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On $k-$weakly primary ideals of $\Gamma-$semirings

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Abstract. The concepts of $k-$prime ideals , $k-$weakly primary ideals, $k-$ weakly prime ideals in $\Gamma-$semiring are introduced and their properties studied. We prove that the intersection of a family of $k-$weakly prime (primary) ideals of $\Gamma-$semiring that are not prime (primary) is a $k-$weakly prime (primary) ideal.

1 Introduction

Semiring, the best algebraic structure which is a common generalization of rings and distributive lattices, was first introduced by American mathematician Vandiver [16] in 1934 but non trivial examples of semirings have appeared in the studies on the theory of commutative ideals of rings by German Mathematician Richard Dedekind in 19th century. Semiring is an universal algebra with two binary operations called addition and multiplication, where one of them distributive over the other. Bounded distributive lattices are commutative semirings which are both additively idempotent and multiplicatively idempotent. Most of the semirings have an order structure in addition to their algebraic structure. A natural example of semiring, which is not a ring, is the set of all natural numbers under usual addition and multiplication of numbers. In particular, if I is the unit interval on the real line then $(I, \max, \min)$ in which 0 is the additive identity and 1 is the multiplicative identity.The theory of rings and the theory of semigroups have considerable impact on the development of the theory of semirings. In structure, semirings lie between semigroups and rings. In semiring multiplicative structure of semiring is not independent of additive structure of semiring. Additive and multiplicative structures of a semiring play an important role in determining the structure of a semiring. Semiring, as the basic algebraic structure, was used in the areas of theoretical computer science as well as in the solutions of graph theory, optimization theory and in particular for studying automata, coding theory and formal languages. Semiring theory has many applications in other branches. Semirings play an important role in studying matrices and determinants.

As a generalization of ring, the notion of a $\Gamma-$ring was introduced by Nobusawa [14] in 1964. In 1981, Sen [15] introduced the notion of a $\Gamma-$semigroup as a generalization of semigroup. The notion of a ternary algebraic system was introduced by Lehmer [8] in 1932. Lister [9] introduced the notion of a ternary ring. Dutta and Kar [6] introduced the notion of regular ternary semirings. In 1995, Murali Krishna Rao [10, 12, 13], introduced the notion of a $\Gamma-$semiring as a generalization of $\Gamma-$ring, ring, ternary semiring and semiring. Murali Krishna Rao and Venkateswarlu [11]introduced the notion of regular $\Gamma-$incline and field $\Gamma-$semiring. The set of all negative integers Z is not a semiring with respect to usual addition and multiplication but Z forms a $\Gamma-$semiring.where $\Gamma = Z$. The important reason for the development of $\Gamma-$semiring is a generalization of results of rings, $\Gamma-$rings, semirings, semigroups, $\Gamma-$semigroups and ternary semirings.

It is well-known that ideals play an important role in the study of any algebraic structures, in particular semirings. Lajos , Iseki characterized the ideals of semigroups and the ideals of semirings respectively. Though semiring is a generalization of a ring, ideals of semiring do not coincide with ring ideals. For example an ideal of a semiring needs not be the kernel of some semiring homomorphism. To solve this problem, Henriksen [7] defined $k-$ideals in semirings to obtain analogues of ring results. Anderson and Smith [2] introduced and studied the concept of a weakly prime ideal of an associative ring with unity. Dubey [5] studied prime and weakly prime ideals in semirings. Atani [3] studied $k-$weakly primary ideals over semirings and Atani et al.[4] studied weakly primary ideals of commutative rings. In this paper, we generalize the results on prime, weakly prime, weakly primary ideal of ring theory and semiring theory studied by several mathematicians to $\Gamma-$semiring theory. We introduce the concept of $k-$weakly prime ideals, $k-$weakly primary ideals in $\Gamma-$semirings, study their properties and relations between them.

2 Preliminaries

In this section, we recall some basic notions of semirings and $\Gamma-$semirings.

Definition 2.1. [1] A set S together with two associative binary operations called addition and multiplication (denoted by $+$ and $\cdot$ respectively) will be called a semiring provided

(i) addition is a commutative operation.

(ii) multiplication distributes over addition both from the left and from the right.

(iii) there exists $0 \in S$ such that $x + 0 = x$ and $x \cdot 0 = 0 \cdot x = 0$ for all $x \in S$.

Definition 2.2. [10] Let $(M, +)$ and $(\Gamma, +)$ be commutative semigroups. If there exists a mapping $M \times \Gamma \times M \to M$ (images to be denoted by $x\alpha y, x, y \in M, \alpha \in \Gamma$) satisfying the following axioms for all $x, y, z \in M$ and $\alpha, \beta \in \Gamma$,

(i) $x\alpha(y + z) = x\alpha y + x\alpha z,$

(ii) $(x + y)\alpha z = x\alpha z + y\alpha z,$

(iii) $x(\alpha + \beta)y = x\alpha y + x\beta y$

(iv) $x\alpha(y\beta z) = (x\alpha y)\beta z,$

then M is called a $\Gamma-$semiring.

Definition 2.3. [10] A $\Gamma-$semiring M is said to have zero element if there exists an element $0 \in M$ such that $0 + x = x = x + 0$ and $0\alpha x = x\alpha 0 = 0,$ for all $x \in M \; and \; \alpha \; \in \; \Gamma.$

Example 2.4. Every semiring M is a $\Gamma-$semiring with $\Gamma = M$ and ternary operation is defined as the usual semiring multiplication

Example 2.5. Let M be the additive semigroup of all $m \times n$ matrices over the set of non negative rational numbers and Γ be the additive semigroup of all $n \times m$ matrices over the set of non negative integers.Define the ternary operation $M \times \Gamma \times M \to M$ by $(a, \alpha, b) \to a\alpha b$, using the usual matrix multiplication M is a $\Gamma-$semiring.

Definition 2.6. [10] A function $f : R \to M$ where R and M are $\Gamma-$semirings is said to be $\Gamma-$semiring homomorphism if $f(a + b) = f(a) + f(b)$ and $f(a\alpha b) = f(a)\alpha f(b)$ for all $a, b \in R \; and \; \alpha \in \Gamma.$

Definition 2.7. [11] Let M be a semiring. An element $1 \in M$ is said to be unity if for each $x \in M$ there exists $\alpha \in \Gamma$ such that $x\alpha 1 = 1\alpha x = x.$

Example 2.8. Let M be the set of all rational numbers and Γ=M is a semigroup with the usual addition. Define the ternary operation $M \times \Gamma \times M \to M$ by $(a, \alpha, b) \to a\alpha b$, using the usual multiplication. Now M is a $\Gamma-$semiring with unity .

Definition 2.9. [10] Let M be a $\Gamma-$semiring and A be a non-empty subset of M. A is called a $\Gamma-$subsemiring of M if A is a sub-semigroup of $(M, +)$ and $A\Gamma A \subseteq A.$

Definition 2.10. [10] Let M be a $\Gamma-$semiring. A subset A of M is called a left(right) ideal of M if A is closed under addition and $M\Gamma A \subseteq A(A\Gamma M \subseteq A)$. A is called an ideal of M if it is both a left ideal and a right ideal.

Definition 2.11. [10] An ideal I of semiring M is called a $k-$ideal if $b \in M, a + b \in I$ and $a \in I$ then $b \in I.$

Definition 2.12. [5] An ideal P of semiring M is called a prime ideal of M if for any $a, b \in M$ and $ab \in P$ then $a \in P$ or $b \in P$.

Definition 2.13. [5] An ideal P of semiring M is said to be $k-$prime ideal of M if P is a $k-$ideal, for any $x, y \in M$ and $xy \in P$ then $x \in P$ or $y \in P$.

Definition 2.14. [5] An ideal P of semiring M is said to be weakly prime ideal of M if $0 \neq x\alpha y \in P, x, y \in M$ then $x \in P$ or $y \in P$.

Every prime ideal of semiring M is a weakly prime ideal.

Definition 2.15. [5] A $k-$ideal P of semiring M is called a $k-$weakly prime ideal if P is a weakly prime ideal of semiring.

Definition 2.16. [4] An ideal P of $\Gamma-$semiring M is said to be primary ideal of M if $xy \in P, x, y \in M$ then $x \in P$ or $y^n \in P, n$ is a positive integer.

Definition 2.17. [4] An ideal P of semiring M is said to be weakly primary ideal of M if $0 \neq xy \in P, x, y \in M$ then $x \in P$ or $y^n \in P, n$ is a positive integer. .

Definition 2.18. [4] A $k-$ideal P of semiring M is called a $k-$weakly primary ideal if P is a weakly primary ideal.

3 $k-$weakly prime ideals

In this section, we introduce the notion of $k-$prime ideal, $k-$weakly prime ideal of $\Gamma-$semirings. Throughout this paper M is a $\Gamma-$semiring with unity element and zero element.

Definition 3.1. An ideal P of $\Gamma-$semiring M is said to be $k-$prime ideal if P is a $k-$ideal, $x\alpha y \in P, \alpha \in \Gamma$ *and* $x, y \in M$ then $x \in P$ or $y \in P$.

Definition 3.2. An ideal P of $\Gamma-$semiring M is said to be weakly prime ideal of M if $0 \neq x\alpha y \in P, \alpha \in \Gamma, x, y \in M$ then $x \in P$ or $y \in P$.

Every $k-$prime ideal of $\Gamma-$semiring M is a $k-$weakly prime ideal.

Definition 3.3. A $k-$ideal I of $\Gamma-$semiring M is called a $k-$weakly prime ideal if I is a weakly prime ideal.

Definition 3.4. Let I be an ideal of $\Gamma-$semiring M. Then radical of I is defined as the set of all elements $x \in M$ such that $(x\alpha)^n x \in I$ for some $n \in Z^+$, for all $\alpha \in \Gamma$ and it is denoted by $rad(I)$.

Definition 3.5. An element x of $\Gamma-$semiring M is said to be nilpotent if there exists a positive integer n such that $(x\alpha)^{n-1}x = 0$, for some $\alpha \in \Gamma$.

We state the following lemmas, proofs of which are easy and straightforward and so we omit the proofs.

Lemma 3.6. *Let I, J be $k-$ideals of $\Gamma-$semiring M. Then $(I : J) = \{r \in M \mid r\alpha j \in I,$ for all $\alpha \in \Gamma, j \in J\}$ is a $k-$ideal.*

Lemma 3.7. *Let I be a $k-$ideal of $\Gamma-$semiring M and $\{0\} \neq A \subseteq M$. Then $I \subseteq (I : A) \subseteq (I : A\Gamma A)$.*

Lemma 3.8. *Let I be a $k-$ideal of $\Gamma-$semiring M and $x \in M$. Then $(I : x) = \{r \in M \mid r\alpha x \in I,$ for all $\alpha \in \Gamma\}$ is a $k-$ideal.*

Lemma 3.9. *Let M be a $\Gamma-$semiring and $x \in M$. Then $(0 : x) = \{r \in M \mid r\alpha x = 0,$ for all $\alpha \in \Gamma\}$ is a $k-$ideal.*

Lemma 3.10. *Let A be a non empty subset of $\Gamma-$semiring M. Then*
$(i).(I : A) = \bigcap\limits_{a \in A} (I : a)$ $\quad (ii).$ *If $A \subseteq I$ then $(I : A) = M$.*

Lemma 3.11. *If an ideal of $\Gamma-$semiring M is the union of two $k-$ideals then it is equal to one of them.*

Theorem 3.12. *Let I be a $k-$ideal of $\Gamma-$semiring M with unity. Then the following statements are equivalent.*

(i) I is a $k-$weakly prime ideal

(ii) If A, B are right ideal and left ideal of $\Gamma-$semiring M respectively such that $\{0\} \neq A\Gamma B \subseteq I$ then $A \subseteq I$ or $B \subseteq I$.

(iii) If $a, b \in M$ such that $\{0\} \neq a\Gamma M\Gamma b \subseteq I$ then $a \in I$ or $b \in I$.

Proof. Let I be a $k-$ideal of $\Gamma-$semiring M with unity.

(i) $\Rightarrow$ (ii) : Suppose I is a $k-$weakly prime ideal of $\Gamma-$semiring M, A and B are right ideal and left ideal of $\Gamma-$semiring M respectively such that $\{0\} \neq A\Gamma B \subseteq I$.
Let $(A), (B)$ be ideals generated by A, B respectively.
Then $\{0\} \neq (A)\Gamma(B) \subseteq I \Rightarrow A \subseteq (A) \subseteq I$ or $B \subseteq (B) \subseteq I$.

(ii) $\Rightarrow$ (iii) : Let $\{0\} \neq a\Gamma M\Gamma b \subseteq I, a, b \in M$.
Then $\{0\} \neq a\Gamma M\Gamma M\Gamma b \subseteq I$
$\Rightarrow a\Gamma M \subseteq I$ or $M\Gamma b \subseteq I$, (by (ii))
$\Rightarrow a \in a\Gamma M \subseteq I$ or $b \in M\Gamma b \subseteq I$, since M has an unity .

(iii) $\Rightarrow$ (i) : Suppose $A\Gamma B \subseteq I$, for ideals A and B of $\Gamma-$semiring $M, A \nsubseteq I$ and $B \nsubseteq I$. Let $a \in A \setminus I, b \in B \setminus I$ and $a' \in A \cap I,\ b' \in B \cap I$.
Then $a + a' \notin I, b + b' \notin I$.
Therefore we have $(a + a')\Gamma M\Gamma(b + b')$ =$\{0\}$ which is a contradiction.
Hence I is a $k-$weakly prime ideal.

□

Theorem 3.13. *Let A be a $k-$ideal of $\Gamma-$semiring M. If any ideals I, J of $\Gamma-$semiring M with $\{0\} \neq I\Gamma J \subseteq A$ and $I \subseteq A$ or $J \subseteq A$ then A is a $k-$weakly prime ideal of $\Gamma-$semiring M.*

Proof. Let A be a $k-$ideal of $\Gamma-$semiring M and ideals I, J of $\Gamma-$semiring M with $\{0\} \neq I\Gamma J \subseteq A$ and $I \subseteq A$ or $J \subseteq A$. Suppose $0 \neq x\alpha y \in A$.

$$\Rightarrow (x\alpha y)\Gamma M \subseteq A \ \text{ and } M\Gamma(x\alpha y) \subseteq A$$
$$\Rightarrow (x\alpha y\Gamma M)\Gamma(M\Gamma x\alpha y) \subseteq A$$
$$\Rightarrow x\Gamma M\Gamma M\Gamma y \subseteq A.$$

Since $x\Gamma M$ and $M\Gamma y$ are right ideal and left ideal respectively,by Theorem 3.12, $x\Gamma M \subseteq A$ or $M\Gamma y \subseteq A$. Since M is a $\Gamma-$semiring with unity, there exist $\alpha, \beta \in \Gamma$ such that $x\alpha 1 = x$ and $1\beta y = y$. Therefore $x \in A$ or $y \in A$. Hence A is a $k-$weakly prime ideal of $\Gamma-$semiring M. □

Theorem 3.14. *Let I be a $k-$weakly prime ideal but not a $k-$prime ideal of $\Gamma-$semiring M. If $a\alpha b = 0$, for some $a, b \in M \setminus I, \alpha \in \Gamma$ then $a\alpha I = I\alpha a = \{0\}$.*

Proof. Let I be a $k-$weakly prime ideal but not a $k-$prime ideal of $\Gamma-$semiring M and $a\alpha b = 0$, for some $a, b \in M \setminus I, \alpha \in \Gamma$.
Suppose $a\alpha i_1 \neq 0$, for some $i_1 \in I_1, \alpha \in \Gamma$. Then $0 \neq a\alpha(b + i_1) \in I$.
Since I is a $k-$weakly prime ideal, we have $a \in I$ or $b + i_1 \in I$
$\Rightarrow a \in I$ or $b \in I$, which is a contradiction.
Hence $a\alpha I = \{0\}$. Similarly we can prove $I\alpha a = \{0\}$. □

Theorem 3.15. *Let I be a $k-$ideal of $\Gamma-$semiring M.If I is a $k-$weakly prime ideal but not a prime ideal then $I\Gamma I = \{0\}$.*

Proof. Let I be a $k-$ideal of $\Gamma-$semiring M.Suppose I is a weakly prime ideal but not a prime ideal and $I\Gamma I \neq \{0\}$. Then there exist $i_1, i_2 \in I, \alpha \in \Gamma$ such that $i_1\alpha i_2 \neq 0$ and $a\alpha b = 0$ for some $a, b \notin I$. By Theorem 3.14, we have $0 \neq (a + i_1)\alpha(b + i_2) = i_1\alpha i_2 \in I$
$\Rightarrow a + i_1 \in I$ or $b + i_2 \in I$
$\Rightarrow a \in I$ or $b \in I$, which is a contradiction.
Hence $I\Gamma I = \{0\}$. □

The following example shows that an ideal I in a $\Gamma-$semiring Msatisfying $I\Gamma I = \{0\}$. need not be a weakly prime ideal

Example 3.16. Let $M = \left\{ \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} \middle| a \in Z_{12}^{+} \right\}$ and $\Gamma = M$. Then M is a commutative $\Gamma-$semiring and $I = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 6 & 0 \\ 0 & 0 \end{pmatrix} \right\}$ is the ideal such that $I\Gamma I = \{0\}$.
We have $\begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 3 & 0 \\ 0 & 0 \end{pmatrix} \in I$. But $\begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 3 & 0 \\ 0 & 0 \end{pmatrix} \notin I$.
Hence I is not a weakly prime ideal of $\Gamma-$semiring M.

Definition 3.17. An element x in a $\Gamma-$semiring M is said to be nilpotent if there exist a positive integer $n, \alpha \in \Gamma$ such that $(x\alpha)^n x = 0$.

Theorem 3.18. *Let A be a $k-$weakly prime ideal of $\Gamma-$semiring M and A be not a prime. Then $rad(A) = rad(0)$.*

Proof. Let A be a $k-$weakly prime ideal of $\Gamma-$semiring M and A not be a prime.
Clearly $rad(0) \subseteq rad(A)$. By Theorem 3.15, we have $A\Gamma A = 0$.
$\Rightarrow A \subseteq rad(0)$.
$\Rightarrow rad(A) \subseteq rad(0)$.
Hence $rad(0) = rad(A)$. □

$Nil\ M$ denotes the set of all nilpotent elements of M.

Corollary 3.19. *Let I be a $k-$weakly prime ideal of $\Gamma-$semiring M. If I is not a prime ideal of M then $I \subseteq Nil\ M$.*

Theorem 3.20. *Every $k-$ideal of $\Gamma-$semiring M is a $k-$weakly prime ideal if and only if for any ideals A, B of $\Gamma-$semiring M, we have $A\Gamma B = A$ or$A\Gamma B = B$ or $A\Gamma B = \{0\}$.*

Proof. Suppose every $k-$ideal of $\Gamma-$semiring M is a $k-$weakly prime ideal.
Let A, B be ideals of $\Gamma-$semiring M and $A\Gamma B \neq \{0\}$. Then $A\Gamma B$ is a $k-$weakly prime ideal.
$\{0\} \neq A\Gamma B \subseteq A\Gamma B$
$\Rightarrow A = A\Gamma B$ or $B = A\Gamma B$.
Converse is obvious. □

Corollary 3.21. *Every $k-$ideal of $\Gamma-$semiring M is a $k-$weakly prime ideal. Then any ideal A of $\Gamma-$semiring M is $A\Gamma A = A$ or $A\Gamma A = \{0\}$.*

Theorem 3.22. *Let I be a $k-$weakly prime ideal and not a prime ideal of commutative $\Gamma-$semiring M. If $x \in NilM$ then either $x \in I$ or $x\Gamma I = \{0\}$.*

Proof. Let I be a $k-$weakly prime ideal and not a prime ideal of commutative $\Gamma-$semiring M, $x \in NilM$ and $x\Gamma I \neq \{0\}$. Then there exist a least positive integer n and $\alpha \in \Gamma$ such that $(x\alpha)^n x = 0$.
$0 \neq x\alpha(i + (x\alpha)^{n-1}x) = x\alpha i \in I$, for some $i \in I$.
Suppose $x \notin I$ then $i + (x\alpha)^{n-1}x \in I \Rightarrow 0 \neq (x\alpha)^{n-1}x \in I \Rightarrow x \in I$.
Therefore for each $x \in NilM, x\Gamma I \neq \{0\}$ then $x \in I$.
Suppose $z \notin I$, for some $z \in NilM$. Then there exists the least positive integer n such that $(z\alpha)^n z = 0$.
Suppose $i\alpha z \neq 0$, for some $\alpha \in \Gamma$ and $i \in I$.
$\Rightarrow ((z\alpha)^{n-2}z + i)\alpha z = i\alpha z \neq 0$
$\Rightarrow (z\alpha)^{n-2}z + i \in I$ or $z \in I$.
In both cases, we have a contradiction. Hence $I\Gamma z = \{0\}$. □

Definition 3.23. An ideal I of $\Gamma-$semiring M is said to be semiprime if $I = rad(I)$.

Theorem 3.24. *A semiprime ideal I of commutative $\Gamma-$semiring M if and only if the quotient $\Gamma-$semiring M/I has no nonzero nilpotent elements.*

Proof. Suppose I is a semiprime ideal of commutative $\Gamma-$semiring M. Let $a + rad(I)$ be a nilpotent element of $M/rad(I)$. Then

$$
\begin{aligned}
&(a + rad(I)\alpha)^{n-1}(a + rad(I)) = rad(I), \text{ for some positive integer } n,\ \alpha \in \Gamma \\
\Rightarrow &(a\alpha)^n a + rad(I) = rad(I) \\
\Rightarrow &(a\alpha)^n a \in rad(I) \\
\Rightarrow &(a\alpha)^{mn} a \in I, \text{ for some positive integer } m \\
\Rightarrow &a \in rad(I) \\
\Rightarrow &a + rad(I) = rad(I).
\end{aligned}
$$

Hence $M/_{rad(I)}$ has no nonzero nilpotent elements.

Conversely suppose that M/I has no nonzero nilpotent elements and $a \in rad(I)$. Then

$$
\begin{aligned}
&(a\alpha)^n a \in I, \text{ for some positive integer } n, \alpha \in \Gamma \\
\Rightarrow &(a\alpha)^n a + I = I \\
\Rightarrow &((a + I)\alpha)^n (a + I) = I \\
\Rightarrow &a + I \text{ is a nilpotent of } M/I \\
\Rightarrow &a + I = I \\
\Rightarrow &a \in I.
\end{aligned}
$$

Therefore $rad(I) \subseteq I$. We have $I \subseteq rad(I)$. Hence $I = rad(I)$. Thus I is a semiprime ideal of $\Gamma-$semiring M. □

Theorem 3.25. *Let I be a proper $k-$ideal of $\Gamma-$semiring M. Then I is a $k-$weakly prime ideal of M if and only if $(I : x) = I \cup (0 : x)$, for $x \in M \setminus I$.*

Proof. Let I be a proper $k-$weakly prime ideal of $\Gamma-$semiring M. Clearly $I \cup (0 : x) \subseteq (I : x)$. Let $y \in (I : x)$. Then $y\alpha x \in I$, for all $\alpha \in \Gamma$.
Suppose $y\alpha x \neq 0$, for all $\alpha \in \Gamma, x \notin I$. Then $y \in I$, since I is weakly prime ideal.
Suppose $y\alpha x = 0$, for all $\alpha \in \Gamma$. Then $y \in (0 : x)$. Therefore $(I : x) \subseteq I \cup (0 : x)$.
Hence for $x \in M \setminus I, (I : x) = I \cup (0 : x)$.

Conversely suppose that $0 \neq x\alpha y \in I, \alpha \in \Gamma$ and $x \in M \setminus I$. Then $y \in (I : x)$ and hence $y \in I$. Therefore I is a $k-$weakly prime ideal of M. □

The following corollary follows from Lemmas 3.8 , 3.9, 3.11 and Theorem 3.25

Corollary 3.26. *If I is a $k-$weakly prime ideal of $\Gamma-$semiring M. Then $(I : x) = I$ or $(I : x) = (0 : x)$, for $x \in M \setminus I$.*

Theorem 3.27. *Let A be a $k-$weakly prime ideal of $\Gamma-$semiring M and A be not a prime. Then $A\Gamma\, rad(0) = \{0\}$.*

Proof. Let A be a $k-$weakly prime ideal of $\Gamma-$semiring M, A be not a prime and $a\alpha b \in A\Gamma\, rad(0)$, where $a \in A, \alpha \in \Gamma$ and $b \in rad(0)$. If $b \in A$ then $a\alpha b \in A\Gamma A = \{0\}$. Hence $a\alpha b = 0$.
Suppose $b \notin A$ then by Corollary 3.26, we have $(A : b) = (0 : b)$ or $(A : b) = A$.
Suppose $(A : b) = (0 : b)$. $\Rightarrow a\alpha b \in A \subseteq (A : b) = (0 : b)$
$\Rightarrow a\alpha b = 0$, for all $\alpha \in \Gamma$.
So suppose $(A : b) = A$. Since $b \in rad(0)$, there exists $n \in Z^+$ such that $(b\alpha)^n b = 0$, for all $\alpha \in \Gamma$
And suppose $(b\alpha)^{n-1} b \neq 0$. Then $0 \neq (b\alpha)^{n-1} b \in (A : b) = A$.
Since A is a weakly prime, $b \in A$, which is a contradiction.
Hence $A\Gamma\, rad(0) = \{0\}$. □

Theorem 3.28. *Let M be a $\Gamma-$semiring and A, B be $k-$weakly prime ideals but are not prime. Then $A\Gamma B = \{0\}$.*

Proof. Let M be a $\Gamma-$semiring, A, B be $k-$weakly prime ideals but are not prime and $a\alpha b \in A\Gamma B$, where $a \in A, \alpha \in \Gamma, b \in B$.
Since $B\Gamma B = \{0\}$, we have $B \subseteq rad(0)$. Then $a\alpha b \in A\Gamma B \subseteq A\Gamma rad(0) = \{0\}$.
Therefore $a\alpha b = 0$. Hence $A\Gamma B = \{0\}$. □

The following theorems proofs are easy so we omit the proofs

Theorem 3.29. *If $f : M \to N$ is a $\Gamma-$semiring homomorphism of $\Gamma-$semirings M and N and P is a $k-$ideal of $\Gamma-$semiring M then $f(P)$ is a $k-$ideal of $\Gamma-$semiring N.*

Theorem 3.30. *If $f : M \to N$ is a $\Gamma-$semiring homomorphism of $\Gamma-$semirings M and N and P is a $k-$ideal of $\Gamma-$semiring N then $f^{-1}(P)$ is a $k-$ideal of $\Gamma-$semiring M.*

Theorem 3.31. *If $f : M \to N$ is a $\Gamma-$semiring homomorphism of $\Gamma-$semirings M and N and P is a $k-$weakly prime ideal of $\Gamma-$semiring N then $f^{-1}(P)$ is a $k-$weakly prime ideal of $\Gamma-$semiring M.*

Proof. Suppose $f : M \to N$ is a $\Gamma-$semiring homomorphism of $\Gamma-$semirings M and N and P is a $k-$weakly prime ideal of $\Gamma-$semiring N. By Theorem 3.30, $f^{-1}(P)$ is a $k-$ideal.

Let $0 \neq x\alpha y \in f^{-1}(P), x, y \in M, \alpha \in \Gamma$. Then

$$\begin{aligned}
&0 \neq f(x\alpha y) \in P \\
\Rightarrow &0 \neq f(x)\alpha f(y) \in P \\
\Rightarrow &f(x) \text{ or } f(y) \in P, \text{ since } P \text{ is a } k-\text{weakly prime ideal} \\
\Rightarrow &x \in f^{-1}(P) \text{ or } y \in f^{-1}(P).
\end{aligned}$$

Hence $f^{-1}(P)$ is a $k-$weakly prime ideal of $\Gamma-$semiring M. □

Theorem 3.32. *Let A be a $k-$weakly prime ideal of $\Gamma-$semiring and A be not a prime. If I and J are ideals of M with $\{0\} \neq I\Gamma J \subseteq A$ then either $I \subseteq A$ or $J \subseteq A$.*

Proof. Let A be a $k-$weakly prime ideal of $\Gamma-$semiring and I, J be ideals of M with $\{0\} \neq I\Gamma J \subseteq A$ such that $I \nsubseteq A$ and $J \nsubseteq A$.
Suppose A is not a prime then $A\Gamma A = \{0\}$. Let $a\alpha b \in I\Gamma J, a \in I, b \in J, \alpha \in \Gamma$.
First suppose that $a \in I \setminus A$ and $a\alpha J \subseteq A$, for all $\alpha \in \Gamma$.
So that $J \subseteq (A : a) \subseteq A$, since A is a $k-$weakly prime ideal.
Since $J \nsubseteq A, a\alpha J = \{ 0 \}$, for all $\alpha \in \Gamma$.
Therefore $a\alpha b = 0$, for all $\alpha \in \Gamma$.
If $a \in A \cap I, b \in A$ then $a\alpha b \in A\Gamma A = \{0\}$.
Similarly if $b \in J \setminus A$, we can prove $a\alpha b = 0$.
Therefore $I\Gamma J = \{0\}$, which is a contradiction.
Hence either $I \subseteq A$ or $J \subseteq A$. □

Theorem 3.33. *Let M be a $\Gamma-$semiring and $\{A_i\}_{i\in I}$ be a family of $k-$weakly prime ideals that are not prime. Then $A = \cap\{A_i\}_{i\in I}$ is a $k-$weakly prime ideal of M.*

Proof. Let M be a $\Gamma-$semiring , $\{A_i\}_{i\in I}$ be a family of $k-$weakly prime ideals that are not prime and $A = \cap\{A_i\}_{i\in I}$. By Theorem 3.18, we have $rad(A_i) = rad(0) \neq M$ and $A_i \subseteq rad(0) \neq M$.
Hence A is a proper ideal of M.
Suppose that $a, b \in M$ such that $0 \neq a\alpha b \in A$ but $b \notin A$.
Then there exists $s \in I$ such that $b \notin A_s$ and $0 \neq a\alpha b \in A_s \subseteq rad(A_s) = rad(0) = rad(A_i)$, for all $i \in I,\ \alpha \in \Gamma$.
Then there exists n such that $0 \neq (a\alpha)^n a \in A_i$, for all $i \in I, \alpha \in \Gamma$ and therefore $a \in A_i$, for all $i \in I$, since A_i is a $k-$weakly prime ideal.
Hence $a \in A$. Thus A is a $k-$weakly prime ideal of M. □

4 $k-$weakly primary ideals

In this section, we introduce the notion of primary ideal and $k-$weakly primary ideal of $\Gamma-$semiring.

Definition 4.1. An ideal P of $\Gamma-$semiring M is said to be primary ideal of M if $x\alpha y \in P, \alpha \in \Gamma, x, y \in M$ then $x \in P$ or $(y\beta)^n y \in P$, for all $\beta \in \Gamma$ for positive integer n.

Definition 4.2. An ideal P of $\Gamma-$semiring M is said to be weakly primary ideal of M if $0 \neq x\alpha y \in P, \alpha \in \Gamma, x, y \in M$ then $x \in P$ or $(y\beta)^n y \in P$, for all $\beta \in \Gamma,\ n$ is a positive integer.

Definition 4.3. A $k-$ideal I of $\Gamma-$semiring M is called a $k-$weakly primary ideal if I is a weakly primary ideal.

Theorem 4.4. *Let I be a proper $k-$ideal of $\Gamma-$semiring M. Then I is a $k-$weakly primary ideal of M if and only if $(I : x) = I \cup (0 : x)$, for $x \in M \setminus rad(I)$.*

Proof. Let I be a proper $k-$weakly ideal of $\Gamma-$semiring M and $x \in M \setminus rad(I)$.
Clearly $I \cup (0 : x) \subseteq (I : x)$. Suppose $y \in (I : x)$.
Then $x\alpha y \in I$, for all $\alpha \in \Gamma$.
If $x\alpha y \neq 0$, for all $\alpha \in \Gamma$, then $y \in I$, since I is a $k-$weakly primary ideal of M.
If $x\alpha y = 0$, for all $\alpha \in \Gamma$ then $y \in (0 : x)$.
Therefore $(I : x) \subseteq I \cup (0 : x)$.
Hence for $x \in M \setminus rad(I), (I : x) = I \cup (0 : x)$. Conversely Suppose that $0 \neq x\alpha y \in rad(I), \alpha \in \Gamma$ and $x \in M \setminus rad(I)$. Then $y \in (I : x)$ and hence $y \in I$. Therefore I is a $k-$weakly primary ideal of M. □

The following corollary follows from Lemmas 3.8 , 3.9, 3.11 and Theorem 4.4

Corollary 4.5. *If I is a $k-$weakly primary ideal of $\Gamma-$semiring M. Then $(I : x) = I$ or $(I : x) = (0 : x)$, for $x \in M \setminus rad(I)$.*

Lemma 4.6. *Let I be a $k-$primary ideal of $\Gamma-$semiring M. If $a \in I$ and $a + b \in rad(I)$ then $b \in rad(I)$.*

Proof. Let I be a $k-$primary ideal of $\Gamma-$semiring $M, a \in I$ and $a + b \in rad(I)$.
Then there exists a positive integer n such that $\{(a + b)\alpha\}^n (a + b) = c + (b\alpha)^n b \in I$,
where $c \in I$, for all $\alpha \in \Gamma$ and hence $(b\alpha)^n b \in I$, for all $\alpha \in \Gamma$.
Hence $b \in rad(I)$. □

Theorem 4.7. *Let I be a $k-$weakly primary ideal of $\Gamma-$semiring M. If I is not a primary then $I\Gamma I = \{0\}$*

Proof. Let I be a $k-$weakly primary ideal of $\Gamma-$semiring M and I be not a primary.
Suppose $I\Gamma I \neq \{0\}$. Then there exist $x, y \in I$ and $\alpha \in \Gamma$ such that $x\alpha y \in I$.
If $x\alpha y = 0, x\alpha I \subseteq I$ then there exists $d \in I$ such that $x\alpha d \neq 0$
and hence $x\alpha d = x\alpha d + x\alpha y = x\alpha(d + y)$.
Therefore $x \in I$ or $((d + y)\gamma)^n (d + y) \in I$, for all $\gamma \in \Gamma$.
Suppose $x \notin I$. Then $d + y \in rad(I)$ and hence $y \in rad(I)$, by Lemma 4.6.
Suppose $x\alpha I = \{0\}, y\alpha I = 0$ and $I\alpha I \neq \{0\}$.
Then there exist $e, f \in I$ such that $e\alpha f \neq 0, \alpha \in \Gamma, 0 \neq e\alpha f = (x + e)\alpha(y + f) \in I$
$\Rightarrow x + e \in I$ or $((y + f)\gamma)^n (y + f) \in I$.
$\Rightarrow x \in I$ or $y + f \in rad(I)$.
$\Rightarrow x \in I$ or $y \in rad(I)$.
$\Rightarrow I$ is a primary, which is a contradiction.
Hence $I\Gamma I = \{0\}$. □

Theorem 4.8. *Let A be a $k-$weakly primary ideal of $\Gamma-$semiring M and A be not a primary. Then $rad(A) = rad(0)$.*

Proof. Let A be a $k-$weakly primary ideal of $\Gamma-$semiring M and A not be a primary.
$rad(0) \subseteq rad(A)$. By Theorem 4.7, we have $A\Gamma A = \{0\}$.
$\Rightarrow A \subseteq rad(0)$.
$\Rightarrow rad(A) \subseteq rad(0)$.
Hence $rad(0) = rad(A)$. □

The following theorem can be verified easily.

Theorem 4.9. *Let M and S be Γ_1 and $\Gamma_2$semirings respectively. If we define*
$(i).(x, y) + (z, w) = (x + z, y + w)$
$(ii).(x, y)(\alpha, \beta)(z, w) = (x\alpha z, y\beta w)$, *for all* $(x, y), (z, w) \in M \times S$ *and* $(\alpha, \beta) \in \Gamma_1 \times \Gamma_2$. *Then* $M \times S$ *is* $\Gamma_1 \times \Gamma_2-$*semiring.*

Definition 4.10. Let $M \times S$ be a $\Gamma_1 \times \Gamma_2-$semiring. An element $(1, 1) \in M \times S$ is said to be unity of $M \times S$ if for each $(a, b) \in M \times S$ there exists $(\alpha, \beta) \in \Gamma_1 \times \Gamma_2$ such that $(a, b)(\alpha, \beta)(1, 1) = (a, b)$.

Proof of the following theorem which is similar to corresponding result in ring theory,so we omit the proof

Theorem 4.11. *Let $M = M_1 \times M_2$ where each $M_i, i = 1, 2$ be a commutative $\Gamma-$semiring with unity element. Then*
(i). If I_1 is an ideal of M_1 then $Rad(I_1 \times M_2) = RadI_1 \times M_2$.
(ii). If I_2 is an ideal of M_2 then $Rad(M_1 \times I_2) = M_1 \times RadI_2$.

Theorem 4.12. *Let $M = M_1 \times M_2$ and $\Gamma = \Gamma_1 \times \Gamma_2$, where M_i is a commutative Γ_i-semiring with unity, $(i = 1, 2)$. If P_1 is a primary ideal of $\Gamma_1$semiring M_1 then $P_1 \times M_2$ is a primary ideal of $\Gamma_1 \times \Gamma_2-$semiring of M.*

Proof. Let $M = M_1 \times M_2$ and $\Gamma = \Gamma_1 \times \Gamma_2$, where M_i is a commutative Γ_i-semiring with unity, $(i = 1, 2)$ and P_1 is a primary ideal of Γ_1-semiring M_1.

$$\begin{aligned}
&Let\ (a, b)(\alpha, \beta)(c, d) = (a\alpha c, b\beta d) \in P_1 \times M_2, \text{ where } (a, b), (c, d) \in M_1 \times M_2 \\
&\quad \text{and}(\alpha, \beta) \in \Gamma_1 \times \Gamma_2. \\
&\Rightarrow a\alpha c \in P_1 \\
&\Rightarrow a \in P_1 \text{ or } c \in RadP_1, \text{ since } P_1 \text{ is a primary} \\
&\Rightarrow (a, b) \in P_1 \times M_2 \text{ or } (c, d) \in RadP_1 \times M_2 = Rad(P_1 \times M_2), \text{ by Theorem 4.11.}
\end{aligned}$$

Thus $P_1 \times M_2$ is a primary ideal of M. □

Corollary 4.13. *Let $M_1 \times M_2$ where M_i is a commutative $\Gamma_i$semiring with unity, $(i = 1, 2)$. If P_2 is a primary ideal of $\Gamma_2$semiring M_2 then $M_1 \times P_2$ is a primary ideal of $\Gamma_1 \times \Gamma_2-$semiring $M_1 \times M_2$.*

Theorem 4.14. *Let $M = M_1 \times M_2$ and $\Gamma = \Gamma_1 \times \Gamma_2$.*
where M_i is a commutative Γ_i-semiring with unity, $(i = 1, 2.)$. If P is a weakly primary ideal of $\Gamma-$semiring M then either $P = \{(0, 0)\}$ or P is a primary.

Proof. Let $P = P_1 \times P_2$ be a weakly primary ideal of M. Suppose that $P \neq \{(0, 0)\}$. Then there exists an element $(a, b) \in P$ with $(a, b) \neq (0, 0)$.

$$\begin{aligned}
&\Rightarrow (0, 0) \neq (a, 1)(\alpha, \beta)(1, b) \in P, (\alpha, \beta) \in \Gamma_1 \times \Gamma_2. \\
&\Rightarrow (a, 1) \text{ or } (1, b) \in RadP.
\end{aligned}$$

If $(a, 1) \in P$ then $P = P_1 \times M_2$. Let $c\alpha d \in P_1$ when $c, d \in M_1, \alpha \in \Gamma_1$

$$\begin{aligned}
&(0, 0) \neq (c, 1)(\alpha, \beta)(d, 1) \\
&\quad (c\alpha d, 1\beta 1) \in P \\
&\Rightarrow (c, 1) \in P \text{ or } (d, 1) \in RadP = Rad(P_1 \times M_2) = RadP_1 \times M_2. \\
&\quad \Rightarrow c \in P_1 \text{ or } d \in RadP_1.
\end{aligned}$$

Therefore P_1 is a primary. Hence by Theorem 4.12, P is a primary.
Now $(1, b) \in RadP = Rad(P_1 \times P_2) = P_1 \times RadP_2$.
$\Rightarrow (1, (b\beta)^{n-1}b) \in P, \beta \in \Gamma$. So $P = M_1 \times P_2$.
Therefore P_2 is a primary.Hence by Corollary 4.13, P is a primary.

□

Theorem 4.15. *Let M be a $\Gamma-$semiring and $\{A_i\}_{i\in I}$ be a family of a $k-$weakly primary ideals that are not primary. Then $A = \cap\{A_i\}_{i\in I}$ is a $k-$weakly primary ideal of M.*

Proof. Let M be a $\Gamma-$semiring,$\{A_i\}_{i\in I}$ be a family of $k-$weakly primary ideals that are not prime and $A = \cap\{A_i\}_{i\in I}$.
By Theorem 4.8, we have $rad(A_i) = rad(0) \neq M$ for each i, so A is a proper ideal of M.
Suppose that $a, b \in M$ such that $0 \neq a\alpha b \in A$ and $b \notin A$.
$\Rightarrow$ there exists $s \in I$, such that $b \notin A_s$ and $0 \neq a\alpha b \in A_s$
$\Rightarrow a \in rad(A_s) = rad(0) = rad(A_i)$, for all $i \in I$
$\Rightarrow$ there exists n such that $0 \neq (a\alpha)^n a \in A_i$, for all $i \in I$
$\Rightarrow (a\alpha)^n a \in A$.
Thus A is a $k-$weakly primary ideal of M. □

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