ADDITIVITY OF MULTIPLICATIVE ISOMORPHISMS IN GAMMA RINGS

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Abstract In this paper, some results given by Martindale III and Rickart are generalized to the $\Gamma$-rings. Using generalized Peirce decomposition of a $\Gamma$-ring given by Mukherjee, it is obtained that any multiplicative isomorphism of $\Gamma$-ring $M$ onto an arbitrary $\Gamma$-ring $N$ is additive.

1 Introduction and Preliminaries

Let $R$ and $S$ be arbitrary associative rings (not necessarily with identity elements). A one-to-one mapping $\sigma$ of $R$ onto $S$ such that $\sigma(xy) = \sigma(x)\sigma(y)$ for all $x, y \in R$ is called a multiplicative isomorphism of $R$ onto $S$. The question of when a multiplicative isomorphism is additive has been considered by Rickart [8] and also by Johnson [3]. Martindale III is generalized the main theorem of Rickart’s paper in [6] and removed a condition from the theorem. Martindale III, using Peirce decomposition of a ring, showed that any multiplicative isomorphism of $R$ onto an arbitrary ring $S$ is additive.

The concept of a $\Gamma$-ring was introduced by Nobusawa in [5] as a generalization of the ring theory and generalized by Barnes [1] as follows: Let $(M, +)$ and $(\Gamma, +)$ be additive Abelian groups. If there exists a mapping $M \times \Gamma \times M \rightarrow M$ (the image of $(a, \alpha, b)$ is denoted by $a\alpha b$ where $a, b \in M$ and $\alpha \in \Gamma$) satisfying the conditions

(i) $(x + y)\alpha z = x\alpha z + y\alpha z$,
(ii) $x\alpha(y + z) = x\alpha y + x\alpha z$,
(iii) $x(\alpha + \beta)z = x\alpha z + x\beta z$,
(iv) $x\alpha(y\beta z) = (x\alpha y)\beta z$

for all $x, y, z \in M$ and $\alpha, \beta \in \Gamma$, then $M$ is called a $\Gamma$-ring.

Every ring is a $\Gamma$-ring and many notions on the ring theory are generalized to the $\Gamma$-ring.

Mukherjee [7] is generalized and extended some results on $\Gamma$-rings obtained by some researchers.

In this paper, some results given by Martindale III and Rickart are generalized to the $\Gamma$-rings. Using generalized Peirce decomposition of a $\Gamma$-ring given by Mukherjee, it is obtained that any multiplicative isomorphism of $\Gamma$-ring $M$ onto an arbitrary $\Gamma$-ring $N$ is additive.

A $\Gamma$-ring $M$ is said to be a prime gamma ring if and only if $a\Gamma M\Gamma b = 0$ for $a, b \in M$ implies $a = 0$ or $b = 0$ and $M$ is called completely prime if and only if $\Gamma b = 0$ implies $a = 0$ or $b = 0$.

Theorem 1.1. [9] Let $M$ be a prime gamma ring. $U$ be a nonzero ideal of $M$. Then, for $a, b \in M$,

(i) if $U\Gamma a = 0$ or $aU\Gamma = 0$ then $a = 0$,
(ii) if $a\Gamma U\Gamma b = 0$ then $a = 0$ or $b = 0$.

An element $e$ in a $\Gamma$-ring is said to be an idempotent, if there exists $\gamma \in \Gamma$ such that $e\gamma e = e$.

In this case we also say that $e$ is $\gamma$-idempotent.

The following result can be termed as generalized Peirce Decomposition of a gamma ring $M$.

Theorem 1.2. [7] If $e$ is an idempotent of $M$ then

$$M = e\gamma M\gamma e \oplus e\gamma M\gamma (1 - e) \oplus (1 - e)\gamma M\gamma e \oplus (1 - e)\gamma M(1 - e).$$

In this Theorem, taking $e_1, e_2$ instead of $e$ and $1 - e$, respectively, we can write the Peirce
Decomposition of a gamma ring $M$ as

$$M = e_1 \gamma M \gamma e_1 \oplus e_1 \gamma M \gamma e_2 \oplus e_2 \gamma M \gamma e_1 \oplus e_2 \gamma M \gamma e_2.$$ 

Then letting $M_{ij} = e_i \gamma M \gamma e_j$, we may write $M$ as

$$M = M_{11} \oplus M_{12} \oplus M_{21} \oplus M_{22}.$$ 

It is also known that $e_i \gamma e_j = e_i$ if $i = j$, and $e_i \gamma e_j = 0$, if $i \neq j$.

2 The Main Part

**Definition 2.1.** Let $M$ and $N$ gamma rings. A one-to-one mapping $\varphi$ of $M$ onto $N$ such that $\varphi(x \gamma y) = \varphi(x) \varphi(y)$ for all $x, y \in M$ will be called a multiplicative isomorphism of $M$ onto $N$.

In this part, $e$ is an idempotent element of $M$ such that $e \neq 0$ and $e \neq 1$ ($M$ need not have an identity) and $\varphi$ is a multiplicative isomorphism of $M$ onto $N$. Also $e_1 = e$ and $e_2 = 1 - e$.

**Theorem 2.1.** Let $M$ and $N$ be two $\Gamma$-rings. Then $\varphi(0) = 0$.

**Proof.** Since $0 \in N$ and $\varphi$ is onto, $\varphi(x) = 0$ for some $x \in M$. Then we have

$$\varphi(0) = \varphi(0 \gamma x) = \varphi(0) \varphi(x) = 0.$$

**Theorem 2.2.** Let $M$ be a prime $\Gamma$-ring, $N$ be a $\Gamma$-ring. Then

(i) $\varphi(x_{11} + x_{12}) = \varphi(x_{11}) + \varphi(x_{12})$,

(ii) $\varphi(x_{11} + x_{21}) = \varphi(x_{11}) + \varphi(x_{21})$,

(iii) $\varphi(x_{22} + x_{12}) = \varphi(x_{22}) + \varphi(x_{12})$,

(iv) $\varphi(x_{22} + x_{21}) = \varphi(x_{22}) + \varphi(x_{21})$

where $x_{ij} \in M_{ij}$.

**Proof.** (i) For $x_{11}, x_{12} \in M$, since $\varphi(x_{11}) + \varphi(x_{12}) \in N$ and $\varphi$ is onto, we have an element $y \in M$ such that $\varphi(y) = \varphi(x_{11}) + \varphi(x_{12})$. Taking $x_{11} = e_1 \gamma m \gamma e_1$, $x_{12} = e_1 \gamma m \gamma e_2$ and $a_{11} = e_1 \gamma m \gamma e_1$, for $a_{11} \in M_{11}$, where $e_1$ is an idempotent element and $e_2 = 1 - e_1$, we have $(x_{11} + x_{12}) \gamma a_{11} = x_{11} \gamma a_{11} + x_{12} \gamma a_{11} = x_{11} \gamma a_{11} + x_{12} \gamma a_{11}$ since $x_{12} \gamma a_{11} = 0$. Then we get,

$$\varphi(y) \varphi(a_{11}) = \varphi(y) \gamma \varphi(a_{11})$$

$$= (\varphi(x_{11}) + \varphi(x_{12})) \gamma \varphi(a_{11})$$

$$= \varphi(x_{11}) \varphi(a_{11}) + \varphi(x_{12}) \gamma \varphi(a_{11})$$

$$= \varphi(x_{11} \gamma a_{11}) + \varphi(x_{12} \gamma a_{11})$$

$$= \varphi((x_{11} + x_{12}) \gamma a_{11}) + \varphi(0)$$

$$= \varphi((x_{11} + x_{12}) \gamma a_{11}).$$

Hence we obtain $y \gamma a_{11} = (x_{11} + x_{12}) \gamma a_{11}$ since $\varphi$ is one to one. Similarly we can see that $y \gamma a_{12} = (x_{11} + x_{12}) \gamma a_{12}$ for $a_{12} \in M_{12}$, $y \gamma a_{21} = (x_{11} + x_{12}) \gamma a_{21}$ for $a_{21} \in M_{21}$, $y \gamma a_{22} = (x_{11} + x_{12}) \gamma a_{22}$ for $a_{22} \in M_{22}$. Hence since $a_{11} + a_{12} + a_{21} + a_{22} = a \in M$, it is obtained that $(y - (x_{11} + x_{12})) \gamma M = 0$. Since $M$ is a prime $\Gamma$-ring, by Theorem 1.1, we have $y - (x_{11} + x_{12}) = 0$ or $y = x_{11} + x_{12}$. That is

$$\varphi(x_{11} + x_{12}) = \varphi(x_{11}) + \varphi(x_{12}).$$

(ii) It is obtained $M \gamma (y - (x_{11} + x_{12})) = 0$ with similar operations. Since $M$ is a prime $\Gamma$-ring, by Theorem 1.1, we get $\varphi(x_{11} + x_{21}) = \varphi(x_{11}) + \varphi(x_{21})$, consequently.

(iii) and (iv) is can be seen similarly.

**Theorem 2.3.** Let $M$ be a prime $\Gamma$-ring, $N$ be a $\Gamma$-ring. Then

$$\varphi(u_{12} + u_{12}) = \varphi(u_{12}) + \varphi(v_{12})$$

for all $u_{12}, v_{12} \in M_{12}$.
Proof: Since \( \varphi(u_1) + \varphi(v_1) \in N \) and \( \varphi \) is onto, we have an element \( y \in M \) such that
\( \varphi(y) = \varphi(u_1) + \varphi(v_1) \). For \( a_1 \in M_{11} \), taking \( a_1 = e_1 \gamma v_1 e_1 \), \( u_1 = e_1 \gamma v_1 e_1 e_2 \) and \( v_1 = e_1 \gamma v_1 e_2 \), we have \( u_1 \gamma v_1 a_1 = 0 \) and \( v_1 \gamma a_1 = 0 \). Hence
\[
\varphi(y) a_1 = \varphi(y) \varphi(a_1) = (\varphi(u_1) + \varphi(v_1)) \varphi(a_1) = \varphi(u_1) \varphi(a_1) + \varphi(v_1) \varphi(a_1) = \varphi(u_1 \gamma a_1) + \varphi(v_1 \gamma a_1) = \varphi(0) + \varphi(0) = 0.
\]
Since \( \varphi \) is one to one, we get \( y \gamma a_1 = 0 \). Similarly, we see that \( y \gamma a_1 = 0 \) for \( a_1 \in M_{12} \). Also for \( a_1 = e_1 \gamma v_1 e_1 \in M_{21} \), using the fact that \( e_1 \gamma a_1 = 0, e_1 \gamma v_1 \gamma a_1 = v_1 \gamma a_2 \) and \( u_1 \gamma v_1 \gamma a_2 = 0 \), we obtain
\[
\varphi(y) a_2 = \varphi(y) \varphi(a_2) = [\varphi(u_1) + \varphi(v_1)] \varphi(a_2) = \varphi(u_1) \varphi(a_2) + \varphi(v_1) \varphi(a_2) = \varphi(u_1 \gamma a_2) + \varphi(v_1 \gamma a_2) = \varphi(u_1 \gamma a_2 + v_1 \gamma a_2), \text{ by Theorem 2.2 (i) and (ii)}
\]
Hence since \( \varphi \) is one to one, we get \( y \gamma a_2 = (u_1 + v_1) \gamma a_2 \). Similarly we see that \( y \gamma a_2 = (u_2 + v_1) \gamma a_2 \). Therefore, it follows that \( (y - (u_2 + v_2)) \gamma M = 0 \), and so by Theorem 1.1, \( y \gamma (u_2 + v_2) = \varphi(u_2) + \varphi(v_2) \).

Theorem 2.4. Let \( M \) be a prime \( \Gamma \)-ring, \( N \) be a \( \Gamma \)-ring. Then \( \varphi(u_1 + v_1) = \varphi(u_1) + \varphi(v_1) \)
for all \( u_1, v_1 \in M_{11} \).

Proof. Since \( \varphi(u_1) + \varphi(v_1) \in N \) and \( \varphi \) is onto, we have an element \( y \in M \) such that
\( \varphi(y) = \varphi(u_1) + \varphi(v_1) \). For \( a_1 \in M_{12} \), we get, since \( u_1 \gamma a_1, v_1 \gamma a_1 \in M_{12} \),
\[
\varphi(y) a_1 = \varphi(y) \varphi(a_1) = (\varphi(u_1) + \varphi(v_1)) \varphi(a_1) = \varphi(u_1) \varphi(a_1) + \varphi(v_1) \varphi(a_1) = \varphi(u_1 \gamma a_1) + \varphi(v_1 \gamma a_1) = \varphi(u_1 \gamma a_1 + v_1 \gamma a_1), \text{ by Theorem 2.3.}
\]
Since \( \varphi \) is one to one, this shows that \( y a_1 = u_1 \gamma a_1 + v_1 \gamma a_1 \). That is, \( (y - (u_1 + v_1)) \gamma M = 0 \). Now let \( y = y_1 + y_2 + y_3 + y_4 \). Then since \( e_1 \gamma v_1 = u_1, e_1 \gamma v_1 = v_1, e_1 \gamma v_1 = y_1, e_1 \gamma v_1 = v_2 \), \( e_1 \gamma v_1 = 0 \) and \( e_1 \gamma v_1 = y_2 \), we obtain
\[
\varphi(y) = \varphi(u_1) + \varphi(v_1) = \varphi(e_1 \gamma u_1 e_1) + \varphi(e_1 \gamma v_1 e_1) = \varphi(e_1) \varphi(u_1) + \varphi(e_1) \varphi(v_1) = \varphi(e_1) \varphi(u_1 + v_1) = \varphi(e_1) \varphi(y) = \varphi(e_1) \varphi(y_1 + y_2 + y_3 + y_4) = \varphi(e_1) \varphi(y_1 + y_2 + y_3 + y_4).
\]
Since \( \varphi \) is one to one, we have \( y = y_1 + y_2 \). Furthermore, we get
\[
\varphi(y) = \varphi(u_1) + \varphi(v_1) = \varphi(x_1 \gamma e_1) + \varphi(x_1 \gamma e_1), \text{ since } u_1 \gamma e_1 = u_1, \text{ and } e_1 \gamma e_1 = v_1
\]
\[
= \varphi(u_1) \varphi(e_1) + \varphi(v_1) \varphi(e_1) = (\varphi(u_1) + \varphi(v_1)) \varphi(e_1) = \varphi(y_1 + y_2) \varphi(e_1) = \varphi(y_1 + y_2) \varphi(e_1).
\]
\[ \psi(y_{11} \gamma e_1 + y_{12} \gamma e_1) = \psi(y_{11}), \text{ since } y_{12} \gamma e_1 = 0. \]

Since \( \psi \) is one to one, we have \( y = y_{11} \in M_{11} \). Therefore \( y - (x_{11} + u_{11}) \in M_{11} \). Then, by theorem 1.1, \( (y - (x_{11} + u_{11})) \gamma M_{12} = 0 \) implies \( y - (u_{11} + v_{11}) = 0 \), that is, \( y = u_{11} + v_{11} \). So we obtain that \( \psi(u_{11} + v_{11}) = \psi(u_{11}) + \psi(v_{11}) \) for all \( u_{11}, v_{11} \in M_{11} \).

**Theorem 2.5.** Let \( M \) be a prime \( \Gamma \)-ring, \( N \) be a \( \Gamma \)-ring and \( \psi : M \rightarrow N \) be multiplicative isomorphism. Then \( \psi \) is additive on \( M_{11} + M_{12} \).

**Proof.** Let \( x, y \in M_{11} + M_{12} \). For any \( a, b \in M_{11} \) and \( c, d \in M_{12} \), we have \( x = a + c \), \( y = b + d \). Then
\[
\psi(x + y) = \psi((a + c) + (b + d)) = \psi((a + b) + (c + d)), a + b \in M_{11} \text{ and } c + d \in M_{12} = \psi(a + b) + \psi(c + d), \text{ by Theorem 2.2. (i), since } a + b \in M_{11}, c + d \in M_{12}
\]
\[ = \psi(a) + \psi(b) + \psi(c) + \psi(d), \text{ by Theorem 2.4. and Theorem 2.3.}
\]
\[ = \psi(a + c) + \psi(b + d), \text{ by Theorem 2.2. (i)}
\]
\[ = \psi(x) + \psi(y).
\]

**Theorem 2.6.** Let \( M \) be a prime \( \Gamma \)-ring, \( N \) be a \( \Gamma \)-ring. Then any multiplicative gamma isomorphism \( \psi \) of \( M \) onto \( N \) is additive.

**Proof:** Since \( \psi(x) + \psi(y) \in N \) for \( x, y \in M \) and \( \psi \) is onto, we have an element \( z \in M \) such that \( \psi(z) = \psi(x) + \psi(y) \).

Let \( t \in e \gamma M \). Since
\[
e \gamma M = e \gamma (e_1 \gamma M e_1 + e_2 \gamma M e_2 + e_2 \gamma M e_2)
\]
\[ = e_1 \gamma M e_1 + e_2 \gamma M e_2
\]
\[ = M_{11} + M_{12},
\]
we obtain
\[
\psi(t \gamma z) = \psi(t) \gamma \psi(z)
\]
\[ = \psi(t) \gamma (\psi(x) + \psi(y))
\]
\[ = \psi(t) \gamma \psi(x) + \psi(t) \gamma \psi(y)
\]
\[ = \psi(t x) + \psi(t y), \text{ by Theorem 2.5.}
\]
So, since \( \psi \) is one-to-one, we have \( t \gamma z = t \gamma x + t \gamma y \). Then \( t \gamma (z - (x + y)) = 0 \) or \( e \gamma M \gamma (z - (x + y)) = 0 \). By Theorem 1.1. (ii), we have \( z = x + y \). Then we obtained that \( \psi(x + y) = \psi(x) + \psi(y) \) for all \( x, y \in M \).

**Definition 2.2.** A gamma ring \( M \) is called a Boolean gamma ring if \( m \gamma m = m \) for all \( m \in M \), \( \gamma \in \Gamma \).

**Theorem 2.7.** Let \( M \) be a Boolean gamma ring. Then \( m = -m \) for all \( m \in M \).

**Proof.** Since \( M \) is a Boolean gamma ring, \( (m + m) \gamma (m + m) = m + m \). Then we have
\[
m + m = (m + m) \gamma (m + m)
\]
\[ = m \gamma m + m \gamma m + m \gamma m + m \gamma m
\]
\[ = m + m + m + m.
\]
Using the cancellation rule in the gamma ring \( M \), we get \( m + m = 0 \) or \( m = -m \).

**Theorem 2.8.** If \( M \) is a Boolean gamma ring, then \( M \) is commutative.

**Proof.** Since \( M \) is Boolean gamma ring, \( (m + n) \gamma (m + n) = m + n \). Then we have
\[
m + n = (m + n) \gamma (m + n)
\]
\[ = m \gamma m + m \gamma n + n \gamma m + n \gamma m
\]
\[ = m + m \gamma n + n \gamma m + n.
\]
Using the cancellation rule in the gamma ring \( M \), we get \( m \gamma n + n \gamma m = 0 \). Hence, by Theorem 2.7, we obtain \( m \gamma n = n \gamma m \).

**Theorem 2.9.** Let \( M \) be a Boolean \( \Gamma \)-ring and \( N \) arbitrary gamma ring. Then any multiplicative isomorphism \( \psi \) of \( M \) onto \( N \) is additive.

**Proof.** Let \( \psi \) multiplicative mapping from \( M \) onto \( N \). Then \( N \) is also a Boolean gamma ring.
Let $x$ and $y$ arbitrary elements in $M$. Since $\varphi(x) + \varphi(y) \in N$ and $\varphi$ is onto, there exist $m \in M$ so that $\varphi(m) = \varphi(x) + \varphi(y)$. The following equations can be obtained using mapping $\varphi$ is multiplicative,

\[
\varphi(x + y) = \varphi(x) + \varphi(y)
\]

(1)

\[
\varphi(x + y) = \varphi(x + y)\gamma\varphi(m)
\]

\[
= \varphi(x + y)\gamma(\varphi(x) + \varphi(y))
\]

\[
= \varphi(x + y)\gamma\varphi(x) + \varphi(x + y)\gamma\varphi(y)
\]

\[
= \varphi(x + y) + \varphi(x + y)\gamma y
\]

\[
= \varphi(x + y) + \varphi(x + y) + \varphi(y)
\]

\[
= \varphi(x + y) + \varphi(y)
\]

(2)

and similarly

\[
\varphi(y) = \varphi(y) + \varphi(x + y)
\]

(3)

Our aim is to show $\varphi(x + y) = \varphi(x) + \varphi(y)$ for all $x, y \in M$. In the above equalities, if $x + y = 0$ (so $y = 0$ by commutativity), we have for (1), (2) and (3)

\[
\varphi(x + y) = \varphi(x) + \varphi(y) = \varphi(m),
\]

(4)

\[
\varphi(x y) = \varphi(x)
\]

(5)

\[
\varphi(y) = \varphi(y)
\]

(6)

respectively. Since the mapping $\varphi$ is one-to-one, equations (4), (5) and (6) imply $x + y = m, y + y = m, x + y = x$ and $y + y = y$. It follows that $m = x + y$ and thus we obtain

\[
\varphi(x + y) = \varphi(x) + \varphi(y)
\]

(7)

If $x + y = y$, then we get the following for (1), (2) and (3), respectively,

\[
\varphi(x + y) = \varphi(x + y) + \varphi(y + y)
\]

(8)

\[
= \varphi(x + y) + \varphi(0)
\]

by Theorem 2.7.

\[
= \varphi(x + y)
\]

\[
\varphi(x + y) = \varphi(x) + \varphi(y) = \varphi(m),
\]

(9)

\[
\varphi(y + y) = \varphi(y + y) = 0.
\]

(10)

Since the mapping is one-to-one, equations (8), (9) and (10) imply

\[
x + y = x + y
\]

and $y + y = y$. Thus, since $m = x + y$, it follows that $\varphi(x + y) = \varphi(x) + \varphi(y)$.

Now, $x + y$ can be written as $x + y = (x + x + y) + (y + y)$ and also we have

\[
(x + x + y)\gamma(y + y) = 0
\]

by Theorem 2.7. So, using the result of the first case in the above, we obtain

\[
\varphi(x + y) = \varphi((x + x + y) + (y + y)) = \varphi(x + x + y) + \varphi(y + x + y)
\]

(11)

Furthermore, since $x + x + y = x + y$ and $y + x + y = x + y$ (by commutativity), using the result of the second case in the above, we have

\[
\varphi(x + x + y) = \varphi(x) + \varphi(x + y), \varphi(y + x + y) = \varphi(y) + \varphi(x + y)
\]

(12)

Substituting the obtained equations in (12) to (11), we obtain

\[
\varphi(x + y) = \varphi(x) + \varphi(y)
\]

for all $x, y \in M$. 

References


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