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On SEMIARTINIAN AND Π-SEMIARTINIAN MODULES

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Abstract All rings will be commutative with identity. A module M is called Π-semiartinian if the direct product M^I is a semiartinian module for every nonempty set I. A ring R is said to be Π-semiartinian if every product of semiartinian R-modules is semiartinian. It is shown that the class of Π-semiartinian R-modules is closed under isomorphic images, submodules, factor modules and extensions. We prove that an R-module M is Π-semiartinian if and only if $R/Ann(M)$ is a semiartinian ring. We also provide a characterization of Π-semiartinian rings.

1 Introduction

Throughout this article, all rings considered are assumed to be commutative rings with an identity and R denotes such a ring. All modules are unital. We denote respectively by $Spec(R)$ and $Max(R)$ the set of all prime ideals of R and the set of all maximal ideals of R. Let M be an R-module and let $x \in M$. By $Ann(x)$ and $Ann(M)$ we denote the *annihilator* of x and M, respectively; i.e. $Ann(x) = \{r \in R \mid rx = 0\}$ and $Ann(M) = \{r \in R \mid rM = 0\}$. The notation $N \subseteq M$ means that N is a subset of M and $N \leq M$ means that N is a submodule of M. If M_1 and M_2 are two R-modules, $Hom_R(M_1, M_2)$ will denote the set of R-homomorphisms from M_1 to M_2. By $\mathbb{Z}$ we denote the ring of integer numbers.

A module M is called *semiartinian* if every nonzero factor module of M has nonzero socle. In Section 2, we investigate some basic properties of semiartinian modules and we provide a new characterization of this kind of modules (Proposition 2.9).

In Section 3, we introduce the notions of Π-semiartinian rings and Π-semiartinian modules. We call a module M Π-*semiartinian* if the direct product M^I is a semiartinian module for every nonempty set I. A ring R is said to be Π-*semiartinian* if every product of semiartinian R-modules is semiartinian. We prove that a module M is Π-semiartinian if and only if $R/Ann(M)$ is a semiartinian ring (Proposition 3.2). It is also shown that the class of Π-semiartinian R-modules is closed under isomorphic images, submodules, factor modules and extensions (Proposition 3.4). We show that the class of Π-semiartinian rings contains the class of semilocal rings R such that $\mathfrak{m}^2 = \mathfrak{m}$ for every $\mathfrak{m} \in Max(R)$ (Proposition 3.5). A characterization of Π-semiartinian rings is provided (Theorem 3.10).

2 Semiartinian Modules

Recall that an R-module M is called *semiartinian* if every nonzero factor module of M has nonzero socle. A ring R is called *semiartinian* if it is semiartinian as an R-module. Note that a ring R is semiartinian if and only if every R-module is semiartinian (see [14, p. 183 Proposition 2.5]). Recall that a subset I of a ring R is called *T-nilpotent* if for every sequence $a_1, a_2, \dots$ in I there exists an integer $n \geq 1$ such that $a_1 \dots a_n = 0$.

We begin with the following lemma which will be useful to our work in this article.

Lemma 2.1. (i) *The class of semiartinian modules is closed under taking isomorphic images, submodules, factor modules, direct sums and module extensions.*

(ii) *A ring R is semiartinian if and only if $Rad(R)$ is T-nilpotent and $R/Rad(R)$ is a semiartinian ring.*

(iii) *If R is a semiartinian ring, then $R/Rad(R)$ is a von Neumann regular ring.*

Proof. (i) See [6, p. 28-29].

(ii) See [14, p. 184 Proposition 2.8].

(iii) See [9, Corollary 3.33E]. □

Example 2.2. From [4, Theorem P], it follows that every perfect ring is semiartinian.

The proof of the following lemma is straightforward and is omitted.

Lemma 2.3. *Let I be a be a proper ideal of a ring R. Then R/I is a semiartinian ring if and only if R/I is a semiartinian R-module.*

The next lemma will be of interest.

Lemma 2.4. *Let R be a commutative ring. Then:*

(i) *If R is a semiartinian ring, then $R/\mathfrak{a}$ is a semiartinian ring for any proper ideal $\mathfrak{a}$ of R.*

(ii) *$R/\mathfrak{a} \cap \mathfrak{b}$ and $R/\mathfrak{a}\mathfrak{b}$ are semiartinian rings whenever $\mathfrak{a}$ and $\mathfrak{b}$ are proper ideals of R such that $R/\mathfrak{a}$ and $R/\mathfrak{b}$ are semiatinian rings.*

Proof. (i) This follows from Lemmas 2.1(i) and 2.3.

(ii) Assume that $R/\mathfrak{a}$ and $R/\mathfrak{b}$ are semiartinian rings. Then $\mathfrak{b}/\mathfrak{a}\mathfrak{b}$ is a semiartinian $R/\mathfrak{a}$-module and hence it is a semiartinian R-module. Moreover, $R/\mathfrak{b}$ is a semiartinian R-module. We have the following exact sequence of R-modules:

$$0 \to \mathfrak{b}/\mathfrak{a}\mathfrak{b} \to R/\mathfrak{a}\mathfrak{b} \to R/\mathfrak{b} \to 0.$$

But the class of semiartinian modules is closed under extensions (Lemma 2.1(i)). So $R/\mathfrak{a}\mathfrak{b}$ is a semiartinian R-module. In addition, since $R/\mathfrak{a} \cap \mathfrak{b}$ is a factor ring of $R/\mathfrak{a}\mathfrak{b}$, $R/\mathfrak{a} \cap \mathfrak{b}$ is also a semiartinian ring by (i). □

Corollary 2.5. *Let $\mathfrak{a}$ be a proper ideal of a ring R. The following conditions are equivalent:*

(i) *$R/\mathfrak{a}$ is a semiartinian ring;*

(ii) *$R/\mathfrak{a}^n$ is a semiatinian ring for every integer $n \geq 1$;*

(iii) *$R/\mathfrak{a}^m$ is a semiatinian ring for some integer $m \geq 1$.*

Proof. (i) $\Rightarrow$ (ii) By induction and using Lemma 2.4(ii).

(ii) $\Rightarrow$ (iii) This is clear.

(iii) $\Rightarrow$ (i) Let $m \geq 1$ such that $R/\mathfrak{a}^m$ is semiartinian. Then $R/\mathfrak{a}$ is a semiartinian ring being a factor ring of $R/\mathfrak{a}^m$ (see Lemma 2.4(i)). □

Corollary 2.6. *The following are equivalent for a ring R:*

(i) *R is a semiartinian ring;*

(ii) *There exists a maximal ideal $\mathfrak{m}$ of R such that $R/Ann(\mathfrak{m})$ is a semiartinian ring;*

(iii) *There exists an ideal $\mathfrak{a}$ of R such that $R/\mathfrak{a}$ and $R/Ann(\mathfrak{a})$ are semiartinian rings.*

Proof. (i) $\Rightarrow$ (ii) This follows from the fact that any factor ring of a semiartinian ring is semiartinian (Lemma 2.4(i)).

(ii) $\Rightarrow$ (iii) It suffices to take $\mathfrak{a} = \mathfrak{m}$.

(iii) $\Rightarrow$ (i) By Lemma 2.4(ii), $R/(\mathfrak{a}Ann(\mathfrak{a}))$ is a semiartinian ring. Therefore R is a semiartinian ring as $\mathfrak{a}Ann(\mathfrak{a}) = 0$. □

Recall that a ring R is said to be *zero dimensional* (or of *Krull dimension zero*), and we write $dim(R) = 0$, if every prime ideal of R is maximal.

Example 2.7. Every commutative semiartinian ring is zero dimensional. To see this, if R is such a ring and if $J = Rad(R)$, then J is T-nilpotent and the ring R/J is von Neumann regular (Lemma 2.1). Hence J is a nil ideal. This yields $J = N$, where N is the nil radical of R. Therefore R/N is a von Neumann regular ring and so $dim(R) = 0$ (see [12, Theorem 1.16]).

For an R-module M and an ideal I of R, I is said to be *T-nilpotent on* M if for every $x \in M$ and every sequence $a_1, a_2, \ldots \in I$ there exists an integer $n \geq 1$ such that $a_1 \ldots a_n x = 0$.

Let M be an R-module. We say that a prime ideal $\mathfrak{p}$ of R is an *associated prime* of M if $\mathfrak{p} = Ann(x)$ for some $x \in M$. The set of associated primes of M is denoted by $Ass(M)$. The support of M is $Supp(M) = \{\mathfrak{p} \in Spec(R) \mid \mathfrak{p} \supseteq Ann(x) \text{ for some } x \in M\}$. We denote by $MaxSupp(M)$ the set of all maximal members in $Supp(M)$. Let $J(M) = \cap_{\mathfrak{m} \in MaxSupp(M)} \mathfrak{m}$. In the case when M is finitely generated, it is easy to see that $Supp(M) = \{\mathfrak{p} \in Spec(R) \mid \mathfrak{p} \supseteq Ann(M)\}$ and hence $Rad(R/Ann(M)) = J(M)/Ann(M)$.

Lemma 2.8. *Let M be a finitely generated R-module. Then:*

(i) $Rad(R/Ann(M)) = J(M)/Ann(M)$.

(ii) *$Rad(R/Ann(M))$ is a T-nilpotent ideal of $R/Ann(M)$ if and only if $J(M)$ is T-nilpotent on M.*

Proof. (i) This is clear.

(ii) (⇒) This follows from (i).

(⇐) Assume that $M = Rx_1 + \cdots + Rx_n$. Note that $J(M)/Ann(M) = Rad(R/Ann(M))$. Let $a_1, a_2, \ldots$ be a sequence in $J(M)$. For every $i \in \{1, \ldots, n\}$, there exists an integer $k_i \geq 1$ such that $a_1 \ldots a_{k_i} x_i = 0$ since $J(M)$ is T-nilpotent on M. Let $k = max(k_1, \ldots, k_n)$. Then $a_1 \ldots a_k x_i = 0$ for every $i \in \{1, \ldots, n\}$. This implies that $a_1 \ldots a_k \in Ann(M)$. Thus $(a_1 \ldots a_k) + Ann(M) = 0 + Ann(M)$. Therefore $Rad(R/Ann(M))$ is a T-nilpotent ideal of $R/Ann(M)$. □

We will say that an R-module M is a *zero dimensional* module, and we write $dimM = 0$, if every prime ideal in $Supp(M)$ is maximal, that is, $Supp(M) = MaxSupp(M)$. It is clear that $dimM = 0$ if and only if $R/Ann(x)$ is a zero dimensional ring for any nonzero element $x \in M$.

Let M be an R-module. An R-module N is called *M-generated* if it is a homomorphic image of a direct sum of copies of M. An R-module N is said to be *subgenerated by M* if N is isomorphic to a submodule of an M-generated module. Let R-Mod denotes the category of all R-modules. We denote by $\sigma[M]$ the full subcategory of R-Mod whose objects are all R-modules subgenerated by M.

Next, we exhibit a characterization of semiartinian modules.

Proposition 2.9. *The following statements are equivalent for a nonzero R-module M:*

(i) *M is a semiartinian R-module;*

(ii) *Every nonzero cyclic submodule of M is semiartinian;*

(iii) *Every nonzero finitely generated submodule N of M is semiartinian;*

(iv) *$R/Ann(x)$ is a semiartinian ring for every $0 \neq x \in M$;*

(v) *$R/Ann(N)$ is a semiartinian ring for every nonzero finitely generated submodule N of M;*

(vi) *$R/J(Rx)$ is semiartinian and $J(Rx)$ is T-nilpotent on Rx for every $0 \neq x \in M$;*

(vii) *$R/J(N)$ is semiartinian and $J(N)$ is T-nilpotent on N for every nonzero finitely generated submodule N of M;*

(viii) *$dim(M) = 0$ and $Ass(N) \neq \emptyset$ for any $0 \neq N \in \sigma[M]$.*

Proof. Since the class of semiartinian modules is closed under submodules, factor modules and direct sums (Lemma 2.1), it follows that for any family $\{N_i\}_{i \in I}$ of submodules of a module M, the sum $\sum_{i \in I} N_i$ is a semiartinian module if and only if each N_i is a semiartinian module (see also [6, p. 29]). From this remark it is easy to deduce the equivalences (i) ⇔ (ii) ⇔ (iii).

(ii) ⇔ (iv) If $0 \neq x \in M$, then $Rx \cong R/Ann(x)$. So, Rx is a semiartinian R-module if and only if $R/Ann(x)$ is a semiartinian ring.

(v) ⇒ (iv) Clear.

(iv) ⇒ (v) Let $N = Rx_1 + \cdots + Rx_n$ be a nonzero finitely generated submodule of M such that $x_i \neq 0$ for every $i \in \{1, \ldots, n\}$. Note that $Ann(N) = \cap_{i=1}^{n} Ann(x_i)$. By hypothesis, $R/Ann(x_i)$ is a semiartinian ring for each $i \in \{1, \ldots, n\}$. By using (ii) of Lemma 2.4 and by induction on n, we see that $R/Ann(N)$ is a semiartinian ring.

(iv) $\Leftrightarrow$ (vi) Let $0 \neq x \in M$. By Lemma 2.1, $A = R/Ann(x)$ is a semiartinian ring if and only if $A/Rad(A)$ is a semiartinian ring and $Rad(A)$ is a T-nilpotent ideal of A. Using Lemma 2.8, we obtain the equivalence.

(v) $\Leftrightarrow$ (vii) Similar to the proof of the equivalence (iv) $\Leftrightarrow$ (vi).

(ii) $\Rightarrow$ (viii) Let $0 \neq x \in M$. Then $R/Ann(x)$ is semiartinian and so it is a zero dimensional ring (Example 2.7). Hence $dim(M) = 0$. Let $0 \neq N \in \sigma[M]$. By (i) of Lemma 2.1, N is also semiartinian. Thus $Soc(N) \neq 0$. Let Ry be a simple submodule of N. Then $Ann(y)$ is a maximal ideal of R and so $Ann(y) \in Ass(N)$.

(viii) $\Rightarrow$ (i) Let $0 \neq N \leq M$. Then $M/N \in \sigma[M]$ and so $Ass(M/N) \neq \emptyset$. Let $\mathfrak{p} \in Ass(M/N)$. Then there exists $0 \neq y \in M/N$ such that $\mathfrak{p} = Ann(y)$. Since $dim(M) = 0$, it is easy to see that $dim(M/N) = 0$. It follows that $\mathfrak{p}$ is a maximal ideal of R. Therefore, Ry is a simple submodule of M/N. This implies that M is a semiartinian module. □

Example 2.10. (i) From Proposition 2.9, it follows immediately that a $\mathbb{Z}$-module M is semiartinian if and only if M is a torsion module.

Corollary 2.11. *Let M be a semiartinian module. Then $J(M)$ is T-nilpotent on M.*

Proof. Let $0 \neq x \in M$. By Proposition 2.9, $J(Rx)$ is T-nilpotent on Rx. But $J(M) \subseteq J(Rx)$. So $J(M)$ is T-nilpotent on Rx. The result follows. □

Remark 2.12. Let M be an R-module. Since $Rad(R) \subseteq J(M)$, it follows that $Rad(R)$ is T-nilpotent on every semiartinian R-module by Corollary 2.11. This shows that Corollary 2.11 is a generalization of [14, Proposition 2.6] in the commuative case.

Recall that a ring R (not necessarily commutative) is said to be *π-regular* if for any $a \in R$, there is an integer $n \geq 1$ and $b \in R$ such that $a^n = a^n b a^n$. The ring R is called *strongly π-regular* if for each $a \in R$, there is an integer $n \geq 1$ and $b \in R$ such that $a^n = a^{n+1}b$. It is shown in [15, Lemma 5.6] that a commutative ring R is π-regular if and only if $dim(R) = 0$.

Corollary 2.13. *Let M be a finitely generated R-module. Then the following statements are equivalent:*

(i) *M is a semiartinian module;*

(ii) *$R/Ann(M)$ is a semiartinian ring;*

(iii) *$R/J(M)$ is semiartinian and $J(M)$ is T-nilpotent on M;*

Moreover, if M is a semiartinian module, then $End_R(M)$ is a strongly π-regular ring.

Proof. (i) $\Leftrightarrow$ (ii) and (i) $\Rightarrow$ (iii) These follow from Proposition 2.9.

(iii) $\Rightarrow$ (ii) Since M is finitely generated, it follows that $J(M)$ is T-nilpotent on M if and only if $Rad(R/Ann(M)) = J(M)/Ann(M)$ is a T-nilpotent ideal of $A = R/Ann(M)$ (Lemma 2.8). In addition, it is clear that $R/J(M)$ is semiartinian if and only if $A/Rad(A)$ is semiartinian. Now apply (ii) of Lemma 2.1.

Now assume that M is a semiartinian R-module. Hence $R/Ann(M)$ is a semiartinian ring. Therefore $dim(R/Ann(M)) = 0$ (Example 2.7). By [2, Theorem 1], $End_R(M)$ is a strongly π-regular ring. □

Proposition 2.14. *Let M be an R-module. Then M is a semiartinian R-module if and only if $M \otimes N$ is a semiartinian R-module for every R-module N.*

Proof. ($\Rightarrow$) Let N be an R-module and let $0 \neq x = \sum_{i=1}^{n} x_i \otimes y_i \in M \otimes N$ where $x_i \in M$ and $y_i \in N$ for every $i \in \{1, \ldots, n\}$. Let $M' = Rx_1 + \cdots + Rx_n$. Therefore $Ann(M') \subseteq Ann(Rx)$ and M' is semiartinian as M' is a submodule of the semiartinian module M. Since $R/Ann(Rx) \cong \frac{R/Ann(M')}{Ann(Rx)/Ann(M')}$ (as rings) and $R/Ann(M')$ is a semiartinian ring (Corollary 2.13), $R/Ann(Rx)$ is also a semiartinian ring by Lemma 2.4(i). Applying Proposition 2.9, we conclude that $M \otimes N$ is a semiartianian R-module.

($\Leftarrow$) This follows by taking $N = R$ and using the fact that $M \otimes R \cong M$. □

3 Π-Semiartinian Modules

In this section, we wish to investigate the class of rings R for which every direct product of semiartinain R-modules is semiartinian. First we introduce the notion of Π-semiartinian modules.

Definition 3.1. An R-module M is called a Π-*semiartinian* module if the direct product M^I is a semiartinian module for every nonempty set I.

Proposition 3.2. *Let M be a nonzero R-module. The following conditions are equivalent:*
(i) *M is a Π-semiartinian R-module;*
(ii) *$R/Ann(M)$ is a semiartinian ring;*
(iii) *$Hom_R(M, M)$ is a semiartinian R-module.*

Proof. (i) ⇒ (ii) Let $\{x_i\}_{i\in I}$ be a familly of generators of M. Consider the map $f : R \to M^I$ defined by $f(a) = (ax_i)_{i\in I}$ for every $a \in R$. Then f is R-linear and $Kerf = Ann(M)$. It follows that $R/Ann(M)$ is isomorphic to a submodule of the semiartinian R-module M^I. Hence $R/Ann(M)$ is semiartinian as R-module. Therefore $R/Ann(M)$ is a semiartinian ring.

(ii) ⇒ (iii) Let $0 \neq f \in Hom_R(M, M)$. Note that $Ann(M) \subseteq Ann(Rf)$. Therefore $R/Ann(Rf) \cong \frac{R/Ann(M)}{Ann(Rf)/Ann(M)}$. But $R/Ann(M)$ is a semiartinian ring. So $R/Ann(Rf)$ is a semiartinian ring (Lemma 2.4(i)). By Proposition 2.9, $Hom_R(M, M)$ is a semiartinian R-module.

(iii) ⇒ (ii) Consider the R-homorphism $\Psi : R \to Hom_R(M, M)$ such that for every $r \in R$, $\Psi(r)$ is the endomorphism of the R-module M defined by $\Psi(r)(x) = rx$ for every $x \in M$. We have $Ker\Psi = Ann(M)$. Hence $R/Ann(M)$ is isomorphic to a submodule of $Hom_R(M, M)$. By Lemma 2.1, $R/Ann(M)$ is a semiartinian R-module and so $R/Ann(M)$ is a semiartinian ring.

(ii) ⇒ (i) Let I be a nonempty set. Since $Ann(M^I) = Ann(M)$ and $R/Ann(M)$ is a semiartinian ring, the direct product M^I is semiartinian as an $R/Ann(M)$-module and hence also as an R-module. This completes the proof. □

Combining Corollary 2.13 and Proposition 3.2, we get the following corollary.

Corollary 3.3. *The following conditions are equivalent for a finitely generated R-module M:*
(i) *M is a Π-semiartinian R-module;*
(ii) *M is a semiartinian R-module;*
(iii) *$Hom_R(M, M)$ is a semiartinian R-module.*

Proposition 3.4. *Let R be a ring. Then the class of Π-semiartinian R-modules is closed under isomorphic images, submodules, factor modules and extensions.*

Proof. It is easy to see that the class of Π-semiartinian R-modules is closed under isomorphic images. Let M be an R-module and let N be a submodule of M.

Assume that M is a Π-semiartinian R-module. It is clear that $Ann(M) \subseteq Ann(N)$ and $Ann(M) \subseteq Ann(M/N)$. Moreover, we have the following two ring isomorphisms:

$$R/Ann(N) \cong \frac{R/Ann(M)}{Ann(N)/Ann(M)} \text{ and } R/Ann(M/N) \cong \frac{R/Ann(M)}{Ann(M/N)/Ann(M)}.$$

But the ring $R/Ann(M)$ is semiartinian by Proposition 3.2. So $R/Ann(N)$ and $R/Ann(M/N)$ are semiartinian rings by Lemma 2.4(i). From Proposition 3.2, it follows that N and M/N are Π-semiartinian R-modules.

Now suppose that N and M/N are Π-semiartinian R-modules and let us show that M is a Π-semiartinian module. By Proposition 3.2, $R/Ann(N)$ and $R/Ann(M/N)$ are semiartinian rings. Applying Lemma 2.4(ii), we deduce that the ring $R/(Ann(N) \cap Ann(M/N))$ is semiartinian. Therefore $R/(Ann(N) \cap Ann(M/N))^2$ is also a semiartinian ring by Corollary 2.5. But $(Ann(N) \cap Ann(M/N)^2 \subseteq Ann(M)$. Hence $R/Ann(M)$ is also a semiartinian ring. By Proposition 3.2, M is a Π-semiartinian R-module. □

We call a ring R Π-*semiartinian* if every product of semiartinian R-modules is semiartinian.

Proposition 3.5. *Let R be a semilocal ring such that $\mathfrak{m}^2 = \mathfrak{m}$ for every maximal ideal $\mathfrak{m}$ of R. Then R is a Π-semiartinian ring.*

Proof. Let $\{M_\lambda\}_{\lambda\in\Lambda}$ be a family of semiartinian R-modules and let $M = \prod_{\lambda\in\Lambda} M_\lambda$. Using [7, Corollary 2.7] and [5, Theorem 5], we conclude that each M_λ $(\lambda \in \Lambda)$ is semisimple. But over a semilocal ring any product of semisimple modules is semisimple (see [1, Proposition 15.17]). Therefore M is a semiartinian module. This proves the proposition. □

Example 3.6. Let F be a field and let R be the ring of polynomials in countably many commuting variables $x_1, x_2, \ldots$, over F subject to the relations $x_1^2 = 0$ and $x_n^2 = x_{n-1}$ for $n \geq 2$. This ring appears in [16] in another context. The ring R is local and its maximal ideal $\mathfrak{m}$ is generated by all the x_i, $(i \geq 1)$. Moreover, we have $\mathfrak{m}^2 = \mathfrak{m}$. From Proposition 3.5, it follows that R is a Π-semiartinian ring.

A ring R is called a *valuation ring* (or a *chain ring*) if any two ideals of R are comparable. It is clear that if R is a valuation ring, then R is local and any finitely generated ideal of R is cyclic.

Lemma 3.7. *Let R be a valuation ring and let $\mathfrak{m}$ be its maximal ideal. If $\mathfrak{m}$ is not finitely generated, then $\mathfrak{m}^2 = \mathfrak{m}$. The converse holds when R is not a field.*

Proof. Suppose that $\mathfrak{m}$ is not finitely generated and let $A = R/\cap_{n\geq 1} \mathfrak{m}^n$. Then A is a valuation ring with $Rad(A) = \mathfrak{m}/\cap_{n\geq 1} \mathfrak{m}^n$ and $\cap_{n\geq 1}(\mathfrak{m}/\cap_{n\geq 1} \mathfrak{m}^n)^n = 0$. By [8, Proposition 5.3], A is a noetherian ring and so $\mathfrak{m}/\cap_{n\geq 1} \mathfrak{m}^n$ is a cyclic ideal generated by an element $a + \cap_{n\geq 1}\mathfrak{m}^n$, where $a \in \mathfrak{m}$. Therefore $\mathfrak{m} = Ra + \cap_{n\geq 1}\mathfrak{m}^n$. If $\cap_{n\geq 1}\mathfrak{m}^n \subseteq Ra$, then $\mathfrak{m} = Ra$ which is impossible since $\mathfrak{m}$ is not finitely generated. So $Ra \subseteq \cap_{n\geq 1}\mathfrak{m}^n$ and hence $\mathfrak{m} = \cap_{n\geq 1}\mathfrak{m}^n$. Consequently, $\mathfrak{m}^2 = \mathfrak{m}$.

Now suppose that R is not a field and $\mathfrak{m}^2 = \mathfrak{m}$. Then $\mathfrak{m}$ is not finitely generated, since otherwise $\mathfrak{m}$ will be a direct summand of R and so R will be a field (see, for example, [1, p. 103 Exercise 12(3)]). □

Corollary 3.8. *Let R be a valuation ring and let $\mathfrak{m}$ be its maximal ideal. If $\mathfrak{m}$ is not finitely generated, then R is a Π-semiartinian ring.*

Proof. By Proposition 3.5 and Lemma 3.7. □

Remark 3.9. Since the class of semiartinian R-modules is closed under submodules, factor modules and direct sums (Lemma 2.1), every module M has a semiartinian submodule $Sa(M)$ which is maximal for this property (see also [6, p. 29]) and we have

$$Sa(M) = \{x \in M \mid Rx \text{ is a semiartinian } R\text{-module}\}.$$

Moreover, we have $Sa(M) = \{x \in M \mid x = 0 \text{ or } R/Ann(x) \text{ is a semiartinian ring}\}$.

Let R be a ring. If I is an ideal of R such that R/I is a semiartinian ring then we call I a *cosemiartinian ideal*. Let $\mathfrak{S}(R)$ denote the intersection of all cosemiartinian ideals of R. Clearly, $\mathfrak{S}(R) \subseteq Rad(R)$. Also, for every maximal ideal $\mathfrak{m}$ of R, we have $\mathfrak{S}(R) \subseteq \cap_{k=1}^{\infty}\mathfrak{m}^k$ (Corollary 2.5).

Let M be an R-module and let $\mathfrak{a}$ be an ideal of R. We will denote by $Ann_M(\mathfrak{a})$ the set $\{m \in M \mid rm = 0 \text{ for every } r \in \mathfrak{a}\}$.

Next, we characterize rings R for which the class of semiartinian R-modules is closed under direct products.

Theorem 3.10. *The following statements are equivalent for a ring R:*

(i) *The class of semiartinian R-modules is closed under direct products (that is, R is a Π-semiartinian ring);*

(ii) *The class of Π-semiartinian R-modules is closed under direct sums;*

(iii) *The class of Π-semiartinian R-modules is closed under direct products;*

(iv) *Every semiartinian R-module is Π-semiartinian;*

(v) *For every nonzero R-module M, M is semiartinian if and only if $R/Ann(M)$ is a semiartinian ring;*

(vi) *For every family $\{I_\lambda\}_{\lambda\in\Lambda}$ of ideals of R such that each R/I_λ ($\lambda \in \Lambda$) is a semiartinian ring, $R/\cap_{\lambda\in\Lambda} I_\lambda$ is a semiartinian ring;*

(vii) *$R/\mathfrak{S}(R)$ is a semiartinian ring;*

(viii) *For every R-module M, $Sa(M) = Ann_M(\mathfrak{S}(R))$.*

Proof. (i) ⇒ (iv) This is evident.

(iv) ⇒ (ii) This follows from the fact that any direct sum of semiartinian modules is semiartinian (Lemma 2.1).

(ii) ⇒ (vi) Let $\{I_\lambda\}_{\lambda\in\Lambda}$ be a family of ideals of R such that R/I_λ is a semiartinian ring for all $\lambda \in \Lambda$. Then R/I_λ is a Π-semiartinian R-module by Corollary 3.3. By hypothesis, $M = \oplus_{\lambda\in\Lambda} R/I_\lambda$ is a Π-semiartinian R-module. Hence $R/Ann(M) = R/\cap_{\lambda\in\Lambda} I_\lambda$ is a semiartinian ring by Proposition 3.2.

(vi) ⇒ (vii) This is immediate.

(vii) ⇒ (viii) Let M be a nonzero R-module and let $0 \neq x \in M$. Thus,

$x \in Sa(M)$ ⇔ $R/Ann(x)$ is a semiartinian ring.
⇔ $\mathfrak{S}(R) \subseteq Ann(x)$ since $R/\mathfrak{S}(R)$ is a semiartinian ring.
⇔ $x \in Ann_M(\mathfrak{S}(R))$.

It follows that $Sa(M) = Ann_M(\mathfrak{S}(R))$.

(viii) ⇒ (i) Let $\{M_\lambda\}_{\lambda\in\Lambda}$ be a family of semiartinian R-modules and let $M = \prod_{\lambda\in\Lambda} M_\lambda$. By hypothesis, we have $\mathfrak{S}(R)M_\lambda = 0$ for all $\lambda \in \Lambda$. This clearly forces $\mathfrak{S}(R)M = 0$. It follows that $Ann_M(\mathfrak{S}(R)) = M$. But $Sa(M) = Ann_M(\mathfrak{S}(R))$. Then $M = Sa(M)$ is a semiartinian module. Consequently, R is a Π-semiartinian ring.

(ii) ⇔ (iii) Let $\{M_\lambda\}_{\lambda\in\Lambda}$ be a familly of Π-semiartinian R-modules. Then

$$Ann(\oplus_{\lambda\in\Lambda} M_\lambda) = Ann(\prod_{\lambda\in\Lambda} M_\lambda) = \cap_{\lambda\in\Lambda} Ann(M_\lambda).$$

Now use Proposition 3.2.

(iv) ⇔ (v) This follows from Proposition 3.2. □

Recall that an R-module M is called a *max module* if every nonzero submodule of M contains a maximal submodule. A ring R is called a *max ring* if every nonzero R-module is a max module. Combining [9, Corollary 3.33E] and [10, Theorem A], we see that any semiartinian ring is a max ring. A module M is called *tall* if it contains some submodule N such that both M/N and N are non-noetherian. A ring R is called *tall* if every non-noetherian R-module is tall. By [13, Corollary 1.2], every max ring is tall. Therefore every semiartinian ring is tall.

Corollary 3.11. *Let R be a Π-semiartinian ring. Then the following hold:*

(i) *$R/Rad(R)$ is a semiartinian von Neumann regular ring.*

(ii) *$R/\cap_{n\geq 1} \mathfrak{m}^n$ is a semiartinian ring for every maximal ideal $\mathfrak{m}$ of R.*

(iii) *R is a tall ring.*

Proof. (i) By Theorem 3.10, $R/\mathfrak{S}(R)$ is a semiartinian ring. But $R/Rad(R)$ is a factor ring of $R/\mathfrak{S}(R)$ since $\mathfrak{S}(R) \subseteq Rad(R)$. Thus $R/Rad(R)$ is a semiartinian ring and hence it is also a von Neumann regular ring (see Lemma 2.1(iii)).

(ii) Let $\mathfrak{m} \in Max(R)$ and let $n \geq 1$. Then $R/\mathfrak{m}^n$ is a semiartinian ring by Corollary 2.5. From Theorem 3.10, it follows that $R/\cap_{n\geq 1} \mathfrak{m}^n$ is a semiartinian ring.

(iii) Let $\mathfrak{m} \in Max(R)$. By (ii), $R/\cap_{n\geq 1} \mathfrak{m}^n$ is a semiartinian ring and hence it is a max ring. By [13, Corollary 1.2], $R/\cap_{n\geq 1} \mathfrak{m}^n$ is a tall ring. Therefore R is a tall ring by [13, Corollary 2.7]. □

Proposition 3.12. *Let R be a ring.*

(i) *Assume that R is noetherian. Then R is Π-semiartinian if and only if R is artinian.*

(ii) *Assume that R is a Prüfer domain. If R is Π-semiartinian, then $\mathfrak{m}^2 = \mathfrak{m}$ for every maximal ideal $\mathfrak{m}$ of R.*

(iii) *Assume that R is a valuation ring with maximal ideal $\mathfrak{m}$. Then R is Π-semiartinian if and only if R is artinian or $\mathfrak{m}$ is not finitely generated.*

Proof. (i) ($\Rightarrow$) By Corollary 3.11, R is a tall ring. Now use [13, Proposition 2.10].

($\Leftarrow$) This follows from the fact that every artinian ring is semiartinian.

(ii) Let $\mathfrak{m} \in Max(R)$. By Corollary 3.11, $A = R/\cap_{n\geq 1} \mathfrak{m}^n$ is a semiartinian ring and so it is zero dimensional (Example 2.7). By [3, Theorem 2.7], we have $\sqrt{\cap_{n\geq 1}\mathfrak{m}^n} = \cap_{n\geq 1}\mathfrak{m}^n$. This implies that A is a reduced ring. Therefore A is a von Neumann regular ring by [12, Theorem 1.16]. Consequently, $(\mathfrak{m}/\cap_{n\geq 1} \mathfrak{m}^n)^2 = \mathfrak{m}/\cap_{n\geq 1} \mathfrak{m}^n$. Thus, $\mathfrak{m}^2 = \mathfrak{m}$.

(iii) ($\Rightarrow$) By Corollary 3.11, R is a tall ring. Now apply [13, Corollary 2.11].

($\Leftarrow$) If R is artinian, then R is semiartinian and so it is Π-semiartinian. Now if $\mathfrak{m}$ is not finitely generated, then R is Π-semiartinian by Corollary 3.8. □

Corollary 3.13. *Let R be a valuation domain which is not a field with maximal ideal $\mathfrak{m}$. Then the following conditions are equivalent:*

(i) *R is a Π-semiartinian ring;*

(ii) *$\mathfrak{m}$ is not finitely generated;*

(iii) $\mathfrak{m}^2 = \mathfrak{m}$.

Proof. This follows easily from Lemma 3.7 and Proposition 3.12(iii). □

Next, we present an example showing that the class of semiartinian rings is larger than that of Π-semiartinian rings.

Remark 3.14. (i) Let $\mathbb{Z}$ be the ring of integers. Since $Rad(\mathbb{Z}) = 0$ and $\mathbb{Z}$ is not a von Neumann regular ring, $\mathbb{Z}$ is not a Π-semiartinian ring by Corollary 3.11.

(ii) It is shown in [11, Example 8.4.8] that there exists a valuation domain R which not a field such that the maximal ideal of R is idempotent. Clearly, the ring R is not semiartinian as $Soc(R) = 0$. On the other hand, R is a Π-semiartinian ring by Corollary 3.13.

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