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On a Class of $\alpha-$para Kenmotsu manifolds with semi-symmetric metric connection

Rajendra Prasad and Sushil Kumar

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Abstract. In this paper we investigate $\alpha-$para Kenmotsu manifold with semi-symmetric metric connection. We have found the relations between curvature tensors, Ricci tensors and scalar curvature of $\alpha-$para Kenmotsu manifold with semi-symmetric metric connection and with metric connection. Also, we have proved some results on quasi-projectively flat, $\xi-$projectively flat, $\phi-$projectively flat, conformally flat and $\xi-$concircularly flat $\alpha-$para Kenmotsu manifolds. We have given two examples of it.

1 Introduction

In 1985, almost paracontact geometry was introduced by Kaneyuki and Williams [10] and then it was continued by many authors. A systematic study of almost paracontact metric manifolds was carried out by Zamkovoy [19]. However such structures were also studied by Buchner and Rosca [[4], [5], [15]], Rossca and Vanhecke [12]. The curvature identities for different classes of almost paracontact metric manifolds were obtained in [6]. Further almost para-Hermitian structures on the tangent bundle of an almost para-coHermitian manifolds was studied by Bejan [1]. A class of $\alpha-$para Kenmotsu manifolds was studied by Srivastava and Srivastava [13] and $\xi-$conformally flat contact metric manifolds was studied by Zhen et al. [20].

We can observe from the form of the concircular curvature tensor that pseudo-Riemannian manifolds with vanishing concircular curvature tensor are of constant curvature [[3], [16]]. Thus one can imagine of the concircular curvature tensor as a measure of the failure of a pseudo-Riemannian manifold to be of constant curvature.

Hayden introduced semi- symmetric linear connections on a Riemannian manifold [9]. Let M be an $n-$dimensional Riemannian manifold of class $C^{\infty}-$endowed with the Riemannian metric g and ∇ be the Levi- Civita connection on M^n.

A linear connection $\bar{\nabla}$ defined on M^n is said to be semi- symmetric [8] if its torsion tensor T is of the form $T(X,Y) = \eta(Y)X - \eta(X)Y$, where ξ is a vector field and η is a $1-$form defined by $g(X,\xi) = \eta(X)$, for all vector fields $X \in \chi(M^n)$, where $\chi(M^n)$ is the set of all differentiable vector fields on M^n. A semi- symmetric connection $\bar{\nabla}$ is called a semi-symmetric metric connection, if it further satisfies $\bar{\nabla} g = 0$. A relation between the semi-symmetric metric connection $\bar{\nabla}$ and the Levi-Civita connection ∇ on M^n has been obtained by Yano [17] which is given by

$$\bar{\nabla}_X Y = \nabla_X Y + \eta(Y)X - g(X,Y)\xi. \tag{1.1}$$

This paper is organized as follows. In Section 3, we have obtained curvature tensors and Ricci tensors of $\alpha-$paracontact Kenmotsu manifold with semi symmetric metric connection. In Section 4, we have found the relation between a second-order parallel tensor and the associated metric on an $\alpha-$para Kenmotsu manifold with semi symmetric metric connection. In Section

5, 6, 7 and 8, we have focussed on some flat conditions for α −para Kenmotsu manifold with semi symmetric metric connection.

2 Preliminaries

A differentiable manifold M^n of dimension n is said to have an almost paracontact $(\phi, \xi, \eta)-$structure if it admits an $(1, 1)$ tensor field ϕ, a unique vector field ξ, $1-$form η such that:

$$\phi^2 = I - \eta \otimes \xi,\ \phi\xi = 0,\ \ \eta \circ \phi = 0\ , \tag{2.1}$$

$$\eta(\xi) = 1, \tag{2.2}$$

for any vector fields X, Y on M^n. The manifold M^n equipped with an almost paracontact structure (ϕ, ξ, η) is called almost paracontact manifold. In addition, if an almost paracontact manifold admits a pseudo-Riemannian metric satisfying:

$$g(X, \xi) = \eta(X), \tag{2.3}$$

$$g(\phi X, \phi Y) = -g(X, Y) + \eta(X)\eta(Y), \tag{2.4}$$

$$g(\phi X, Y) = -g(X, \phi Y), \tag{2.5}$$

for any vector fields X, Y on M^n, then (ϕ, ξ, η, g), is called an almost paracontact metric structure and the manifold M^n equipped with an almost paracontact metric structure is called an almost paracontact metric manifold. Further in addition, if the structure (ϕ, ξ, η, g), satisfies

$$d\eta(X, Y) = g(X, \phi Y), \tag{2.6}$$

for any vector fields X, Y on M^n. Then the manifold is called paracontact metric manifold and the corresponding structure (ϕ, ξ, η, g) is called a paracontact structure with the associated metric g [19].

On an almost paracontact metric manifold, one defines the $(1, 2)$ tensor field N_ϕ by

$$N_\phi = [\phi, \phi] - 2d\eta \otimes \xi, \tag{2.7}$$

where $[\phi, \phi]$ is the Nijenhuis tensor of ϕ. If N_ϕ vanishes identically, then we say that the manifold M^n is a normal almost paracontact metric manifold. The normality condition implies that the almost paracomplex structure J defined on $M^n \times R$ by

$$J(X, \lambda\frac{d}{dt}) = (\phi X + \lambda\xi, \eta(X)\frac{d}{dt}),$$

is integrable. Here X is tangent to M^n, t is the coordinate on R and λ is a differentiable function on $M^n \times R$.

For an almost paracontact metric 3−dimensional manifold M^3, the following three conditions are mutually equivalent [14]:

(i) there exist smooth functions α, β on M^3 such that

$$(\nabla_X\phi)Y = \beta(g(X, Y)\xi - \eta(Y)X) + \alpha(g(\phi X, Y)\xi - \eta(Y)\phi X), \tag{2.8}$$

(ii) M^3 is normal,

(iii) there exist smooth functions α, β on M^3 such that

$$\nabla_X\xi = \alpha(X - \eta(X)\xi) + \beta\phi X, \tag{2.9}$$

where ∇ is the Levi-Civita connection of pseudo−Riemannian metric g.

A normal almost paracontact metric 3−dimensional manifold is called

(A) Para-Cosymlectic manifold if $\alpha = \beta = 0$ [6],

(B) quasi-para Sasakian manifold if and only if $\alpha = 0$ and $\beta \neq 0$ [7],

(C) $\beta-$para Sasakian manifold if and only if $\alpha = 0$ and β is a non-zero constant, in particular para Sasakian manifold if $\beta = -1$ [19],

(D) $\alpha-$para Kenmotsu manifold if α is a non-zero constant and $\beta = 0$ [20], in particular para Kenmotsu manifold if $\alpha = 1$ [2].

For a $3-$dimensional manifold M^3 with an almost para$-$contact metric structure (ϕ, ξ, η, g) one can also construct a local orthonormal basis as follows:

Let U be coordinate neighbourhood on M and e_1 any vector field on U orthogonal to ξ. Then ϕe_1 is a vector field orthogonal to both e_1, ξ and $||\phi e_1||^2 = -1$. So, we have $g(e_1, e_1) = 1, g(\phi e_1, \phi e_1) = -1$ and $g(\xi, \xi) = 1$. Hence we obtain orthonormal basis $\{e_1, \phi e_1, \xi\}$ called a $\phi-$basis [19].

Remark 2.1. Since the Ricci tensor of Levi-Civita connection ∇ is given by

$$S(Y, Z) = g(R(e_1, Y)Z, e_1) - g(R(\phi e_1, Y)Z, \phi e_1) + g(R(\xi, Y)Z, \xi).$$

On an $n-$dimensional connected almost paracontact pseudo$-$Riemannian manifold M^n the curvature tensor R [11] and the projective curvature tensor P [18] are defined by

$$R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z, \tag{2.10}$$

$$P(X, Y)Z = R(X, Y)Z - \frac{1}{n-1}[g(QY, Z)X - g(QX, Z)Y], \tag{2.11}$$

where Q denotes the Ricci operator.

Let $M^3(\phi, \xi, \eta, g)$ be an $\alpha-$para Kenmotsu manifold [13], then we have

$$\begin{aligned} R(X, Y)Z \;\; = \;\; & (\frac{r}{2} + 2\alpha^2)\{g(Y, Z)X - g(X, Z)Y\} \\ & -(\frac{r}{2} + 3\alpha^2)\{\eta(X)g(Y, Z) - \eta(Y)g(X, Z)\}\xi \\ & +(\frac{r}{2} + 3\alpha^2)\{\eta(X)Y - \eta(Y)X\}\eta(Z). \end{aligned} \tag{2.12}$$

Replace $Z = \xi$ in equation (2.12), we get

$$R(X, Y)\xi = \alpha^2\{\eta(X)Y - \eta(Y)X\}, \tag{2.13}$$

$$S(Y, Z) = (\frac{r}{2} + \alpha^2)g(Y, Z) - (\frac{r}{2} + 3\alpha^2)\eta(Y)\eta(Z), \tag{2.14}$$

$$S(Y, \xi) = -2\alpha^2\eta(Y), \tag{2.15}$$

$$S(\xi, \xi) = -2\alpha^2,$$

$$(\nabla_X \phi)Y = \alpha(g(\phi X, Y)\xi - \eta(Y)\phi X), \tag{2.16}$$

$$\nabla_X \xi = \alpha(X - \eta(X)\xi). \tag{2.17}$$

From equation (1.1), we have

$$\bar{\nabla}_X \xi = (1 + \alpha)(X - \eta(X)\xi). \tag{2.18}$$

3 Curvature tensor on $\alpha-$para Kenmotsu manifold with semi-symmetric metric connection

Let M^3 be a 3−dimensional $\alpha-$para Kenmotsu manifold. The curvature tensor $\bar{R}$ of M^3 with respect to the semi-symmetric metric connection $\bar{\nabla}$ is defined by

$$\bar{R}(X,Y)Z = \bar{\nabla}_X \bar{\nabla}_Y Z - \bar{\nabla}_Y \bar{\nabla}_X Z - \bar{\nabla}_{[X,Y]} Z. \tag{3.1}$$

By using equations (1.1), (2.2), (2.3), (2.17) and (2.18), we get

$$\begin{aligned} \bar{R}(X,Y)Z &= R(X,Y)Z - (1+2\alpha)[g(Y,Z)X - g(X,Z)Y] \\ &\quad +(1+\alpha)[\eta(Y)X - \eta(X)Y]\eta(Z) \\ &\quad +(1+\alpha)[\eta(X)g(Y,Z) - \eta(Y)g(X,Z)]\xi. \end{aligned} \tag{3.2}$$

From equation (3.2), we obtain that the curvature tensor $\bar{R}$ satisfies:

$$\bar{R}(X,Y)Z + \bar{R}(Y,Z)X + \bar{R}(Z,X)Y = 0, \tag{3.3}$$

Taking inner product of equation (3.2) with U and using equation (2.3), we have

$$\begin{aligned} & g(\bar{R}(X,Y)Z,U) \\ = \; & g(R(X,Y)Z,U) - (1+2\alpha)[g(Y,Z)g(X,U) - g(X,Z)g(Y,U)] \\ & +(1+\alpha)[\eta(Y)g(X,U) - \eta(X)g(Y,U)]\eta(Z) \\ & +(1+\alpha)[\eta(X)g(Y,Z) - \eta(Y)g(X,Z)]\eta(U). \end{aligned} \tag{3.4}$$

Let $\{e_1, \phi e_1, \xi\}$ be a local orthonormal $\phi-$basis of vector fields on $\alpha-$para Kenmotsu manifold M^3. Then, we get

$$\bar{S}(Y,Z) = (-1 + \frac{r}{2} - 3\alpha + \alpha^2)g(Y,Z) + (1 - \frac{r}{2} + \alpha - 3\alpha^2)\eta(Y)\eta(Z). \tag{3.5}$$

From equation (3.5), we have

$$\bar{r} = -2 + r - 8\alpha, \tag{3.6}$$

where $\bar{r}$ scalar curvature with semi-symmetric metric connection.

Replace $Y = \xi$ in equation (3.5), using (2.2) and (2.3), we get

$$\bar{S}(Y,\xi) = -2\alpha(1+\alpha)\eta(Y). \tag{3.7}$$

From equation (3.2) in interchange X to Y, we have

$$\begin{aligned} \bar{R}(Y,X)Z &= R(Y,X)Z - (1+2\alpha)[g(X,Z)Y - g(Y,Z)X] \\ &\quad +(1+\alpha)[\eta(X)Y - \eta(Y)X]\eta(Z) \\ &\quad +(1+\alpha)[\eta(Y)g(X,Z) - \eta(X)g(Y,Z)]\xi. \end{aligned} \tag{3.8}$$

From equations (3.2) and (3.8), we get

$$\bar{R}(Y,X)Z = -\bar{R}(X,Y)Z, \tag{3.9}$$

where $R(X,Y)Z = -R(Y,X)Z$.

Replace $Z = \xi$ in equation (3.2), using equations (2.3) and (2.13), we have

$$\bar{R}(X,Y)\xi = \alpha(1+\alpha)(\eta(X)Y - \eta(Y)X). \tag{3.10}$$

Replace $X = \xi$ in equation (3.10) and using equation (2.3), we get

$$\bar{R}(\xi,Y)\xi = \alpha(1+\alpha)(Y - \eta(Y)\xi). \tag{3.11}$$

4 Second-Order Parallel Tensor Field

Definition 4.1. A tensor T of second order is said to be a second-order parallel tensor if $\bar{\nabla}T = 0$, where $\bar{\nabla}$ denotes the operator of covariant differentiation with respect to the associated semi-symmetric metric connection.

Here, we give the following result which established the relation between a second-order parallel tensor and the associated metric on an $\alpha-$para Kenmotsu manifold with semi-symmetric metric connection.

Theorem 4.2. *On an $\alpha-$para Kenmotsu manifold M^3 with semi-symmetric metric connection a second-order parallel tensor is a constant multiple of the associated metric g .*

Proof. Let h denote a symmetric $(0,2)-$tensor field $\alpha-$para Kenmotsu manifold with semi-symmetric metric connection on M^3 such that $\bar{\nabla}h = 0$.

Then the condition satisfies

$$\begin{aligned} \bar{R}(X,Y).h &= 0, \\ \bar{R}(X,Y).h(Z,U) &= 0. \end{aligned}$$

Then, we have

$$h(\bar{R}(X,Y)Z,U) + h(Z,\bar{R}(X,Y)U) = 0, \tag{4.1}$$

for any vector fields $X,Y,Z,U \in \chi(M^3)$. Substituting $X = Z = U = \xi$ in equation (4.1), we obtain

$$h(\bar{R}(\xi,Y)\xi,\xi) + h(\xi,\bar{R}(\xi,Y)\xi) = 0. \tag{4.2}$$

Using equation (3.2), we get

$$h(Y,\xi) = \eta(Y)h(\xi,\xi). \tag{4.3}$$

Differentiating equation (4.3) with respect to semi-symmetric metric connection along an arbitrary $X \in \chi(M^3)$, using equations (2.18) and (4.3), we get

$$h(X,Y) = g(X,Y)h(\xi,\xi). \tag{4.4}$$

Again, Differentiating equation (4.4) with semi-symmetric metric connection covariantly along any vector field on M^3 it can be easily seen that $h(\xi,\xi)$ is constant. □

Let us suppose that h is a parallel $2-$form on M^3 $\alpha-$para Kenmotsu manifold with semi-symmetric metric, that is

$$h(X,Y) = -h(Y,X) \quad \text{and } \bar{\nabla}h = 0. \tag{4.5}$$

Theorem 4.3. *Let M^3 be an $\alpha-$para Kenmotsu manifold with semi-symmetric metric connection. Then non-zero parallel $2-$forms h cannot occur on M^3.*

Proof. For h the parallel form, we have from equation (4.5) that

$$h(\xi,\xi)=0. \tag{4.6}$$

Differentiating equation (4.6) covariantly with semi-symmetric metric connection along arbitrary $X \in \chi(M)$ and using equations (2.18) and (4.6), we have

$$h(X,\xi)=0. \tag{4.7}$$

Next, differentiating equation (4.7) covariantly with semi-symmetric metric connection along any arbitrary $Y \in \chi(M)$ and using equation (2.18) and (4.7), we have

$$h(X,Y)=0. \tag{4.8}$$

□

5 Quasi-Projectively flat and $\xi-$Projectively flat $\alpha-$para Kenmotsu manifold with semi-symmetric metric connection

Let M^n be an $n-$dimensional $\alpha-$para Kenmotsu manifold. The Projective curvature tensor $\bar{P}$ of type $(1,3)$ with semi-symmetric metric connection is defined by

$$\bar{P}(X,Y)Z=\bar{R}(X,Y)Z-\frac{1}{(n-1)}[\bar{S}(Y,Z)X-\bar{S}(X,Z)Y]. \tag{5.1}$$

(i) An $\alpha-$para Kenmotsu manifold M^n is said to be quasi-Projectively flat with semi-symmetric metric connection, if

$$g(\bar{P}(\phi X,Y)Z,\phi U)=0. \tag{5.2}$$

(ii)An $\alpha-$para Kenmotsu manifold M^n is said to be $\xi-$Projectively flat with semi-symmetric metric connection, if the condition satisfies

$$\bar{P}(X,Y)\xi=0.$$

Theorem 5.1. *A $3-$dimensional quasi-Projectively flat $\alpha-$para Kenmotsu manifold M^3 with semi-symmetric metric connection is $\eta-$Einstein manifold.*

Proof. From equation (5.1), we have

$$\bar{P}(X,Y)Z=\bar{R}(X,Y)Z-\frac{1}{2}[\bar{S}(Y,Z)X-\bar{S}(X,Z)Y].$$

Taking inner product of above equation with U, we get

$$g(\bar{P}(X,Y)Z,U)=g(\bar{R}(X,Y)Z,U)-\frac{1}{2}[\bar{S}(Y,Z)g(X,U)-\bar{S}(X,Z)g(Y,U)]. \tag{5.3}$$

Replace $X=\phi X$ and $U=\phi U$ in equation (5.3), we get

$$\begin{aligned} & g(\bar{P}(\phi X,Y)Z,\phi U) \\ = \; & g(\bar{R}(\phi X,Y)Z,\phi U)-\frac{1}{2}[\bar{S}(Y,Z)g(\phi X,\phi U)-\bar{S}(\phi X,Z)g(Y,\phi U)]. \end{aligned} \tag{5.4}$$

From equations (5.2) and (5.4), using equations (3.2) and (3.5), we get

$$\begin{aligned} & g(R(\phi X,Y)Z,\phi U) \qquad (5.5) \\ = \; & (\frac{1}{2}+\frac{r}{4}+\frac{\alpha}{2}+\frac{\alpha^2}{2})g(Y,Z)g(\phi X,\phi U)-(\frac{1}{2}+\frac{r}{4}+\frac{\alpha}{2}+\frac{\alpha^2}{2})g(\phi X,Z)g(Y,\phi U) \\ & -(\frac{1}{2}+\frac{r}{4}+\frac{\alpha}{2}+\frac{3\alpha^2}{2})\eta(Y)\eta(Z)g(\phi X,\phi U). \end{aligned}$$

Let $\{e_1,\phi e_1,\xi\}$ be a local orthonormal basis of vector fields on $\alpha-$para Kenmotsu manifold M^3. Then, we get

$$S(Y,Z)=(1+\frac{r}{2}+\alpha+\alpha^2)g(Y,Z)-\frac{3}{2}(1+\frac{r}{2}+\alpha+3\alpha^2)\eta(Y)\eta(Z). \qquad (5.6)$$

□

Theorem 5.2. *If M^3 be an $\alpha-$para Kenmotsu manifold with semi-symmetric metric connection, then M^3 is $\xi-$Projectively flat.*

Proof. Putting $Z=\xi$ in equation (5.1), using equations (2.13), (3.2) and (3.7), we get

$$\bar{P}(X,Y)\xi=0.$$

Hence the theorem is proved. □

6 $\phi-$Projectively flat $\alpha-$para Kenmotsu manifold with semi-symmetric metric connection

Let M^n be an $n-$dimensional $\alpha-$para Kenmotsu manifold with semi-symmetric metric connection is said to be $\phi-$Projectively flat, if $\phi^2(\bar{P}(\phi X,\phi Y)\phi Z)=0$, where $\bar{P}$ is the Projective curvature tensor of $\alpha-$para Kenmotsu manifold with semi-symmetric metric connection. Suppose M^n be a $\phi-$Projectively flat $\alpha-$para Kenmotsu manifold with semi-symmetric metric connection. It is known that

$$\phi^2(\bar{P}(\phi X,\phi Y)\phi Z)=0 \text{ holds if and only if } g(\bar{P}(\phi X,\phi Y)\phi Z,\phi U))=0 \qquad (6.1)$$

for any vector fields $X,Y,Z,U\in TM^n$.

Theorem 6.1. *A $3-$dimensional $\phi-$Projectively flat $\alpha-$para Kenmotsu manifold M^3 with semi-symmetric metric connection is $\eta-$Einstein manifold.*

Proof. We take equation (5.4), replace $Y=\phi Y$ and $U=\phi U$, using equation (6.1), then

$$\begin{aligned} & g(\bar{P}(\phi X,\phi Y)\phi Z,\phi U) \qquad (6.2) \\ = \; & \frac{1}{2}[\bar{S}(\phi Y,\phi Z)g(\phi X,\phi U)-\bar{S}(\phi X,\phi Z)g(\phi Y,\phi U)]. \end{aligned}$$

Using equations (3.2) and (3.5), we get

$$\begin{aligned} g(R(\phi X,\phi Y)\phi Z,\phi U) = \; & (\frac{1}{2}+\frac{r}{4}+\frac{\alpha}{2}+\frac{\alpha^2}{2})g(\phi Y,\phi Z)g(\phi X,\phi U) \qquad (6.3) \\ & -(\frac{1}{2}+\frac{r}{4}+\frac{\alpha}{2}+\frac{\alpha^2}{2})g(\phi X,\phi Z)g(\phi Y,\phi U). \end{aligned}$$

Let $\{e_1,\phi e_1,\xi\}$ be a local orthonormal basis of vector fields on $\alpha-$para Kenmotsu manifold M^3. Then, we get

$$S(Y,Z)=(1+\frac{r}{2}+\frac{\alpha}{2}+\alpha^2)g(Y,Z)-(1+\frac{r}{2}+\frac{\alpha}{2}+\alpha^2)\eta(Y)\eta(Z). \qquad (6.4)$$

□

7 Weyl conformal flat curvature tensor on $\alpha-$para Kenmotsu manifold with semi-symmetric metric connection

The Weyl conformal curvature tensor $\bar{C}$ of type $(1,3)$ of M^n an $n-$dimensional $\alpha-$para Kenmotsu manifold with semi-symmetric metric connection is given by

$$\begin{aligned} & \bar{C}(X,Y)Z \qquad (7.1) \\ = \quad & \bar{R}(X,Y)Z - \frac{1}{(n-2)}[\bar{S}(Y,Z)X - \bar{S}(X,Z)Y + g(Y,Z)\bar{Q}X \\ & -g(X,Z)\bar{Q}Y] + \frac{\bar{r}}{(n-1)(n-2)}[g(Y,Z)X - g(X,Z)Y], \end{aligned}$$

where $\bar{Q}$ Ricci operator with respect to the semi-symmetric metric connection.

An $\alpha-$para Kenmotsu manifold M^n is said to be Weyl conformal flat with semi-symmetric metric connection, if $\bar{C} = 0$

Theorem 7.1. *Let M^3 be a 3-dimensional Weyl conformal flat $\alpha-$para Kenmotsu manifold M^3 with semi-symmetric metric connection is $\eta-$Einstein manifold*

Proof. Taking inner product equation (7.1) with U, we get

$$\begin{aligned} & g(\bar{C}(X,Y)Z,U) \qquad (7.2) \\ = \quad & g(\bar{R}(X,Y)Z,U) - [\bar{S}(Y,Z)g(X,U) - \bar{S}(X,Z)g(Y,U) \\ & +g(Y,Z)g(\bar{Q}X,U) - g(X,Z)g(\bar{Q}Y,U)] \\ & +\frac{\bar{r}}{2}[g(Y,Z)g(X,U) - g(X,Z)g(Y,U)]. \end{aligned}$$

An $\alpha-$para Kenmotsu manifold M^3 is said to be Weyl conformal flat with semi-symmetric metric connection, if $g(\bar{C}(X,Y)Z,U) = 0$ and using equations (2.12), (2.14), (2.15), (2.16), (3.2), (3.4), (3.5) and (3.6), we get

$$S(Y,Z) = (\frac{r}{2} + \alpha^2)g(Y,Z) - (\frac{r}{2} + 3\alpha^2)\eta(Y)\eta(Z).$$

□

8 $\xi-$concircularly flat $\alpha-$para Kenmotsu manifold with semi-symmetric metric connection

Let (M^n, g) be an n-dimensional $\alpha-$para Kenmotsu manifold with semi-symmetric metric connection. The con-circular curvature tensor $\bar{L}$ [16] of M^ndefined by

$$\bar{L}(X,Y)Z = \bar{R}(X,Y)Z - \frac{\bar{r}}{n(n-1)}[g(Y,Z)X - g(X,Z)Y], \qquad (8.1)$$

for vector fields $X, Y, Z \in TM^n$.

A $\alpha-$para Kenmotsu manifold M^n is said to be $\xi-$concircularly flat with semi-symmetric metric connection, if the condition satisfies $\bar{L}(X,Y)\xi = 0$.

Theorem 8.1. *Let M^3 be an $\alpha-$para Kenmotsu manifold with semi-symmetric metric connection. Then M^3 is $\xi-$con-circularly flat if and only if $r = (1 - 2\alpha - 6\alpha^2)$.*

Proof. From equation (8.1), we have

$$\bar{L}(X,Y)Z = \bar{R}(X,Y)Z - \frac{\bar{r}}{6}[g(Y,Z)X - g(X,Z)Y].$$

Putting $Z = \xi$ in above equation, using (2.14) and (3.2), we get

$$\bar{L}(X,Y)\xi = (\frac{-1 + r + 2\alpha + 6\alpha^2}{6})[\eta(X)Y - \eta(Y)X]. \tag{8.2}$$

This implies that $\bar{L}(X,Y)\xi = 0$ if and only if $r = (1 - 2\alpha - 6\alpha^2)$. □

Example 8.2. Let 3−dimensional manifold $M^3 = R^2 \times R\_ \subset R^3$ with the standard Cartesian coordinates (x,y,z). Define the almost paracontact structure (ϕ,ξ,η) with semi-symmetric metric connection on M^3 by

$$\phi e_1 = e_2,\ \ \phi e_2 = e_1,\ \ \phi e_3 = 0,\ \ \xi = e_3,\ \eta = dZ \tag{8.3}$$

where $e_1 = \frac{\partial}{\partial x}$, $e_2 = \frac{\partial}{\partial y}$, and $e_3 = \frac{\partial}{\partial z}$. By calculations,

$$[\phi,\phi](e_i,e_j) - 2d\eta(e_i,e_j) = 0;\ \ 1 \le i < j \le 3 \tag{8.4}$$

which implies that the structure is normal.
Let g be the pseudo-Riemannian metric defined by

$$\begin{aligned} g(e_1,e_1) &= \exp(2z),\ \ g(e_2,e_2) = -\exp(2z),\ \ g(e_3,e_3) = 1, \\ g(e_1,e_2) &= 0,\ \ g(e_1,e_3) = 0,\ \ g(e_2,e_3) = 0 \end{aligned} \tag{8.5}$$

Let ∇ Levi-Civita connection with metric g, then we given

$$[e_1,e_2] = 0,\quad [e_1,e_3] = 0,\quad [e_2,e_3] = 0$$

For Levi-Civita connection ∇ of the metric g is given by

$$\begin{aligned} 2g(\nabla_X Y, Z) &= Xg(Y,Z) + Yg(Z,X) - Zg(X,Y) + g([X,Y],Z) \\ &\quad - g([Y,Z],X) + g([Z,X],Y) \end{aligned}$$

which is known as Koszuls formula, we have

$$\begin{aligned} \nabla_{e_1} e_1 &= -\exp(2z)e_3,\ \ \nabla_{e_1} e_2 = 0,\ \ \nabla_{e_1} e_3 = e_1 \\ \nabla_{e_2} e_1 &= 0,\ \ \nabla_{e_2} e_2 = \exp(2z)e_3,\ \ \nabla_{e_2} e_3 = e_2 \\ \nabla_{e_3} e_1 &= e_1,\ \ \nabla_{e_3} e_2 = e_2,\ \ \nabla_{e_3} e_3 = 0 \end{aligned} \tag{8.6}$$

Therefore, the semi-symmetric metric connection on M is given by

$$\begin{aligned} \bar{\nabla}_{e_1} e_1 &= -2\exp(2z)e_3,\ \ \bar{\nabla}_{e_1} e_2 = 0,\ \ \bar{\nabla}_{e_1} e_3 = 2e_1 \\ \bar{\nabla}_{e_2} e_1 &= 0,\ \ \bar{\nabla}_{e_2} e_2 = 2\exp(2z)e_3,\ \ \bar{\nabla}_{e_2} e_3 = 2e_2 \\ \bar{\nabla}_{e_3} e_1 &= e_1,\ \ \bar{\nabla}_{e_3} e_2 = e_2,\ \ \bar{\nabla}_{e_3} e_3 = 0 \end{aligned} \tag{8.7}$$

Now, for $\xi = e_3$, above results satisfies

$$\bar{\nabla}_X \xi = (1+\alpha)(X - \eta(X)\xi)$$

with $\alpha = 1$. Consequently $M^3(\phi,\xi,\eta,g)$ is $\alpha-$para Kenmotsu manifold with semi-symmetric metric connection.

Example 8.3. Let 3−dimensional manifold $M^3 = R^2 \times R\_ \subset R^3$ with the standard Cartesian coordinates (x, y, z). Define the almost paracontact structure (ϕ, ξ, η) with semi-symmetric metric connection on M^3 by

$$\phi e_1 = e_2, \ \ \phi e_2 = -e_1, \ \ \phi e_3 = 0, \ \ \xi = e_3, \ \eta = dZ \tag{8.8}$$

where $e_1 = x\frac{\partial}{\partial x}$, $e_2 = y\frac{\partial}{\partial y}$, and $e_3 = \frac{\partial}{\partial z}$. By calculations,

$$[\phi, \phi](e_i, e_j) - 2d\eta(e_i, e_j) = 0; \ \ 1 \le i < j \le 3 \tag{8.9}$$

which implies that the structure is normal. Let g be the pseudo-Riemannian metric defined by

$$\begin{aligned} g(e_1, e_1) &= \exp(z), \quad g(e_2, e_2) = -\exp(z), \quad g(e_3, e_3) = 1, \\ g(e_1, e_2) &= 0, \qquad\quad g(e_1, e_3) = 0, \qquad\quad g(e_2, e_3) = 0 \end{aligned} \tag{8.10}$$

Let ∇ Levi-Civita connection with metric g, then we given

$$[e_1, e_2] = 0, \quad [e_1, e_3] = 0, \quad [e_2, e_3] = 0.$$

We have

$$\begin{aligned} \nabla_{e_1} e_1 &= -\frac{1}{2}\exp(z)e_3, \ \ \nabla_{e_1} e_2 = 0, \qquad \nabla_{e_1} e_3 = \frac{e_1}{2} \\ \nabla_{e_2} e_1 &= 0, \qquad \nabla_{e_2} e_2 = \frac{1}{2}\exp(z)e_3, \ \ \nabla_{e_2} e_3 = \frac{e_2}{2} \\ \nabla_{e_3} e_1 &= \frac{e_1}{2}, \qquad \nabla_{e_3} e_2 = \frac{e_2}{2}, \qquad \nabla_{e_3} e_3 = 0 \end{aligned} \tag{8.11}$$

Therefore, the semi-symmetric metric connection on M is given by

$$\begin{aligned} \overline{\nabla}_{e_1} e_1 &= -\frac{3}{2}\exp(z)e_3, \quad \overline{\nabla}_{e_1} e_2 = 0, \quad \overline{\nabla}_{e_1} e_3 = \frac{3}{2}e_1 \\ \overline{\nabla}_{e_2} e_1 &= 0, \quad \overline{\nabla}_{e_2} e_2 = \frac{3}{2}\exp(z)e_3, \quad \overline{\nabla}_{e_2} e_3 = \frac{3}{2}e_2 \\ \overline{\nabla}_{e_3} e_1 &= \frac{e_1}{2}, \quad \overline{\nabla}_{e_3} e_2 = \frac{e_2}{2}, \qquad \overline{\nabla}_{e_3} e_3 = 0 \end{aligned} \tag{8.12}$$

Now, for $\xi = e_3$, above results satisfies

$$\bar{\nabla}_X \xi = (1 + \alpha)(X - \eta(X)\xi)$$

with $\alpha = \frac{1}{2}$. Consequently $M^3(\phi, \xi, \eta, g)$ is $\alpha-$para Kenmotsu manifold with semi-symmetric metric connection.

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Author information

Rajendra Prasad and Sushil Kumar, Department of Mathematics and Astronomy, Universiy of Lucknow, Lucknow, India.
E-mail: rp.manpur@rediffmail.com and sushilmath20@gmail.com

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