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A SUBCLASS OF HARMONIC UNIVALENT FUNCTIONS DEFINED BY A LINEAR OPERATOR

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Abstract. In the present paper, we investigate some basic properties of a subclass of harmonic functions defined by multiplier transformations. Such as, coefficient inequalities, distortion bounds and extreme points.

1 Introduction

Let H denote the family of continuous complex valued harmonic functions which are harmonic in the open unit disk $U = \{z : |z| < 1\}$ and let A be the subclass of H consisting of functions which are analytic in U. A function harmonic in U may be written as $f = h+\overline{g}$, where h and g are members of A. We call h the analytic part and g co-analytic part of f. A necessary and sufficient condition for f to be locally univalent and sense-preserving in U is that $|h'(z)| > |g'(z)|$ (see Clunie and Sheil-Small [4]). To this end, without loss of generality, we may write

$$h(z) = z + \sum_{k=2}^{\infty} a_k z^k \quad \text{and} \quad g(z) = \sum_{k=1}^{\infty} b_k z^k. \tag{1.1}$$

Let SH denote the family of functions $f = h + \overline{g}$ which are harmonic, univalent, and sense-preserving in U for which $f(0) = f_z(0) - 1 = 0$. One shows easily that the sense-preserving property implies that $|b_1| < 1$. The subclass SH^0 of SH consists of all functions in SH which have the additional property $f_{\bar{z}}(0) = 0$.

In 1984 Clunie and Sheil-Small [4] investigated the class SH as well as its geometric subclasses and obtained some coefficient bounds. Since then, there have been several related papers on SH and its subclasses. Also note that SH reduces to the class S of normalized analytic univalent functions in U, if the co-analytic part of f is identically zero.

For $f \in S$, the differential operator D^n $(n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\})$ of f was introduced by Salagean [8]. For $f = h + \overline{g}$ given by (1.1), Jahangiri et al. [7] defined the modified Salagean operator of f as

$$D^n f(z) = D^n h(z) + (-1)^n \overline{D^n g(z)},$$

where

$$D^n h(z) = z + \sum_{k=2}^{\infty} k^n a_k z^k \quad \text{and} \quad D^n g(z) = \sum_{k=1}^{\infty} k^n b_k z^k.$$

Next, for functions $f \in A$, Cho and Srivastava [2] defined multiplier transformations. For $f = h + \overline{g}$ given by (1.1), we define the modified multiplier transformation of f

$$I^0_{\gamma,\beta} f(z) = D^0 f(z) = h(z) + \overline{g(z)},$$

$$I^1_{\gamma,\beta} f(z) = \frac{\gamma D^0 f(z) + \beta D^1 f(z)}{\gamma + \beta} = \frac{\gamma(h(z) + \overline{g(z)}) + \beta(zh'(z) - \overline{zg'(z)})}{\gamma + \beta}, \tag{1.2}$$

$$I^n_{\gamma,\beta} f(z) = I^1_{\gamma,\beta}\left(I^{n-1}_{\gamma,\beta} f(z)\right). \; (n \in \mathbb{N}_0) \tag{1.3}$$

Where $\beta \geq \gamma \geq 0$. If f is given by (1.1), then from (1.2) and (1.3) we see that

$$I^n_{\gamma,\beta} f(z) = z + \sum_{k=2}^{\infty} \left(\frac{\beta k + \gamma}{\gamma + \beta}\right)^n a_k z^k + (-1)^n \sum_{k=1}^{\infty} \left(\frac{\beta k - \gamma}{\gamma + \beta}\right)^n \overline{b_k z^k}. \tag{1.4}$$

Also if f is given by (1.1), then we have

$$\begin{aligned} I^n_{\gamma,\beta} f(z) \quad &: \quad = f * \underbrace{\left(\phi_1(z) + \overline{\phi_2(z)}\right) * \ldots * \left(\phi_1(z) + \overline{\phi_2(z)}\right)}_{n \ times} \\ &= \quad h * \underbrace{\phi_1(z) * \ldots * \phi_1(z)}_{n \ times} + \overline{g * \underbrace{\phi_2(z) * \ldots * \phi_2(z)}_{n \ times}}, \end{aligned}$$

where "$*$" denotes the usual Hadamard product or convolution of power series and

$$\phi_1(z) = \frac{(\gamma + \beta)z - \gamma z^2}{(\gamma + \beta)(1 - z)^2}, \quad \phi_2(z) = \frac{(\gamma - \beta)z - \gamma z^2}{(\gamma + \beta)(1 - z)^2}.$$

By specializing the parametres γ and n, we obtain the following operators studied by various authors:

for $f \in A$,
(i) $I^n_{0,1} f(z) = D^n f(z)$ ([8]),
(ii) $I^n_{\lambda} f(z)$ ([2], [3],[5]),
(iii) $I^n_{1,1} = I^n f(z)$ ([11]),
for $f \in H$,
(iv) $I^n_{0,1} f(z) = D^n f(z)$ ([7]),
(v) $I^n_{\gamma,1} f(z) = I^n_{\gamma} f(z)$ ([12]).

Denote by $SH(\gamma, \beta, n, \alpha)$ the subclass of SH consisting of functions f of the form (1.1) that satisfy the condition

$$\Re \left(\frac{I^{n+1}_{\gamma,\beta} f(z)}{I^n_{\gamma,\beta} f(z)}\right) \geq \alpha, \quad 0 \leq \alpha < 1 \tag{1.5}$$

where $I^n_{\gamma,\beta} f(z)$ is defined by (1.4).

We let the subclass $\overline{SH}(\gamma, \beta, n, \alpha)$ consisting of harmonic functions $f_n = h + \overline{g}_n$ in SH so that h and g_n are of the form

$$h(z) = z - \sum_{k=2}^{\infty} a_k z^k, \ g_n(z) = (-1)^n \sum_{k=1}^{\infty} b_k z^k, \quad a_k, \ b_k \geq 0. \tag{1.6}$$

By suitably specializing the parameters, the classes $SH(\gamma, \beta, n, \alpha)$ reduces to the various subclasses of harmonic univalent functions. Such as,
(i) $SH(0, 1, 0, 0) = SH^*(0)$ ([1], [9], [10]),
(ii) $SH(0, 1, 0, \alpha) = SH^*(\alpha)$ ([6]),
(iii) $SH(0, 1, 1, 0) = KH(0)$ ([1], [9], [10]),
(iv) $SH(0, 1, 1, \alpha) = KH(\alpha)$ ([6]),
(v) $SH(0, 1, n, \alpha) = H(n, \alpha)$ ([7]),
(vi) $SH(\gamma, 1, n, \alpha) = SH(\gamma, n, \alpha)$ ([12]).

Define $SH^0(\gamma, \beta, n, \alpha) := SH(\gamma, \beta, n, \alpha) \cap SH^0$ and

$$\overline{SH}^0(\gamma, \beta, n, \alpha) := \overline{SH}(\gamma, \beta, n, \alpha) \cap SH^0.$$

2 Main results

Theorem 2.1. *Let $f = h + \overline{g}$ be so that h and g are given by (1.1) with $b_1 = 0$. Furthermore, let*

$$\sum_{k=2}^{\infty} \left(\frac{\beta k + \gamma}{\gamma + \beta}\right)^n \left(\frac{\beta k + \gamma}{\gamma + \beta} - \alpha\right) |a_k| + \sum_{k=2}^{\infty} \left(\frac{\beta k - \gamma}{\gamma + \beta}\right)^n \left(\frac{\beta k - \gamma}{\gamma + \beta} + \alpha\right) |b_k| \leq 1 - \alpha, \tag{2.1}$$

where $0 \leq \gamma \leq \beta/2$, $n \in \mathbb{N}_0$, $\frac{\gamma}{\gamma+\beta} \leq \alpha \leq \frac{\beta}{\gamma+\beta}$. Then f is sense-preserving, harmonic univalent in U and $f \in SH^0(\gamma, \beta, n, \alpha)$.

Proof. If $z_1 \neq z_2$,

$$\begin{aligned}
\left|\frac{f(z_1)-f(z_2)}{h(z_1)-h(z_2)}\right| &\geq 1-\left|\frac{g(z_1)-g(z_2)}{h(z_1)-h(z_2)}\right| = 1-\left|\frac{\sum_{k=2}^{\infty} b_k\left(z_1^k-z_2^k\right)}{(z_1-z_2)+\sum_{k=2}^{\infty} a_k\left(z_1^k-z_2^k\right)}\right| \\
&> 1-\frac{\sum_{k=2}^{\infty} k\,|b_k|}{1-\sum_{k=2}^{\infty} k\,|a_k|} \geq 1-\frac{\sum_{k=2}^{\infty}\frac{\left(\frac{\beta k-\gamma}{\gamma+\beta}\right)^n\left(\frac{\beta k-\gamma}{\gamma+\beta}+\alpha\right)}{1-\alpha}|b_k|}{1-\sum_{k=2}^{\infty}\frac{\left(\frac{\beta k+\gamma}{\gamma+\beta}\right)^n\left(\frac{\beta k+\gamma}{\gamma+\beta}-\alpha\right)}{1-\alpha}|a_k|} \geq 0,
\end{aligned}$$

which proves univalence. Note that f is sense preserving in U. This is because

$$\begin{aligned}
|h'(z)| &\geq 1-\sum_{k=2}^{\infty} k\,|a_k|\,|z|^{k-1} > 1-\sum_{k=2}^{\infty}\frac{\left(\frac{\beta k+\gamma}{\gamma+\beta}\right)^n\left(\frac{\beta k+\gamma}{\gamma+\beta}-\alpha\right)}{1-\alpha}|a_k| \\
&\geq \sum_{k=2}^{\infty}\frac{\left(\frac{\beta k-\gamma}{\gamma+\beta}\right)^n\left(\frac{\beta k-\gamma}{\gamma+\beta}+\alpha\right)}{1-\alpha}|b_k| > \sum_{k=2}^{\infty} k\,|b_k|\,|z|^{k-1} \geq |g'(z)|\,.
\end{aligned}$$

Using the fact that $\Re(w) \geq \alpha$ if and only if $|1-\alpha+w| \geq |1+\alpha-w|$, it suffices to show that

$$\left|(1-\alpha)I_{\gamma,\beta}^{n} f(z)+I_{\gamma,\beta}^{n+1} f(z)\right|-\left|(1+\alpha)I_{\gamma,\beta}^{n} f(z)-I_{\gamma,\beta}^{n+1} f(z)\right| \geq 0. \tag{2.2}$$

Substituting for $I_{\gamma,\beta}^{n} f(z)$ and $I_{\gamma,\beta}^{n+1} f(z)$ in (2.2), we obtain

$$\begin{aligned}
&\left|(1-\alpha)I_{\gamma,\beta}^{n} f(z)+I_{\gamma,\beta}^{n+1} f(z)\right|-\left|(1+\alpha)I_{\gamma,\beta}^{n} f(z)-I_{\gamma,\beta}^{n+1} f(z)\right| \\
\geq\ & 2(1-\alpha)\,|z|-\sum_{k=2}^{\infty}\left(\frac{\beta k+\gamma}{\gamma+\beta}\right)^n\left(\frac{\beta k+\gamma}{\gamma+\beta}+1-\alpha\right)|a_k|\,|z|^k \\
&-\sum_{k=2}^{\infty}\left(\frac{\beta k-\gamma}{\gamma+\beta}\right)^n\left(\frac{\beta k-\gamma}{\gamma+\beta}-1+\alpha\right)|b_k|\,|z|^k \\
&-\sum_{k=2}^{\infty}\left(\frac{\beta k+\gamma}{\gamma+\beta}\right)^n\left(\frac{\beta k+\gamma}{\gamma+\beta}-1-\alpha\right)|a_k|\,|z|^k \\
&-\sum_{k=2}^{\infty}\left(\frac{\beta k-\gamma}{\gamma+\beta}\right)^n\left(\frac{\beta k-\gamma}{\gamma+\beta}+1+\alpha\right)|b_k|\,|z|^k \\
>\ & 2(1-\alpha)\,|z|\left\{1-\sum_{k=2}^{\infty}\left(\frac{\beta k+\gamma}{\gamma+\beta}\right)^n\left(\frac{\beta k+\gamma}{\gamma+\beta}-\alpha\right)|a_k|\right. \\
&\left.-\sum_{k=2}^{\infty}\left(\frac{\beta k-\gamma}{\gamma+\beta}\right)^n\left(\frac{\beta k-\gamma}{\gamma+\beta}+\alpha\right)|b_k|\right\}.
\end{aligned}$$

This last expression is non-negative by (2.1), and so the proof is complete. □

Theorem 2.2. *Let* $f_n = h+\overline{g}_n$ *be given by (1.6) with* $b_1 = 0$. *Then* $f_n \in \overline{SH}^0(\gamma,\beta,n,\alpha)$ *if and only if*

$$\sum_{k=2}^{\infty}\left(\frac{\beta k+\gamma}{\gamma+\beta}\right)^n\left(\frac{\beta k+\gamma}{\gamma+\beta}-\alpha\right)a_k+\sum_{k=2}^{\infty}\left(\frac{\beta k-\gamma}{\gamma+\beta}\right)^n\left(\frac{\beta k-\gamma}{\gamma+\beta}+\alpha\right)b_k \leq 1-\alpha, \tag{2.3}$$

where $0 \leq \gamma \leq \beta/2$, $n \in \mathbb{N}_0$, $\frac{\gamma}{\gamma+\beta} \leq \alpha \leq \frac{\beta}{\gamma+\beta}$.

Proof. The "if" part follows from Theorem 1 upon noting that $\overline{SH}^0(\gamma,\beta,n,\alpha) \subset SH^0(\gamma,\beta,n,\alpha)$. For the "only if" part, we show that $f_n \notin \overline{SH}^0(\gamma,\beta,n,\alpha)$ if the condition (2.3) does not hold. Note that a necessary and sufficient condition for $f_n = h + \overline{g}_n$ given by (1.6), to be in $\overline{SH}^0(\gamma,\beta,n,\alpha)$ is that the condition (1.5) to be satisfied. This is equivalent to

$$\Re\left\{\frac{(1-\alpha)z - \sum\limits_{k=2}^{\infty}\left(\frac{\beta k+\gamma}{\gamma+\beta}\right)^n\left(\frac{\beta k+\gamma}{\gamma+\beta}-\alpha\right)a_k z^k - \sum\limits_{k=2}^{\infty}\left(\frac{\beta k-\gamma}{\gamma+\beta}\right)^n\left(\frac{\beta k-\gamma}{\gamma+\beta}+\alpha\right)b_k \overline{z}^k}{z - \sum\limits_{k=2}^{\infty}\left(\frac{\beta k+\gamma}{\gamma+\beta}\right)^n a_k z^k + \sum\limits_{k=2}^{\infty}\left(\frac{\beta k-\gamma}{\gamma+\beta}\right)^n b_k \overline{z}^k}\right\} \geq 0.$$

The above condition must hold for all values of z, $|z| = r < 1$. Upon choosing the values of z on the positive real axis where $0 \leq z = r < 1$ we must have

$$\frac{(1-\alpha) - \sum\limits_{k=2}^{\infty}\left(\frac{\beta k+\gamma}{\gamma+\beta}\right)^n\left(\frac{\beta k+\gamma}{\gamma+\beta}-\alpha\right)a_k r^{k-1} - \sum\limits_{k=2}^{\infty}\left(\frac{\beta k-\gamma}{\gamma+\beta}\right)^n\left(\frac{\beta k-\gamma}{\gamma+\beta}+\alpha\right)b_k r^{k-1}}{1 - \sum\limits_{k=2}^{\infty}\left(\frac{\beta k+\gamma}{\gamma+\beta}\right)^n a_k r^{k-1} + \sum\limits_{k=2}^{\infty}\left(\frac{\beta k-\gamma}{\gamma+\beta}\right)^n b_k r^{k-1}} \geq 0 \quad (2.4)$$

If the condition (2.3) does not hold, then the numerator in (2.4) is negative for r sufficiently close to 1. Hence there exist $z_0 = r_0$ in $(0,1)$ for which the quotient in (2.4) is negative. This contradicts the required condition for $f_n \in \overline{SH}^0(\gamma,\beta,n,\alpha)$ and so the proof is complete. □

Theorem 2.3. *Let f_n be given by (1.6). Then $f_n \in \overline{SH}^0(\gamma,\beta,n,\alpha)$ if and only if*

$$f_n(z) = \sum_{k=1}^{\infty}\left(X_k h_k(z) + Y_k g_{n_k}(z)\right),$$

where

$$h_1(z) = z, \quad h_k(z) = z - \frac{1-\alpha}{\left(\frac{\beta k+\gamma}{\gamma+\beta}\right)^n\left(\frac{\beta k+\gamma}{\gamma+\beta}-\alpha\right)}z^k \ (k = 2,3,...),$$

and

$$g_{n_1}(z) = z, \quad g_{n_k}(z) = z + (-1)^n\frac{1-\alpha}{\left(\frac{\beta k-\gamma}{\gamma+\beta}\right)^n\left(\frac{\beta k-\gamma}{\gamma+\beta}+\alpha\right)}\overline{z}^k \ (k = 2,3,...),$$

$$X_k \geq 0, Y_k \geq 0, \sum_{k=1}^{\infty}(X_k + Y_k) = 1, \ 0 \leq \gamma \leq \beta/2, n \in \mathbb{N}_0, \frac{\gamma}{\gamma+\beta} \leq \alpha \leq \frac{\beta}{\gamma+\beta}.$$

In particular, the extreme points of $\overline{SH}^0(\gamma,\beta,n,\alpha)$ are $\{h_k\}$ and $\{g_{n_k}\}$.

Proof. For functions f_n of the form (1.6) we have

$$\begin{aligned} f_n(z) &= \sum_{k=1}^{\infty}\left(X_k h_k(z) + Y_k g_{n_k}(z)\right) \\ &= \sum_{k=1}^{\infty}(X_k + Y_k)z - \sum_{k=2}^{\infty}\frac{1-\alpha}{\left(\frac{\beta k+\gamma}{\gamma+\beta}\right)^n\left(\frac{\beta k+\gamma}{\gamma+\beta}-\alpha\right)}X_k z^k \\ &\quad + (-1)^n\sum_{k=2}^{\infty}\frac{1-\alpha}{\left(\frac{\beta k-\gamma}{\gamma+\beta}\right)^n\left(\frac{\beta k-\gamma}{\gamma+\beta}+\alpha\right)}Y_k \overline{z}^k. \end{aligned}$$

Then

$$\begin{aligned} &\sum_{k=2}^{\infty}\frac{\left(\frac{\beta k+\gamma}{\gamma+\beta}\right)^n\left(\frac{\beta k+\gamma}{\gamma+\beta}-\alpha\right)}{1-\alpha}\left(\frac{1-\alpha}{\left(\frac{\beta k+\gamma}{\gamma+\beta}\right)^n\left(\frac{\beta k+\gamma}{\gamma+\beta}-\alpha\right)}X_k\right) \\ &+\sum_{k=2}^{\infty}\frac{\left(\frac{\beta k-\gamma}{\gamma+\beta}\right)^n\left(\frac{\beta k-\gamma}{\gamma+\beta}+\alpha\right)}{1-\alpha}\left(\frac{1-\alpha}{\left(\frac{\beta k-\gamma}{\gamma+\beta}\right)^n\left(\frac{\beta k-\gamma}{\gamma+\beta}+\alpha\right)}Y_k\right) \\ = &\sum_{k=2}^{\infty}X_k + \sum_{k=2}^{\infty}Y_k = 1 - X_1 - Y_1 \leq 1 \end{aligned}$$

and so $f_n \in \overline{SH}^0(\gamma,\beta,n,\alpha)$. Conversely, if $f_n \in \overline{SH}^0(\gamma,\beta,n,\alpha)$, then

$$a_k \leq \frac{1-\alpha}{\left(\frac{\beta k+\gamma}{\gamma+\beta}\right)^n\left(\frac{\beta k+\gamma}{\gamma+\beta}-\alpha\right)}$$

and

$$b_k \leq \frac{1-\alpha}{\left(\frac{\beta k-\gamma}{\gamma+\beta}\right)^n\left(\frac{\beta k-\gamma}{\gamma+\beta}+\alpha\right)}.$$

Set

$$X_k = \frac{\left(\frac{\beta k+\gamma}{\gamma+\beta}\right)^n\left(\frac{\beta k+\gamma}{\gamma+\beta}-\alpha\right)}{1-\alpha}a_k,\ (k=2,3,...)$$

$$Y_k = \frac{\left(\frac{\beta k-\gamma}{\gamma+\beta}\right)^n\left(\frac{\beta k-\gamma}{\gamma+\beta}+\alpha\right)}{1-\alpha}b_k,\ (k=2,3,...)$$

and

$$X_1+Y_1 = 1-\left(\sum_{k=2}^{\infty}X_k+Y_k\right)$$

where $X_k,\ Y_k \geq 0$. Then, as required, we obtain

$$f_n(z) = (X_1+Y_1)z+\sum_{k=2}^{\infty}X_kh_k(z)+\sum_{k=2}^{\infty}Y_kg_{n_k}(z) = \sum_{k=1}^{\infty}\left(X_kh_k(z)+Y_kg_{n_k}(z)\right).$$

□

Theorem 2.4. *Let $f_n \in \overline{SH}^0(\gamma,\beta,n,\alpha)$. Then for $|z| = r < 1$ and $0 \leq \gamma \leq \beta/2$, $n \in \mathbb{N}_0$, $\frac{\gamma}{\gamma+\beta} \leq \alpha \leq \frac{\beta}{\gamma+\beta}$ we have*

$$|f_n(z)| \leq r+\frac{(1-\alpha)}{\left(\frac{2\beta+\gamma}{\gamma+\beta}\right)^n\left(\frac{2\beta+\gamma}{\gamma+\beta}-\alpha\right)}r^2,$$

and

$$|f_n(z)| \geq r-\frac{(1-\alpha)}{\left(\frac{2\beta+\gamma}{\gamma+\beta}\right)^n\left(\frac{2\beta+\gamma}{\gamma+\beta}-\alpha\right)}r^2.$$

Proof. We only prove the right hand inequality. The proof for the left hand inequality is similar and will be omitted. Let $f_n \in \overline{SH}^0(\gamma,\beta,n,\alpha)$. Taking the absolute value of f_n we have

$$\begin{aligned}
|f_n(z)| &\leq r+\sum_{k=2}^{\infty}(a_k+b_k)\,r^2 \\
&\leq r+\frac{(1-\alpha)\,r^2}{\left(\frac{2\beta+\gamma}{\gamma+\beta}\right)^n\left(\frac{2\beta+\gamma}{\gamma+\beta}-\alpha\right)}\sum_{k=2}^{\infty}\left\{\frac{\left(\frac{\beta k+\gamma}{\gamma+\beta}\right)^n\left(\frac{\beta k+\gamma}{\gamma+\beta}-\alpha\right)}{1-\alpha}a_k+\frac{\left(\frac{\beta k-\gamma}{\gamma+\beta}\right)^n\left(\frac{\beta k-\gamma}{\gamma+\beta}+\alpha\right)}{1-\alpha}b_k\right\} \\
&\leq r+\frac{(1-\alpha)}{\left(\frac{2\beta+\gamma}{\gamma+\beta}\right)^n\left(\frac{2\beta+\gamma}{\gamma+\beta}-\alpha\right)}r^2.
\end{aligned}$$

The following covering result follows from the left hand inequality in Theorem 2.4. □

Corollary 2.5. *Let f_n of the form (1.6) be so that $f_n \in \overline{SH}^0(\gamma,\beta,n,\alpha)$, where $0 \leq \gamma \leq \beta/2$, $n \in \mathbb{N}_0$, $\frac{\gamma}{\gamma+\beta} \leq \alpha \leq \frac{\beta}{\gamma+\beta}$. Then*

$$\left\{w : |w| < 1-\frac{(1-\alpha)}{\left(\frac{2\beta+\gamma}{\gamma+\beta}\right)^n\left(\frac{2\beta+\gamma}{\gamma+\beta}-\alpha\right)}\right\} \subset f_n(U).$$

Theorem 2.6. *The class* $\overline{SH}^0(\gamma,\beta,n,\alpha)$ *is closed under convex combinations.*

Proof. Let $f_{n_i}\in\overline{SH}^0(\gamma,\beta,n,\alpha)$ for $i=1,2,...$, where f_{n_i} is given by

$$f_{n_i}(z)=z-\sum_{k=2}^{\infty}a_{k_i}z^k+(-1)^n\sum_{k=2}^{\infty}b_{k_i}\overline{z}^k.$$

Then by (2.3),

$$\sum_{k=2}^{\infty}\frac{\left(\frac{\beta k+\gamma}{\gamma+\beta}\right)^n\left(\frac{\beta k+\gamma}{\gamma+\beta}-\alpha\right)}{1-\alpha}a_{k_i}+\sum_{k=2}^{\infty}\frac{\left(\frac{\beta k-\gamma}{\gamma+\beta}\right)^n\left(\frac{\beta k-\gamma}{\gamma+\beta}+\alpha\right)}{1-\alpha}b_{k_i}\le 1. \tag{2.5}$$

For $\sum_{i=1}^{\infty}t_i=1$, $0<t_i<1$, the convex combination of f_{n_i} may be written as

$$\sum_{i=1}^{\infty}t_if_{n_i}(z)=z-\sum_{k=2}^{\infty}\left(\sum_{i=1}^{\infty}t_ia_{k_i}\right)z^k+(-1)^n\sum_{k=2}^{\infty}\left(\sum_{i=1}^{\infty}t_ib_{k_i}\right)\overline{z}^k.$$

Then by (2.5),

$$\begin{aligned}&\sum_{k=2}^{\infty}\frac{\left(\frac{\beta k+\gamma}{\gamma+\beta}\right)^n\left(\frac{\beta k+\gamma}{\gamma+\beta}-\alpha\right)}{1-\alpha}\left(\sum_{i=1}^{\infty}t_ia_{k_i}\right)+\sum_{k=2}^{\infty}\frac{\left(\frac{\beta k-\gamma}{\gamma+\beta}\right)^n\left(\frac{\beta k-\gamma}{\gamma+\beta}+\alpha\right)}{1-\alpha}\left(\sum_{i=1}^{\infty}t_ib_{k_i}\right)\\&=\sum_{i=1}^{\infty}t_i\left\{\sum_{k=2}^{\infty}\frac{\left(\frac{\beta k+\gamma}{\gamma+\beta}\right)^n\left(\frac{\beta k+\gamma}{\gamma+\beta}-\alpha\right)}{1-\alpha}a_{k_i}+\frac{\left(\frac{\beta k-\gamma}{\gamma+\beta}\right)^n\left(\frac{\beta k-\gamma}{\gamma+\beta}+\alpha\right)}{1-\alpha}b_{k_i}\right\}\\&\le\sum_{i=1}^{\infty}t_i=1.\end{aligned}$$

This is the condition required by (2.3) and so $\sum_{i=1}^{\infty}t_if_{n_i}(z)\in\overline{SH}^0(\gamma,\beta,n,\alpha)$. □

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