

On SEMIARTINIAN AND Π -SEMIARTINIAN MODULES

Farid Kourki and Rachid Tribak

Communicated by Najib Mahdou

MSC 2010 Classifications: Primary 16D10, 16D99; Secondary 13C99.

Keywords and phrases: Semiartinian modules (rings), Π -semiartinian modules (rings).

Abstract All rings will be commutative with identity. A module M is called Π -semiartinian if the direct product M^I is a semiartinian module for every nonempty set I . A ring R is said to be Π -semiartinian if every product of semiartinian R -modules is semiartinian. It is shown that the class of Π -semiartinian R -modules is closed under isomorphic images, submodules, factor modules and extensions. We prove that an R -module M is Π -semiartinian if and only if $R/Ann(M)$ is a semiartinian ring. We also provide a characterization of Π -semiartinian rings.

1 Introduction

Throughout this article, all rings considered are assumed to be commutative rings with an identity and R denotes such a ring. All modules are unital. We denote respectively by $Spec(R)$ and $Max(R)$ the set of all prime ideals of R and the set of all maximal ideals of R . Let M be an R -module and let $x \in M$. By $Ann(x)$ and $Ann(M)$ we denote the *annihilator* of x and M , respectively; i.e. $Ann(x) = \{r \in R \mid rx = 0\}$ and $Ann(M) = \{r \in R \mid rM = 0\}$. The notation $N \subseteq M$ means that N is a subset of M and $N \leq M$ means that N is a submodule of M . If M_1 and M_2 are two R -modules, $Hom_R(M_1, M_2)$ will denote the set of R -homomorphisms from M_1 to M_2 . By \mathbb{Z} we denote the ring of integer numbers.

A module M is called *semiartinian* if every nonzero factor module of M has nonzero socle. In Section 2, we investigate some basic properties of semiartinian modules and we provide a new characterization of this kind of modules (Proposition 2.9).

In Section 3, we introduce the notions of Π -semiartinian rings and Π -semiartinian modules. We call a module M Π -*semiartinian* if the direct product M^I is a semiartinian module for every nonempty set I . A ring R is said to be Π -*semiartinian* if every product of semiartinian R -modules is semiartinian. We prove that a module M is Π -semiartinian if and only if $R/Ann(M)$ is a semiartinian ring (Proposition 3.2). It is also shown that the class of Π -semiartinian R -modules is closed under isomorphic images, submodules, factor modules and extensions (Proposition 3.4). We show that the class of Π -semiartinian rings contains the class of semilocal rings R such that $\mathfrak{m}^2 = \mathfrak{m}$ for every $\mathfrak{m} \in Max(R)$ (Proposition 3.5). A characterization of Π -semiartinian rings is provided (Theorem 3.10).

2 Semiartinian Modules

Recall that an R -module M is called *semiartinian* if every nonzero factor module of M has nonzero socle. A ring R is called *semiartinian* if it is semiartinian as an R -module. Note that a ring R is semiartinian if and only if every R -module is semiartinian (see [14, p. 183 Proposition 2.5]). Recall that a subset I of a ring R is called *T-nilpotent* if for every sequence a_1, a_2, \dots in I there exists an integer $n \geq 1$ such that $a_1 \dots a_n = 0$.

We begin with the following lemma which will be useful to our work in this article.

Lemma 2.1. (i) *The class of semiartinian modules is closed under taking isomorphic images, submodules, factor modules, direct sums and module extensions.*

(ii) *A ring R is semiartinian if and only if $Rad(R)$ is T-nilpotent and $R/Rad(R)$ is a semiartinian ring.*

(iii) *If R is a semiartinian ring, then $R/Rad(R)$ is a von Neumann regular ring.*

- Proof.** (i) See [6, p. 28-29].
(ii) See [14, p. 184 Proposition 2.8].
(iii) See [9, Corollary 3.33E]. \square

Example 2.2. From [4, Theorem P], it follows that every perfect ring is semiartinian.

The proof of the following lemma is straightforward and is omitted.

Lemma 2.3. *Let I be a proper ideal of a ring R . Then R/I is a semiartinian ring if and only if R/I is a semiartinian R -module.*

The next lemma will be of interest.

Lemma 2.4. *Let R be a commutative ring. Then:*

- (i) *If R is a semiartinian ring, then R/\mathfrak{a} is a semiartinian ring for any proper ideal \mathfrak{a} of R .*
(ii) *$R/\mathfrak{a} \cap \mathfrak{b}$ and $R/\mathfrak{a}\mathfrak{b}$ are semiartinian rings whenever \mathfrak{a} and \mathfrak{b} are proper ideals of R such that R/\mathfrak{a} and R/\mathfrak{b} are semiartinian rings.*

Proof. (i) This follows from Lemmas 2.1(i) and 2.3.

(ii) Assume that R/\mathfrak{a} and R/\mathfrak{b} are semiartinian rings. Then $\mathfrak{b}/\mathfrak{a}\mathfrak{b}$ is a semiartinian R/\mathfrak{a} -module and hence it is a semiartinian R -module. Moreover, R/\mathfrak{b} is a semiartinian R -module. We have the following exact sequence of R -modules:

$$0 \rightarrow \mathfrak{b}/\mathfrak{a}\mathfrak{b} \rightarrow R/\mathfrak{a}\mathfrak{b} \rightarrow R/\mathfrak{b} \rightarrow 0.$$

But the class of semiartinian modules is closed under extensions (Lemma 2.1(i)). So $R/\mathfrak{a}\mathfrak{b}$ is a semiartinian R -module. In addition, since $R/\mathfrak{a} \cap \mathfrak{b}$ is a factor ring of $R/\mathfrak{a}\mathfrak{b}$, $R/\mathfrak{a} \cap \mathfrak{b}$ is also a semiartinian ring by (i). \square

Corollary 2.5. *Let \mathfrak{a} be a proper ideal of a ring R . The following conditions are equivalent:*

- (i) *R/\mathfrak{a} is a semiartinian ring;*
(ii) *R/\mathfrak{a}^n is a semiartinian ring for every integer $n \geq 1$;*
(iii) *R/\mathfrak{a}^m is a semiartinian ring for some integer $m \geq 1$.*

Proof. (i) \Rightarrow (ii) By induction and using Lemma 2.4(ii).

(ii) \Rightarrow (iii) This is clear.

(iii) \Rightarrow (i) Let $m \geq 1$ such that R/\mathfrak{a}^m is semiartinian. Then R/\mathfrak{a} is a semiartinian ring being a factor ring of R/\mathfrak{a}^m (see Lemma 2.4(i)). \square

Corollary 2.6. *The following are equivalent for a ring R :*

- (i) *R is a semiartinian ring;*
(ii) *There exists a maximal ideal \mathfrak{m} of R such that $R/\text{Ann}(\mathfrak{m})$ is a semiartinian ring;*
(iii) *There exists an ideal \mathfrak{a} of R such that R/\mathfrak{a} and $R/\text{Ann}(\mathfrak{a})$ are semiartinian rings.*

Proof. (i) \Rightarrow (ii) This follows from the fact that any factor ring of a semiartinian ring is semiartinian (Lemma 2.4(i)).

(ii) \Rightarrow (iii) It suffices to take $\mathfrak{a} = \mathfrak{m}$.

(iii) \Rightarrow (i) By Lemma 2.4(ii), $R/(\mathfrak{a}\text{Ann}(\mathfrak{a}))$ is a semiartinian ring. Therefore R is a semiartinian ring as $\mathfrak{a}\text{Ann}(\mathfrak{a}) = 0$. \square

Recall that a ring R is said to be *zero dimensional* (or of *Krull dimension zero*), and we write $\dim(R) = 0$, if every prime ideal of R is maximal.

Example 2.7. Every commutative semiartinian ring is zero dimensional. To see this, if R is such a ring and if $J = \text{Rad}(R)$, then J is T-nilpotent and the ring R/J is von Neumann regular (Lemma 2.1). Hence J is a nil ideal. This yields $J = N$, where N is the nil radical of R . Therefore R/N is a von Neumann regular ring and so $\dim(R) = 0$ (see [12, Theorem 1.16]).

For an R -module M and an ideal I of R , I is said to be *T-nilpotent on M* if for every $x \in M$ and every sequence $a_1, a_2, \dots \in I$ there exists an integer $n \geq 1$ such that $a_1 \dots a_n x = 0$.

Let M be an R -module. We say that a prime ideal \mathfrak{p} of R is an *associated prime* of M if $\mathfrak{p} = \text{Ann}(x)$ for some $x \in M$. The set of associated primes of M is denoted by $\text{Ass}(M)$. The support of M is $\text{Supp}(M) = \{\mathfrak{p} \in \text{Spec}(R) \mid \mathfrak{p} \supseteq \text{Ann}(x) \text{ for some } x \in M\}$. We denote by $\text{MaxSupp}(M)$ the set of all maximal members in $\text{Supp}(M)$. Let $J(M) = \bigcap_{\mathfrak{m} \in \text{MaxSupp}(M)} \mathfrak{m}$. In the case when M is finitely generated, it is easy to see that $\text{Supp}(M) = \{\mathfrak{p} \in \text{Spec}(R) \mid \mathfrak{p} \supseteq \text{Ann}(M)\}$ and hence $\text{Rad}(R/\text{Ann}(M)) = J(M)/\text{Ann}(M)$.

Lemma 2.8. *Let M be a finitely generated R -module. Then:*

- (i) $\text{Rad}(R/\text{Ann}(M)) = J(M)/\text{Ann}(M)$.
- (ii) $\text{Rad}(R/\text{Ann}(M))$ is a *T-nilpotent ideal* of $R/\text{Ann}(M)$ if and only if $J(M)$ is *T-nilpotent on M* .

Proof. (i) This is clear.

(ii) (\Rightarrow) This follows from (i).

(\Leftarrow) Assume that $M = Rx_1 + \dots + Rx_n$. Note that $J(M)/\text{Ann}(M) = \text{Rad}(R/\text{Ann}(M))$. Let a_1, a_2, \dots be a sequence in $J(M)$. For every $i \in \{1, \dots, n\}$, there exists an integer $k_i \geq 1$ such that $a_1 \dots a_{k_i} x_i = 0$ since $J(M)$ is *T-nilpotent on M* . Let $k = \max(k_1, \dots, k_n)$. Then $a_1 \dots a_k x_i = 0$ for every $i \in \{1, \dots, n\}$. This implies that $a_1 \dots a_k \in \text{Ann}(M)$. Thus $(a_1 \dots a_k) + \text{Ann}(M) = 0 + \text{Ann}(M)$. Therefore $\text{Rad}(R/\text{Ann}(M))$ is a *T-nilpotent ideal* of $R/\text{Ann}(M)$. \square

We will say that an R -module M is a *zero dimensional module*, and we write $\dim M = 0$, if every prime ideal in $\text{Supp}(M)$ is maximal, that is, $\text{Supp}(M) = \text{MaxSupp}(M)$. It is clear that $\dim M = 0$ if and only if $R/\text{Ann}(x)$ is a zero dimensional ring for any nonzero element $x \in M$.

Let M be an R -module. An R -module N is called *M -generated* if it is a homomorphic image of a direct sum of copies of M . An R -module N is said to be *subgenerated by M* if N is isomorphic to a submodule of an M -generated module. Let $R\text{-Mod}$ denotes the category of all R -modules. We denote by $\sigma[M]$ the full subcategory of $R\text{-Mod}$ whose objects are all R -modules subgenerated by M .

Next, we exhibit a characterization of semiartinian modules.

Proposition 2.9. *The following statements are equivalent for a nonzero R -module M :*

- (i) M is a semiartinian R -module;
- (ii) Every nonzero cyclic submodule of M is semiartinian;
- (iii) Every nonzero finitely generated submodule N of M is semiartinian;
- (iv) $R/\text{Ann}(x)$ is a semiartinian ring for every $0 \neq x \in M$;
- (v) $R/\text{Ann}(N)$ is a semiartinian ring for every nonzero finitely generated submodule N of M ;
- (vi) $R/J(Rx)$ is semiartinian and $J(Rx)$ is *T-nilpotent on Rx* for every $0 \neq x \in M$;
- (vii) $R/J(N)$ is semiartinian and $J(N)$ is *T-nilpotent on N* for every nonzero finitely generated submodule N of M ;
- (viii) $\dim(M) = 0$ and $\text{Ass}(N) \neq \emptyset$ for any $0 \neq N \in \sigma[M]$.

Proof. Since the class of semiartinian modules is closed under submodules, factor modules and direct sums (Lemma 2.1), it follows that for any family $\{N_i\}_{i \in I}$ of submodules of a module M , the sum $\sum_{i \in I} N_i$ is a semiartinian module if and only if each N_i is a semiartinian module (see also [6, p. 29]). From this remark it is easy to deduce the equivalences (i) \Leftrightarrow (ii) \Leftrightarrow (iii).

(ii) \Leftrightarrow (iv) If $0 \neq x \in M$, then $Rx \cong R/\text{Ann}(x)$. So, Rx is a semiartinian R -module if and only if $R/\text{Ann}(x)$ is a semiartinian ring.

(v) \Rightarrow (iv) Clear.

(iv) \Rightarrow (v) Let $N = Rx_1 + \dots + Rx_n$ be a nonzero finitely generated submodule of M such that $x_i \neq 0$ for every $i \in \{1, \dots, n\}$. Note that $\text{Ann}(N) = \bigcap_{i=1}^n \text{Ann}(x_i)$. By hypothesis, $R/\text{Ann}(x_i)$ is a semiartinian ring for each $i \in \{1, \dots, n\}$. By using (ii) of Lemma 2.4 and by induction on n , we see that $R/\text{Ann}(N)$ is a semiartinian ring.

(iv) \Leftrightarrow (vi) Let $0 \neq x \in M$. By Lemma 2.1, $A = R/Ann(x)$ is a semiartinian ring if and only if $A/Rad(A)$ is a semiartinian ring and $Rad(A)$ is a T-nilpotent ideal of A . Using Lemma 2.8, we obtain the equivalence.

(v) \Leftrightarrow (vii) Similar to the proof of the equivalence (iv) \Leftrightarrow (vi).

(ii) \Rightarrow (viii) Let $0 \neq x \in M$. Then $R/Ann(x)$ is semiartinian and so it is a zero dimensional ring (Example 2.7). Hence $dim(M) = 0$. Let $0 \neq N \in \sigma[M]$. By (i) of Lemma 2.1, N is also semiartinian. Thus $Soc(N) \neq 0$. Let Ry be a simple submodule of N . Then $Ann(y)$ is a maximal ideal of R and so $Ann(y) \in Ass(N)$.

(viii) \Rightarrow (i) Let $0 \neq N \leq M$. Then $M/N \in \sigma[M]$ and so $Ass(M/N) \neq \emptyset$. Let $\mathfrak{p} \in Ass(M/N)$. Then there exists $0 \neq y \in M/N$ such that $\mathfrak{p} = Ann(y)$. Since $dim(M) = 0$, it is easy to see that $dim(M/N) = 0$. It follows that \mathfrak{p} is a maximal ideal of R . Therefore, Ry is a simple submodule of M/N . This implies that M is a semiartinian module. \square

Example 2.10. (i) From Proposition 2.9, it follows immediately that a \mathbb{Z} -module M is semiartinian if and only if M is a torsion module.

Corollary 2.11. *Let M be a semiartinian module. Then $J(M)$ is T-nilpotent on M .*

Proof. Let $0 \neq x \in M$. By Proposition 2.9, $J(Rx)$ is T-nilpotent on Rx . But $J(M) \subseteq J(Rx)$. So $J(M)$ is T-nilpotent on Rx . The result follows. \square

Remark 2.12. Let M be an R -module. Since $Rad(R) \subseteq J(M)$, it follows that $Rad(R)$ is T-nilpotent on every semiartinian R -module by Corollary 2.11. This shows that Corollary 2.11 is a generalization of [14, Proposition 2.6] in the commutative case.

Recall that a ring R (not necessarily commutative) is said to be π -regular if for any $a \in R$, there is an integer $n \geq 1$ and $b \in R$ such that $a^n = a^n b a^n$. The ring R is called *strongly π -regular* if for each $a \in R$, there is an integer $n \geq 1$ and $b \in R$ such that $a^n = a^{n+1} b$. It is shown in [15, Lemma 5.6] that a commutative ring R is π -regular if and only if $dim(R) = 0$.

Corollary 2.13. *Let M be a finitely generated R -module. Then the following statements are equivalent:*

- (i) M is a semiartinian module;
- (ii) $R/Ann(M)$ is a semiartinian ring;
- (iii) $R/J(M)$ is semiartinian and $J(M)$ is T-nilpotent on M ;

Moreover, if M is a semiartinian module, then $End_R(M)$ is a strongly π -regular ring.

Proof. (i) \Leftrightarrow (ii) and (i) \Rightarrow (iii) These follow from Proposition 2.9.

(iii) \Rightarrow (ii) Since M is finitely generated, it follows that $J(M)$ is T-nilpotent on M if and only if $Rad(R/Ann(M)) = J(M)/Ann(M)$ is a T-nilpotent ideal of $A = R/Ann(M)$ (Lemma 2.8). In addition, it is clear that $R/J(M)$ is semiartinian if and only if $A/Rad(A)$ is semiartinian. Now apply (ii) of Lemma 2.1.

Now assume that M is a semiartinian R -module. Hence $R/Ann(M)$ is a semiartinian ring. Therefore $dim(R/Ann(M)) = 0$ (Example 2.7). By [2, Theorem 1], $End_R(M)$ is a strongly π -regular ring. \square

Proposition 2.14. *Let M be an R -module. Then M is a semiartinian R -module if and only if $M \otimes N$ is a semiartinian R -module for every R -module N .*

Proof. (\Rightarrow) Let N be an R -module and let $0 \neq x = \sum_{i=1}^n x_i \otimes y_i \in M \otimes N$ where $x_i \in M$ and $y_i \in N$ for every $i \in \{1, \dots, n\}$. Let $M' = Rx_1 + \dots + Rx_n$. Therefore $Ann(M') \subseteq Ann(Rx)$ and M' is semiartinian as M' is a submodule of the semiartinian module M . Since $R/Ann(Rx) \cong \frac{R/Ann(M')}{Ann(Rx)/Ann(M')}$ (as rings) and $R/Ann(M')$ is a semiartinian ring (Corollary 2.13), $R/Ann(Rx)$ is also a semiartinian ring by Lemma 2.4(i). Applying Proposition 2.9, we conclude that $M \otimes N$ is a semiartinian R -module.

(\Leftarrow) This follows by taking $N = R$ and using the fact that $M \otimes R \cong M$. \square

3 Π -Semiartinian Modules

In this section, we wish to investigate the class of rings R for which every direct product of semiartinian R -modules is semiartinian. First we introduce the notion of Π -semiartinian modules.

Definition 3.1. An R -module M is called a Π -semiartinian module if the direct product M^I is a semiartinian module for every nonempty set I .

Proposition 3.2. Let M be a nonzero R -module. The following conditions are equivalent:

- (i) M is a Π -semiartinian R -module;
- (ii) $R/Ann(M)$ is a semiartinian ring;
- (iii) $Hom_R(M, M)$ is a semiartinian R -module.

Proof. (i) \Rightarrow (ii) Let $\{x_i\}_{i \in I}$ be a family of generators of M . Consider the map $f : R \rightarrow M^I$ defined by $f(a) = (ax_i)_{i \in I}$ for every $a \in R$. Then f is R -linear and $Ker f = Ann(M)$. It follows that $R/Ann(M)$ is isomorphic to a submodule of the semiartinian R -module M^I . Hence $R/Ann(M)$ is semiartinian as R -module. Therefore $R/Ann(M)$ is a semiartinian ring.

(ii) \Rightarrow (iii) Let $0 \neq f \in Hom_R(M, M)$. Note that $Ann(M) \subseteq Ann(Rf)$. Therefore $R/Ann(Rf) \cong \frac{R/Ann(M)}{Ann(Rf)/Ann(M)}$. But $R/Ann(M)$ is a semiartinian ring. So $R/Ann(Rf)$ is a semiartinian ring (Lemma 2.4(i)). By Proposition 2.9, $Hom_R(M, M)$ is a semiartinian R -module.

(iii) \Rightarrow (ii) Consider the R -homomorphism $\Psi : R \rightarrow Hom_R(M, M)$ such that for every $r \in R$, $\Psi(r)$ is the endomorphism of the R -module M defined by $\Psi(r)(x) = rx$ for every $x \in M$. We have $Ker \Psi = Ann(M)$. Hence $R/Ann(M)$ is isomorphic to a submodule of $Hom_R(M, M)$. By Lemma 2.1, $R/Ann(M)$ is a semiartinian R -module and so $R/Ann(M)$ is a semiartinian ring.

(ii) \Rightarrow (i) Let I be a nonempty set. Since $Ann(M^I) = Ann(M)$ and $R/Ann(M)$ is a semiartinian ring, the direct product M^I is semiartinian as an $R/Ann(M)$ -module and hence also as an R -module. This completes the proof. \square

Combining Corollary 2.13 and Proposition 3.2, we get the following corollary.

Corollary 3.3. The following conditions are equivalent for a finitely generated R -module M :

- (i) M is a Π -semiartinian R -module;
- (ii) M is a semiartinian R -module;
- (iii) $Hom_R(M, M)$ is a semiartinian R -module.

Proposition 3.4. Let R be a ring. Then the class of Π -semiartinian R -modules is closed under isomorphic images, submodules, factor modules and extensions.

Proof. It is easy to see that the class of Π -semiartinian R -modules is closed under isomorphic images. Let M be an R -module and let N be a submodule of M .

Assume that M is a Π -semiartinian R -module. It is clear that $Ann(M) \subseteq Ann(N)$ and $Ann(M) \subseteq Ann(M/N)$. Moreover, we have the following two ring isomorphisms:

$$R/Ann(N) \cong \frac{R/Ann(M)}{Ann(N)/Ann(M)} \text{ and } R/Ann(M/N) \cong \frac{R/Ann(M)}{Ann(M/N)/Ann(M)}.$$

But the ring $R/Ann(M)$ is semiartinian by Proposition 3.2. So $R/Ann(N)$ and $R/Ann(M/N)$ are semiartinian rings by Lemma 2.4(i). From Proposition 3.2, it follows that N and M/N are Π -semiartinian R -modules.

Now suppose that N and M/N are Π -semiartinian R -modules and let us show that M is a Π -semiartinian module. By Proposition 3.2, $R/Ann(N)$ and $R/Ann(M/N)$ are semiartinian rings. Applying Lemma 2.4(ii), we deduce that the ring $R/(Ann(N) \cap Ann(M/N))$ is semiartinian. Therefore $R/(Ann(N) \cap Ann(M/N))^2$ is also a semiartinian ring by Corollary 2.5. But $(Ann(N) \cap Ann(M/N))^2 \subseteq Ann(M)$. Hence $R/Ann(M)$ is also a semiartinian ring. By Proposition 3.2, M is a Π -semiartinian R -module. \square

We call a ring R Π -semiartinian if every product of semiartinian R -modules is semiartinian.

Proposition 3.5. Let R be a semilocal ring such that $\mathfrak{m}^2 = \mathfrak{m}$ for every maximal ideal \mathfrak{m} of R . Then R is a Π -semiartinian ring.

Proof. Let $\{M_\lambda\}_{\lambda \in \Lambda}$ be a family of semiartinian R -modules and let $M = \prod_{\lambda \in \Lambda} M_\lambda$. Using [7, Corollary 2.7] and [5, Theorem 5], we conclude that each M_λ ($\lambda \in \Lambda$) is semisimple. But over a semilocal ring any product of semisimple modules is semisimple (see [1, Proposition 15.17]). Therefore M is a semiartinian module. This proves the proposition. \square

Example 3.6. Let F be a field and let R be the ring of polynomials in countably many commuting variables x_1, x_2, \dots , over F subject to the relations $x_1^2 = 0$ and $x_n^2 = x_{n-1}$ for $n \geq 2$. This ring appears in [16] in another context. The ring R is local and its maximal ideal \mathfrak{m} is generated by all the x_i , ($i \geq 1$). Moreover, we have $\mathfrak{m}^2 = \mathfrak{m}$. From Proposition 3.5, it follows that R is a Π -semiartinian ring.

A ring R is called a *valuation ring* (or a *chain ring*) if any two ideals of R are comparable. It is clear that if R is a valuation ring, then R is local and any finitely generated ideal of R is cyclic.

Lemma 3.7. *Let R be a valuation ring and let \mathfrak{m} be its maximal ideal. If \mathfrak{m} is not finitely generated, then $\mathfrak{m}^2 = \mathfrak{m}$. The converse holds when R is not a field.*

Proof. Suppose that \mathfrak{m} is not finitely generated and let $A = R / \bigcap_{n \geq 1} \mathfrak{m}^n$. Then A is a valuation ring with $\text{Rad}(A) = \mathfrak{m} / \bigcap_{n \geq 1} \mathfrak{m}^n$ and $\bigcap_{n \geq 1} (\mathfrak{m} / \bigcap_{n \geq 1} \mathfrak{m}^n)^n = 0$. By [8, Proposition 5.3], A is a noetherian ring and so $\mathfrak{m} / \bigcap_{n \geq 1} \mathfrak{m}^n$ is a cyclic ideal generated by an element $a + \bigcap_{n \geq 1} \mathfrak{m}^n$, where $a \in \mathfrak{m}$. Therefore $\mathfrak{m} = Ra + \bigcap_{n \geq 1} \mathfrak{m}^n$. If $\bigcap_{n \geq 1} \mathfrak{m}^n \subseteq Ra$, then $\mathfrak{m} = Ra$ which is impossible since \mathfrak{m} is not finitely generated. So $Ra \subseteq \bigcap_{n \geq 1} \mathfrak{m}^n$ and hence $\mathfrak{m} = \bigcap_{n \geq 1} \mathfrak{m}^n$. Consequently, $\mathfrak{m}^2 = \mathfrak{m}$.

Now suppose that R is not a field and $\mathfrak{m}^2 = \mathfrak{m}$. Then \mathfrak{m} is not finitely generated, since otherwise \mathfrak{m} will be a direct summand of R and so R will be a field (see, for example, [1, p. 103 Exercise 12(3)]). \square

Corollary 3.8. *Let R be a valuation ring and let \mathfrak{m} be its maximal ideal. If \mathfrak{m} is not finitely generated, then R is a Π -semiartinian ring.*

Proof. By Proposition 3.5 and Lemma 3.7. \square

Remark 3.9. Since the class of semiartinian R -modules is closed under submodules, factor modules and direct sums (Lemma 2.1), every module M has a semiartinian submodule $Sa(M)$ which is maximal for this property (see also [6, p. 29]) and we have

$$Sa(M) = \{x \in M \mid Rx \text{ is a semiartinian } R\text{-module}\}.$$

Moreover, we have $Sa(M) = \{x \in M \mid x = 0 \text{ or } R/\text{Ann}(x) \text{ is a semiartinian ring}\}$.

Let R be a ring. If I is an ideal of R such that R/I is a semiartinian ring then we call I a *cosemiartinian ideal*. Let $\mathfrak{S}(R)$ denote the intersection of all cosemiartinian ideals of R . Clearly, $\mathfrak{S}(R) \subseteq \text{Rad}(R)$. Also, for every maximal ideal \mathfrak{m} of R , we have $\mathfrak{S}(R) \subseteq \bigcap_{k=1}^{\infty} \mathfrak{m}^k$ (Corollary 2.5).

Let M be an R -module and let \mathfrak{a} be an ideal of R . We will denote by $\text{Ann}_M(\mathfrak{a})$ the set $\{m \in M \mid rm = 0 \text{ for every } r \in \mathfrak{a}\}$.

Next, we characterize rings R for which the class of semiartinian R -modules is closed under direct products.

Theorem 3.10. *The following statements are equivalent for a ring R :*

- (i) *The class of semiartinian R -modules is closed under direct products (that is, R is a Π -semiartinian ring);*
- (ii) *The class of Π -semiartinian R -modules is closed under direct sums;*
- (iii) *The class of Π -semiartinian R -modules is closed under direct products;*
- (iv) *Every semiartinian R -module is Π -semiartinian;*

- (v) For every nonzero R -module M , M is semiartinian if and only if $R/\text{Ann}(M)$ is a semiartinian ring;
- (vi) For every family $\{I_\lambda\}_{\lambda \in \Lambda}$ of ideals of R such that each R/I_λ ($\lambda \in \Lambda$) is a semiartinian ring, $R/\bigcap_{\lambda \in \Lambda} I_\lambda$ is a semiartinian ring;
- (vii) $R/\mathfrak{S}(R)$ is a semiartinian ring;
- (viii) For every R -module M , $Sa(M) = \text{Ann}_M(\mathfrak{S}(R))$.

Proof. (i) \Rightarrow (iv) This is evident.

(iv) \Rightarrow (ii) This follows from the fact that any direct sum of semiartinian modules is semiartinian (Lemma 2.1).

(ii) \Rightarrow (vi) Let $\{I_\lambda\}_{\lambda \in \Lambda}$ be a family of ideals of R such that R/I_λ is a semiartinian ring for all $\lambda \in \Lambda$. Then R/I_λ is a Π -semiartinian R -module by Corollary 3.3. By hypothesis, $M = \bigoplus_{\lambda \in \Lambda} R/I_\lambda$ is a Π -semiartinian R -module. Hence $R/\text{Ann}(M) = R/\bigcap_{\lambda \in \Lambda} I_\lambda$ is a semiartinian ring by Proposition 3.2.

(vi) \Rightarrow (vii) This is immediate.

(vii) \Rightarrow (viii) Let M be a nonzero R -module and let $0 \neq x \in M$. Thus,

$$\begin{aligned} x \in Sa(M) &\Leftrightarrow R/\text{Ann}(x) \text{ is a semiartinian ring.} \\ &\Leftrightarrow \mathfrak{S}(R) \subseteq \text{Ann}(x) \text{ since } R/\mathfrak{S}(R) \text{ is a semiartinian ring.} \\ &\Leftrightarrow x \in \text{Ann}_M(\mathfrak{S}(R)). \end{aligned}$$

It follows that $Sa(M) = \text{Ann}_M(\mathfrak{S}(R))$.

(viii) \Rightarrow (i) Let $\{M_\lambda\}_{\lambda \in \Lambda}$ be a family of semiartinian R -modules and let $M = \prod_{\lambda \in \Lambda} M_\lambda$. By hypothesis, we have $\mathfrak{S}(R)M_\lambda = 0$ for all $\lambda \in \Lambda$. This clearly forces $\mathfrak{S}(R)M = 0$. It follows that $\text{Ann}_M(\mathfrak{S}(R)) = M$. But $Sa(M) = \text{Ann}_M(\mathfrak{S}(R))$. Then $M = Sa(M)$ is a semiartinian module. Consequently, R is a Π -semiartinian ring.

(ii) \Leftrightarrow (iii) Let $\{M_\lambda\}_{\lambda \in \Lambda}$ be a family of Π -semiartinian R -modules. Then

$$\text{Ann}(\bigoplus_{\lambda \in \Lambda} M_\lambda) = \text{Ann}(\prod_{\lambda \in \Lambda} M_\lambda) = \bigcap_{\lambda \in \Lambda} \text{Ann}(M_\lambda).$$

Now use Proposition 3.2.

(iv) \Leftrightarrow (v) This follows from Proposition 3.2. \square

Recall that an R -module M is called a *max module* if every nonzero submodule of M contains a maximal submodule. A ring R is called a *max ring* if every nonzero R -module is a max module. Combining [9, Corollary 3.33E] and [10, Theorem A], we see that any semiartinian ring is a max ring. A module M is called *tall* if it contains some submodule N such that both M/N and N are non-noetherian. A ring R is called *tall* if every non-noetherian R -module is tall. By [13, Corollary 1.2], every max ring is tall. Therefore every semiartinian ring is tall.

Corollary 3.11. *Let R be a Π -semiartinian ring. Then the following hold:*

- (i) $R/\text{Rad}(R)$ is a semiartinian von Neumann regular ring.
(ii) $R/\bigcap_{n \geq 1} \mathfrak{m}^n$ is a semiartinian ring for every maximal ideal \mathfrak{m} of R .
(iii) R is a tall ring.

Proof. (i) By Theorem 3.10, $R/\mathfrak{S}(R)$ is a semiartinian ring. But $R/\text{Rad}(R)$ is a factor ring of $R/\mathfrak{S}(R)$ since $\mathfrak{S}(R) \subseteq \text{Rad}(R)$. Thus $R/\text{Rad}(R)$ is a semiartinian ring and hence it is also a von Neumann regular ring (see Lemma 2.1(iii)).

(ii) Let $\mathfrak{m} \in \text{Max}(R)$ and let $n \geq 1$. Then R/\mathfrak{m}^n is a semiartinian ring by Corollary 2.5. From Theorem 3.10, it follows that $R/\bigcap_{n \geq 1} \mathfrak{m}^n$ is a semiartinian ring.

(iii) Let $\mathfrak{m} \in \text{Max}(R)$. By (ii), $R/\bigcap_{n \geq 1} \mathfrak{m}^n$ is a semiartinian ring and hence it is a max ring. By [13, Corollary 1.2], $R/\bigcap_{n \geq 1} \mathfrak{m}^n$ is a tall ring. Therefore R is a tall ring by [13, Corollary 2.7]. \square

Proposition 3.12. *Let R be a ring.*

- (i) Assume that R is noetherian. Then R is Π -semiartinian if and only if R is artinian.

(ii) Assume that R is a Prüfer domain. If R is Π -semiartinian, then $\mathfrak{m}^2 = \mathfrak{m}$ for every maximal ideal \mathfrak{m} of R .

(iii) Assume that R is a valuation ring with maximal ideal \mathfrak{m} . Then R is Π -semiartinian if and only if R is artinian or \mathfrak{m} is not finitely generated.

Proof. (i) (\Rightarrow) By Corollary 3.11, R is a tall ring. Now use [13, Proposition 2.10].

(\Leftarrow) This follows from the fact that every artinian ring is semiartinian.

(ii) Let $\mathfrak{m} \in \text{Max}(R)$. By Corollary 3.11, $A = R / \bigcap_{n \geq 1} \mathfrak{m}^n$ is a semiartinian ring and so it is zero dimensional (Example 2.7). By [3, Theorem 2.7], we have $\sqrt{\bigcap_{n \geq 1} \mathfrak{m}^n} = \bigcap_{n \geq 1} \mathfrak{m}^n$. This implies that A is a reduced ring. Therefore A is a von Neumann regular ring by [12, Theorem 1.16]. Consequently, $(\mathfrak{m} / \bigcap_{n \geq 1} \mathfrak{m}^n)^2 = \mathfrak{m} / \bigcap_{n \geq 1} \mathfrak{m}^n$. Thus, $\mathfrak{m}^2 = \mathfrak{m}$.

(iii) (\Rightarrow) By Corollary 3.11, R is a tall ring. Now apply [13, Corollary 2.11].

(\Leftarrow) If R is artinian, then R is semiartinian and so it is Π -semiartinian. Now if \mathfrak{m} is not finitely generated, then R is Π -semiartinian by Corollary 3.8. \square

Corollary 3.13. Let R be a valuation domain which is not a field with maximal ideal \mathfrak{m} . Then the following conditions are equivalent:

- (i) R is a Π -semiartinian ring;
- (ii) \mathfrak{m} is not finitely generated;
- (iii) $\mathfrak{m}^2 = \mathfrak{m}$.

Proof. This follows easily from Lemma 3.7 and Proposition 3.12(iii). \square

Next, we present an example showing that the class of semiartinian rings is larger than that of Π -semiartinian rings.

Remark 3.14. (i) Let \mathbb{Z} be the ring of integers. Since $\text{Rad}(\mathbb{Z}) = 0$ and \mathbb{Z} is not a von Neumann regular ring, \mathbb{Z} is not a Π -semiartinian ring by Corollary 3.11.

(ii) It is shown in [11, Example 8.4.8] that there exists a valuation domain R which not a field such that the maximal ideal of R is idempotent. Clearly, the ring R is not semiartinian as $\text{Soc}(R) = 0$. On the other hand, R is a Π -semiartinian ring by Corollary 3.13.

References

- [1] F. W. Anderson and K. R. Fuller, *Rings and Categories of Modules*, Graduate Texts in Mathematics **13**, Springer-Verlag, New York (1974).
- [2] E. P. Armendariz, Modules with artinian prime factors, *Proc. Amer. Math. Soc.* **78**(3), 311-314 (1980).
- [3] J. T. Arnold and R. Gilmer, Idempotent ideals and unions of nets of Prüfer domains, *J. Sci. Hiroshima Univ. Ser. A-I* **31**, 131-145 (1967).
- [4] H. Bass, Finitistic dimension and a homological generalization of semi-primary rings, *Trans. Amer. Math. Soc.* **95**, 466-488 (1960).
- [5] T. J. Cheatham and J. R. Smith, Regular and semisimple modules, *Pacific J. Math.* **65**(2), 315-323 (1976).
- [6] N. V. Dung and P. F. Smith, On Semi-artinian V-modules, *J. Pure Appl. Algebra* **82**, 27-37 (1992).
- [7] S. E. Dickson, Decomposition of modules: II. Rings without chain conditions, *Math. Z.* **104**, 349-357 (1968).
- [8] A. Facchini, *Module Theory: Endomorphism rings and direct sum decompositions in some classes of modules*, Birkhäuser, Basel (1998).
- [9] C. Faith, *Rings and Things and a fine array of twentieth century associative algebra*, Amer. Math. Soc., Providence (1999).
- [10] C. Faith, Locally perfect commutative rings are those whose modules have maximal submodules, *Comm. Algebra* **23**(13), 4885-4886 (1995).
- [11] M. Fontana, J. A. Huckaba and I. J. Papick, *Prüfer Domains*, Marcel Dekker Inc., New York (1997).
- [12] K. R. Goodearl, *Von Neumann Regular Rings*, Pitman, London (1979).
- [13] T. Penk and J. Žemlička, Commutative tall rings, *J. Algebra Appl.* **13**(4), 1350129 (11 pages) (2014).
- [14] B. Stenström, *Rings of Quotients: An Introduction to Methods of Ring Theory*, **217**, Springer-Verlag, New York (1975).

-
- [15] H. H. Storrer, Epimorphismen von kommutativen ringen, *Comment. Math. Helv.* **43**, 387-401 (1968).
[16] H. H. Storrer, On Goldman's primary decomposition, *Lecture Notes in Math.* **246**, 617-661 (1972).

Author information

Farid Kourki, Centre Régional des Métiers de l'Education et de la Formation (CRMEF)-Tanger, Annexe de Larache, B.P. 4063, Larache, Morocco.

E-mail: kourkifarid@hotmail.com

Rachid Tribak, Centre Régional des Métiers de l'Education et de la Formation (CRMEF)-Tanger, Avenue My Abdelaziz, Souani, B.P. 3117, Tangier, Morocco.

E-mail: tribak12@yahoo.com