begin{pmatrix} r & m \\ 0 & r \end{pmatrix}$, where $r \in A$ and $m \in E$.

Corollary 2.3. *Let A be a reduced ring. Then the trivial extension $A \propto A$ is a power serieswise Armendariz ring.*

Proof. Notice that $A \propto A$ is isomorphic to S=$\{\begin{pmatrix} a & b & 0 \\ 0 & a & 0 \\ 0 & 0 & a \end{pmatrix} / a, b \in A\}$ (It is easy to see that the mapping defined via $\varphi(\begin{pmatrix} a & b \\ 0 & a \end{pmatrix}) = \begin{pmatrix} a & b & 0 \\ 0 & a & 0 \\ 0 & 0 & a \end{pmatrix}$ is a ring isomorphism) and that each subring of a power serieswise Armendariz is also power serieswise Armendariz. Thus $A \propto A$ is a power serieswise Armendariz ring by Proposition 2.1. □

From Corollary 2.3, one can may suspect that if A is power serieswise Armendariz then $A \propto A$ is power serieswise Armendariz. But the following example eliminates this possibility.

Example 2.4. Let T be a reduced ring. Then $R = \{\begin{pmatrix} r & m \\ 0 & r \end{pmatrix} / r, m \in T\}$ is a power serieswise Armendariz by Corollary 2.3. Let $S = \{\begin{pmatrix} A & B \\ 0 & A \end{pmatrix} / A, B \in R\}$ and let

$$f(x) = \begin{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \end{pmatrix} + \begin{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \\ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \end{pmatrix} x \quad \text{and,}$$

$$g(x) = \begin{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \end{pmatrix} + \begin{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \end{pmatrix} x$$

be polynomials in $S[[x]]$. Then

$f(x)g(x) = 0$, and

$$\begin{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \end{pmatrix} \begin{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \end{pmatrix} \neq 0 .$$

Thus S is not power serieswise Armendariz.

But we may have an affirmative answer to this situation, taking a condition "(A, M) is a local ring and E a bimodule such that $ME = 0$ and $EM = 0$".

Theorem 2.5. *Let A be a ring, E be a nonzero bimodule. Then: Assume that (A,M) is a local ring and E a bimodule such that $ME=0$ and $EM=0$. Then, $A\propto E$ is a power serieswise Armendariz ring if and only if so is A.*

Proof. If $A\propto E$ is a power serieswise Armendariz, then so is A since A is a subring of $A\propto E$. Conversely, assume that A is power serieswise Armendariz and let $f=\sum_{i=0}^{\infty}(a_i,e_i)x^i$, $g=\sum_{j=0}^{\infty}(b_j,f_j)x^j$ in $(A\propto E)[[x]]$ such that $fg=0$. It remains to show that $(a_ib_j,a_if_j+e_ib_j)=0$ for all i,j. For this purpose, we set $f_A=\sum_{i=0}^{\infty}a_ix^i$ and $g_A=\sum_{j=0}^{\infty}b_jx^j$ in $A[[x]]$. We have $f_Ag_A=0$ since $fg=0$, then $a_ib_j=0$ for all i,j since A is power serieswise Armendariz. So it suffices to show that $a_if_j+e_ib_j=0$. Two cases are possible:

1^{st} case: $a_i,b_j\in M$ for all i,j. Then $a_if_j+e_ib_j=0$ for all i,j since $ME=0$ and $EM=0$.

2^{nd} case: One of $a_i,b_j\notin M$.
Without loss of generality, we may assume that $a_k\notin M$ for some positive integer k. Let i_0 be the smallest integer such that $a_{i_0}\notin M$, that is a_{i_0} is invertible (since (A,M) is local).
Note since that $a_{i_0}b_j=0$ and since a_{i_0} is invertible, then $b_j=0$ for all j. Consequently, it suffices to show that $a_if_j=0$ for all i,j.
Remark that $fg=0$ implies that $\sum_{i+j=k}a_if_j=0$ for every positive integer k.
For $k=i_0$, we have $a_{i_0}f_0+a_{i_0-1}f_1+...+a_0f_{i_0}=a_{i_0}f_0=0$, then $f_0=0$ since a_{i_0} is invertible.
For $k=i_0+1$, we have $a_{i_0+1}f_0+a_{i_0}f_1+...+a_0f_{i_0+1}=a_{i_0}f_1=0$, then $f_1=0$.
By induction we have $f_j=0$ for all j. Consequently, $a_if_j=0$, as desired.
Hence, in both cases $a_if_j+b_je_i=0$ for all i,j making $A\propto E$ a power serieswise Armendariz ring and this completes the proof of Theorem 2.5. □

Example 2.6. Let K be a field, $K[[x]]$ is a local power serieswise Armendariz ring and (x) is the unique ideal maximal. Then, $K[[x]]\propto K$ is power serieswise Armendariz (which is never reduced) by Theorem 2.5 since $(x)S=0$, $S(x)=0$ and $K[[x]]$ is power serieswise Armendariz, where $S=K[[x]]/(x)\simeq K$.

Now we study the transfer property of power serieswise Armendariz to the direct product of rings.

Theorem 2.7. *Let $(A_i)_{i=1,2,\ldots,n}$ be a family of rings and let $A:=\prod_{i=1}^{n}A_i$. Then, A is power serieswise Armendariz ring if and only if so is A_i for each $i=1,\ldots,n$.*

Proof. Assume that $A_1\times A_2$ is a power serieswise Armendariz ring and we must to show that A_1 is a power serieswise Armendariz ring (it is the same for A_2). Let $f=\sum_{i=0}^{\infty}a_ix^i$ and $g=\sum_{j=0}^{\infty}b_jx^j$ in $A_1[[x]]$ such that $fg=0$ and set $f_1:=\sum_{i=0}^{\infty}(a_i,0)x^i$ and $g_1:=\sum_{j=0}^{\infty}(b_j,0)x^j\in(A_1\times A_2)[[x]]$.
Hence, $f_1g_1=0$ (since $fg=0$) and so $(a_ib_j,0)=0$ since $(A_1\times A_2)$ is a power serieswise Armendariz ring. Therefore, $a_ib_j=0$, and this means that A_1 is a power serieswise Armendariz ring.
Conversely, assume that A_1 and A_2 are power serieswise Armendariz rings and let $f=\sum_{i=0}^{\infty}(a_i,e_i)x^i$ and $g=\sum_{j=0}^{\infty}(b_j,f_j)x^j\in(A_1\times A_2)[[x]]$ such that $fg=0$. Set $f_1:=\sum_{i=0}^{\infty}a_ix^i\in A_1[[x]]$, $f_2:=\sum_{i=0}^{\infty}e_ix^i\in A_2[[x]]$, $g_1:=\sum_{j=0}^{\infty}b_jx^j\in A_1[[x]]$ and $g_2:=\sum_{j=0}^{\infty}f_jx^j\in A_2[[x]]$. Then, $f_1g_1=0$ and $f_2g_2=0$ since $fg=0$. Hence $a_ib_j=0$ and $e_if_j=0$ since A_1 and A_2 are power serieswise Armendariz rings. Therefore, $(a_i,e_i)(b_j,f_j)=0$, and this means that $A_1\times A_2$ is a power serieswise Armendariz ring. □

Corollary 2.8. *Let A be a ring and let $n\in\mathbb{N}-\{0\}$ be an integer. Then, A^n is power srieswise Armendariz if and only if so is A.*

The following example show that the implication “A/I and I are power serieswise Armendariz imply that so is A (where I is an ideal of A)" is false, in general.

Example 2.9. Let F be a field and consider the ring $A = \begin{pmatrix} F & F \\ 0 & F \end{pmatrix}$. Then A is not Armendariz (by [18, Examples 1]) and so A is not power serieswise Armendariz. Now we claim that A/I and I are power serieswise Armendariz for any nonzero ideal I of A. Note that the only nonzero proper ideals of A are $\begin{pmatrix} F & F \\ 0 & 0 \end{pmatrix}$,$\begin{pmatrix} 0 & F \\ 0 & F \end{pmatrix}$ and $\begin{pmatrix} 0 & F \\ 0 & 0 \end{pmatrix}$. First, let $I = \begin{pmatrix} F & F \\ 0 & 0 \end{pmatrix}$. Then $A/I \simeq F$ and so A/I is power serieswise Armendariz.
It remains to show that I is power serieswise Armendariz. Let $f(x) = \sum_{i=0}^{\infty} \alpha_i x^i$, $g(x) = \sum_{j=0}^{\infty} \alpha_j x^j$ in $I[[x]]$ such that $f(x)g(x) = 0$ and set $\alpha_i = \begin{pmatrix} a_i & b_i \\ 0 & 0 \end{pmatrix}$ and $\beta j = \begin{pmatrix} c_j & d_j \\ 0 & 0 \end{pmatrix}$.
Assume that $\alpha_0 \neq 0$ and $\beta_0 \neq 0$. Then $a_0c_0 = a_0d_0 = 0$. If $a_0 \neq 0$, then $c_0 = 0$ and $d_0 = 0$, which is a contradiction. So $a_0 = 0$ and hence $b_0 \neq 0$. This implies that $\alpha_0\beta_j = 0$ for all j. Hence the coefficient of x in $f(x)g(x) = 0$ is $\alpha_1\beta_0 = 0$. Then $a_1c_0 = a_1d_0 = 0$. If $a_1 \neq 0$, then $c_0 = 0$ and $d_0 = 0$, which is a contradiction. So $\alpha_1\beta_j = 0$ for all j.
Continuing this process, we show that $\alpha_i\beta_j = 0$ for all i, j. Therefore, I is a power serieswise Armendariz.

Next let $J = \begin{pmatrix} 0 & F \\ 0 & F \end{pmatrix}$. Then $A/J \simeq F$ and so A/J is power serieswise Armendariz. By the same method, we have that J is power serieswise Armendariz.
Finally, let $K = \begin{pmatrix} 0 & F \\ 0 & 0 \end{pmatrix}$. Then $A/K \simeq F \bigoplus F$ and so A/K is power serieswise Armendariz. Also $K^2 = 0$ and so K is power serieswise Armendariz.

Under the condition "I is reduced", we show that we have an affirmative answer to the above implication.

Theorem 2.10. *Let I be a reduced ideal of a ring A such that A/I is a power serieswise Armendariz. Then A is power serieswise Armendariz.*

Proof. Let $f(x) = \sum_{i \geq 0} a_i x^i$ and $g(x) = \sum_{j \geq 0} b_j x^j$ in $A[[x]]$ such that $f(x)g(x) = 0$.
Set $\overline{f}(x) = \sum_{i \geq 0} \overline{a_i} x^i$ and $\overline{g}(x) = \sum_{j \geq 0} \overline{b_j} x^j$ in $(A/I)[[x]]$.
Remark that $f(x)g(x) = \sum_{k \geq 0}(\sum_{i+j=k} a_ib_j)x^k = 0$ implies that $\sum_{i+j=k} a_ib_j = 0$ for all k. Also, $\overline{f}(x)\overline{g}(x) = 0$ imply that $\overline{a_ib_j} = 0$ since A/I is power serieswise Armendariz. Hence $a_ib_j \in I$ for all i, j.
We will show that $a_ib_j = 0$ by induction on $i + j$.
If $i + j = 0$ then $a_0b_0 = 0$.
Now suppose that k is a positive integer such that $a_ib_j = 0$ when $i + j < k$. We will show that $a_ib_j = 0$ when $i + j = k$.
By the hypothesis, $a_0b_{k-1} = 0$, then $(b_{k-1}a_0)^2 = 0$.
Thus
$$((a_1b_{k-1})(a_0b_k)^2a_1)(b_{k-1}a_0)^2(b_{k-1}(a_0b_k)^2) = 0.$$

Since
$$\begin{cases} ((a_1b_{k-1})(a_0b_k)^2a_1)(b_{k-1}a_0) \in I \\ (b_{k-1}a_0)(b_{k-1}(a_0b_k)^2) \in I \\ b_k(a_0b_k)a_1 \in I \end{cases}$$

and I is semicommutative (reduced), it follows that
$((a_1b_{k-1})(a_0b_k)^2a_1)(b_{k-1}a_0)(b_k(a_0b_k)a_1)(b_{k-1}a_0)(b_{k-1}(a_0b_k)^2) = 0$,
that is $[(a_1b_{k-1})(a_0b_k)^2]^2a_1(b_{k-1}a_0)(b_{k-1}(a_0b_k)^2) = 0$.
Continuing this procedure yields that
$$[(a_1b_{k-1})(a_0b_k)^2]^4 = 0.$$

Thus $(a_1 b_{k-1})(a_0 b_k)^2 = 0$ since I is reduced.
Similarly we can show that $(a_i b_{k-i})(a_0 b_k)^2 = 0$ for $i = 2, 3, ..., k$.
We have $\sum_{i+j=k} a_i b_j = 0$, if we multiply the last equation on the right side by $(a_0 b_k)^2$, then

$$(a_0 b_k)^3 = -\sum_{i=1}^{k} (a_i b_{k-i})(a_0 b_k)^2 = 0,$$

which implies that $a_0 b_k = 0$ since I is reduced.
We have $(a_1 b_{k-1}) \in I$, by analogy with the above proof, we have

$$(a_i b_{k-i})(a_1 b_{k-1})^2 = 0$$

for $i = 2, 3, ..., k$. If we multiply the equation $\sum_{i+j=k} a_i b_j = 0$ on the right side by $(a_1 b_{k-1})^2$, then

$$(a_1 b_{k-1})^3 = -\sum_{i=2}^{k} (a_i b_{k-i})(a_1 b_{k-1})^2 - (a_0 b_k)(a_1 b_{k-1})^2 = 0$$

which implies that $(a_1 b_{k-1}) = 0$. Similarly, we can show that $a_2 b_{k-2} = 0$,..., $a_k b_0 = 0$.

Thus $a_i b_j = 0$ when $i + j = k$. Therefore, by induction, we have $a_i b_j = 0$ for all i, j and this shows that A is power serieswise Armendariz. □

Corollary 2.11. *Let A be a ring. Then:*

(i) A is power serieswise Armendariz if and only if so is $A[x]$.

(ii) A is power serieswise Armendariz if and only if so is $A[[x]]$.

Proof. (i) If $A[x]$ is power serieswise Armendariz then so is A since A is a subring of $A[x]$. Conversely, we have $A \simeq A[x]/(x)$ and (x) is reduced.

(ii) If $A[[x]]$ is power serieswise Armendariz then so is A since A is a subring of $A[[x]]$. Conversely, we have $A \simeq A[[x]]/(x)$ and (x) is reduced.

□

In a ring containing a central idempotent element, we have:

Theorem 2.12. *Let A be a ring containing a central idempotent element e. Then, A is power serieswise Armendariz if and only if so are eA and $(1-e)A$.*

Proof. If A is power serieswise Armendariz, then so are eA and $(1-e)A$ since eA and $(1-e)A$ are subring of A.
Conversely, assume that eA and $(1-e)A$ are power serieswise Armendariz for a central idempotent element e and consider $f(x) = \sum_{i=0}^{\infty} a_i x^i$, $g(x) = \sum_{j=0}^{\infty} b_j x^j \in A[[x]]$ such that $f(x)g(x) = 0$. Let $f_1(x) = e\sum_{i=0}^{\infty} a_i x^i$, $g_1(x) = e\sum_{j=0}^{\infty} b_j x^j \in eA[[x]]$
$f_2(x) = (1-e)\sum_{i=0}^{\infty} a_i x^i$, $g_2(x) = (1-e)\sum_{j=0}^{\infty} b_j x^j \in (1-e)A[[x]]$.
We have $f_1(x)g_1(x) = ef(x)g(x) = 0$ and $f_2(x)g_2(x) = (1-e)f(x)g(x) = 0$. By the conditions we have that $ea_i b_j = 0$ and $(1-e)a_i b_j = 0$ for every i, j.
Hence $a_i b_j = ea_i b_j + (1-e)a_i b_j = 0$ for every i,j. Thus A is power serieswise Armendariz.

□

Corollary 2.13. *For an abelian ring A, the following statements are equivalent:*

(i) A is power serieswise Armendariz.

(ii) eA and $(1-e)A$ are power serieswise Armendariz for every idempotent e of A.

(iii) eA and $(1-e)A$ are power serieswise Armendariz for some idempotent e of A.

Proof. (1) $\Rightarrow$ (2) is obvious since eA and $(1-e)A$ are subring of A.
(2) $\Rightarrow$ (3) Straightforward.
(3) $\Rightarrow$ (1) Let $f(x)=\sum_{i=0}^{\infty}a_ix^i$, $g(x)=\sum_{j=0}^{\infty}b_jx^j\in A[[x]]$ such that $f(x)g(x)=0$. For some $e=e^2\in A$, let $f_1(x)=e\sum_{i=0}^{\infty}a_ix^i$, $g_1(x)=e\sum_{j=0}^{\infty}b_jx^j\in eA[[x]]$ and $f_2(x)=(1-e)\sum_{i=0}^{\infty}a_ix^i$, $g_2(x)=(1-e)\sum_{j=0}^{\infty}b_jx^j\in(1-e)A[[x]]$.
We have $f_1(x)g_1(x)=ef(x)g(x)=0$ and $f_2(x)g_2(x)=(1-e)f(x)g(x)=0$. By the conditions we have that $ea_ib_j=0$ and $(1-e)a_ib_j=0$ for every i,j.
Hence $a_ib_j=ea_ib_j+(1-e)a_ib_j=0$ for every i,j. Thus A is power serieswise Armendariz. □

References

[1] D. D. Anderson and V. Camillo, Armendariz rings and Gaussian rings, Comm. Algebra. (26) (1998), 2265-2272.

[2] R. Antoine, Nilpotent elements and Armendariz rings, J. Algebra, 319 (2008), 3128-3140.

[3] R. Antoine, Examples of Armendariz rings, Comm. Algebra 38 (11) (2010), 4130-4143.

[4] E. Armendariz, A note on extensions of Baer and P. P. rings, J. Austral. Math. Soc.(18) (1974), 470-473.

[5] C. Bakkari, and N. Mahdou, On Armendariz Rings, Contributions to Algebra and Geometry. (50) (2009), 363-368.

[6] G. Birkenmeier and H. Heatherly, Permutation identity rings and the medial radical, Non commutative ring theory. Lectures notes in mathematics, 1448 (1990), 125-138

[7] M. Boisen and P. B. Sheldon, CPI-extension over rings of integral domains with special prime spectrum, Canad. J. Math. (29) (1977), 722-737.

[8] P. Cohn, Reversible rings, Bull. London Math. Soc. 31 (1999), 641-648.

[9] D. E. Fields, Zero divisors and nilpotent elements in power series rings, Proc. Amer. Math. Soc., 27 (3) (1971), 427-433.

[10] S. Glaz, Commutative coherent rings, Lectures notes in Mathematics, 1371. Springer-Verlag, Berlin, 1989.

[11] S. Hizem, A note on nil power serieswise Armendariz rings, Rendiconti del Circolo Matematico di Palermo 59 (2010), 87-99.

[12] C. Y. Hong, N. K. Kim, T. K. Kwak, Y. Lee, Extensions of zip rings, J. Pure Appl. Algebra 195 (2005), 231-242.

[13] J. A. Huckaba, Commutative Rings with Zero-Divisors, Marcel Dekker, New York, 1988.

[14] C. Huh, H. K. Kim and Y. Lee, Questions On 2-primal rings, Comm. Algebra, 26 (1998), 595-600.

[15] C. Huh, CO. Kim, E. J. Kim, HK. Kim and Y. Lee, Nilradicals of power series rings and nil power series rings, J. Korean Math. Soc., 42 (2005), 1003-1015.

[16] C. Huh, Y. Lee and A. Smoktunowicz, Armendariz and semicommutative ring, Comm. Algebra. 30 (2) (2002), 751-761.

[17] S. Kabbaj and N. Mahdou, Trivial extension defined by coherent-like conditions, Comm. Algebra. 32 (10) (2004), 3937-3953.

[18] N. K. Kim and Y. Lee, Armendariz rings and reduced rings, J. Algebra 223 (2000), 477-488.

[19] N. K. Kim and Y. Lee, Extensions of reversible rings, J. Pure Appl. Algebra. 185 (2003), 207-223.

[20] N. K. Kim, K. H Lee and Y. Lee, Power series rings satisfying a zero divisor property, Comm. Algebra. 34 (2006), 2205-2218.

[21] T. K. Lee and T. L. Wong, On Armendariz rings, Houston J. Math. 28 (3) (2003), 583-593.

[22] L. Liang, L. Wang and Z. Ziu, On a generalization of semi commutative rings, Taiwanese J. Math., 11,5 (2007), 1359-1368.

[23] Z. Liu and R. Zhao, On weak Armendariz rings, Comm. Algebra. 34 (2006), 2607-2616.

[24] M. Nagata, Local Rings, Interscience, New York. 1962.

[25] M. B. Rege and S. Chhawchharia, Armendariz rings, Proc. Japan Acad. Ser. A Math. Sci. 73 (1997), 14-17.

[26] G. Shin, Prime ideal and sheaf representation of a pseudo symmetric rings, Trans. Amer. Math. Soc.(1973), 184:43-60.

[27] A. Smoktunowicz, Polynomial rings over nil rings need not be nil, J. Algebra. 233 (2000), 427-436.

Author information

Mounir El Ouarrachi, Department of Mathematics, Faculty of Science and Technology of Fez, Box 2202, University S. M. Ben Abdellah Fez, Morocco, Morocco.
E-mail: m.elouarrachi@gmail.com

Najib Mahdou, Department of Mathematics, Faculty of Science and Technology of Fez, Box 2202, University S. M. Ben Abdellah Fez, Morocco, Morocco.
E-mail: mahdou@hotmail.com