Palestine Journal of Mathematics

Vol. 6(Special Issue: II) (2017) , 348–354

SOME PROPERTIES OF GENERALIZED SEMI PSEUDO RICCI SYMMETRIC MANIFOLD

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Communicated by Zafar Ahsan

Abstract. Object of this paper is to find some properties of generalized semi pseudo Ricci symmetric manifold (denoted by $G(SPRS)_n$). At last we have given an example of this manifold.

MSC 2010 Classifications: 53C25 .

Keywords: generalised semi pseudo Ricci symmetric manifold, semi pseudo Ricci symmetric manifold, pseudo Ricci symmetric manifold, Ricci tensor of Codazzi type, conformal curvature tensor, torse forming vector field.

1. Introduction

The notion of locally symmetric and Ricci symmetric Riemannian manifold began with work of Cartan[10] and Eisenhert[8] respectively. A Riemannian manifold is said to be locally symmetric if its curvature tensor R satisfies the relation

$$\nabla R = 0 \tag{0.1}$$

where ∇ is the operator of covariant differentiation w.r.t. the metric tensor g. Again a Ricci symmetric manifold is a Riemannian manifold with the Ricci tensor S of type (0,2) satisfying

$$\nabla S = 0. \tag{0.2}$$

After them these notions have flowed in several branches such as recurrent manifold, Ricci-recurrent manifold, semi-symmetric manifold, pseudo-symmetric manifold[4], pseudo Ricci-symmetric manifold[6] and so on.

A non flat Riemannian manifold $(M^n, g), (n > 2)$ is said to be pseudo Ricci symmetric manifold $((PRS)_n)$[5] if Ricci tensor S is not identically zero and satisfies

$$(\nabla_X S)(Y, Z) = 2A(X)S(Y, Z) + A(Y)S(X, Z) + A(Z)S(X, Y) \tag{0.3}$$

where A is nonzero 1-form satisfying

$$g(X, U) = A(X) \tag{0.4}$$

for a particular vector field U.

A non flat Riemannian manifold $(M^n, g), (n > 2)$ is said to be semi pseudo Ricci symmetric manifold $((SPRS)_n)$[2] if Ricci tensor S is not identically zero and satisfies

$$(\nabla_X S)(Y, Z) = A(Y)S(X, Z) + A(Z)S(X, Y) \tag{0.5}$$

where A is nonzero 1-form satisfying

$$g(X, U) = A(X) \tag{0.6}$$

for a particular vector field U.

A non flat Riemannian manifold $(M^n, g), (n > 2)$ is said to be generalised semi pseudo Ricci symmetric manifold $(G(SPRS)_n)$[1] if Ricci tensor S is not identically zero and satisfies

$$(\nabla_X S)(Y, Z) = A(Y)S(X, Z) + B(Z)S(X, Y) \tag{0.7}$$

where A, B are nonzero 1-forms satisfying

$$g(X, V) = A(X) \tag{0.8}$$

$$g(X, W) = B(X) \tag{0.9}$$

for particular vector fields V, W respectively.

From the above definition we observe that when $\delta = A - B$ is identically zero, $G(SPRS)_n$ reduces $(SPRS)_n$.

A vector field is said to be a torse forming vector field[8] if there is a nonzero scalar a and a nonzero 1-form ω such that

$$\nabla_X P = aX + \omega(X)P \tag{0.10}$$

where $X \in \chi(M)$.

The Ricci tensor of a Riemannian manifold is said to be of Codazzi type if it satisfies the following

$$(\nabla_X S)(Y, Z) - (\nabla_Y S)(X, Z) = 0 \tag{0.11}$$

where $X, Y, Z \in \chi(M)$.
The above relations will be used in the followings.

2. $G(SPRS)_n$ and its scalar curvature

Let Q be the symmetric endomorphism of the tangent space at each point of a $G(SPRS)_n$ corresponding to the Ricci tensor S. Then

$$g(QX, Y) = S(X, Y). \tag{0.12}$$

Now from (7) we can get

$$(\nabla_X S)(Y, Z) - (\nabla_Y S)(X, Z) = A(Y)S(X, Z) - A(X)S(Y, Z). \tag{0.13}$$

Contracting above with respect to Y and Z, we have

$$dr(X) = 2\bar{A}(X) - 2A(X)r \tag{0.14}$$

where $\bar{A}(X) = A(QX)$.

Similarly we can obtain

$$dr(X) = 2\bar{B}(X) - 2B(X)r \tag{0.15}$$

where $\bar{B}(X) = B(QX)$.

Again contracting (7) with respect to Y and Z, we get

$$dr(X) = \bar{A}(X) + \bar{B}(X). \tag{0.16}$$

Hence from (14) and (16), we get

$$\bar{\delta}(X) = 2A(X)r \tag{0.17}$$

where $\bar{\delta}(X) = \delta(QX)$.

Similarly we have

$$\bar{\delta}(X) = 2B(X)r. \tag{0.18}$$

Since $A(X), B(X) \neq 0$,

$$\bar{\delta} = 0 \Leftrightarrow r = 0. \tag{0.19}$$

So we can conclude that,

Theorem 1: *In a $G(SPRS)_n$ $\bar{\delta}$ is identically zero iff its scalar curvature is zero.*

Now if r is constant then $dr(X) = 0$. So from (14) we have

$$\bar{A}(X) = A(X)r \tag{0.20}$$

that is,

$$S(X, V) = rg(X, V). \tag{0.21}$$

Similarly,

$$S(X, W) = rg(X, W) \tag{0.22}$$

Thus we can state,

Theorem 2: *If the scalar curvature r of $G(SPRS)_n$ is constant, r is an eigen value of the Ricci tensor corresponding to the eigen vector V and W.*

Again r is zero implies from (14),

$$\bar{A}(X) = 0 \tag{0.23}$$

$\forall X \in \chi(M)$.

But since $A(X)$ is non-zero 1-form, then Q must not be surjective.

So our conclusion is,

Theorem 3: *If the scalar curvature r of $G(SPRS)_n$ is zero, the symmetric endomorphism of the tangent space of the manifold at each point corresponding to the Ricci tensor S never be surjective.*

Now $ddr(X, Y) = 0$ implies from (14) and (16),

$$d\bar{A}(X, Y) = rdA(X, Y) \tag{0.24}$$

$$d\bar{A}(X, Y) + d\bar{B}(X, Y) = 0 \tag{0.25}$$

which give us the following theorems,

Theorem 4: *(i) $\bar{A}$ is closed $\Leftrightarrow$ $\bar{B}$ is closed.*
(ii) If $r = 0$ or A is closed then $\bar{A}$ and $\bar{B}$ are both closed.
(iii) If r is non-zero constant, any of $\bar{A}$ and $\bar{B}$ is closed implies A is closed.

Again since S is symmetric then, from (7) we can obtain,

$$\delta(Y)S(X, Z) = \delta(Z)S(X, Y). \tag{0.26}$$

Now contracting above with respect to X, Z we have,

$$\delta(Y)r = \delta(QY). \tag{0.27}$$

Then we have,

$$S(X,Z) = rT(X)T(Z) \tag{0.28}$$

where $T(X) = \frac{\delta(X)}{\sqrt{\delta(U)}}$, $\delta(X) = g(X,U)$, $U = V - W$.

So we can state,

Theorem 5: *In a $G(SPRS)_n$ (which is not a $(SPRS)_n$)) the Ricci tensor is of the form $S(X,Y) = rT(X)T(Y)$ where r is the scalar curvature and T is a 1-form such that $T(X) = g(X,\xi)$ for some unit vector ξ.*

Again from (27),

$$rg(X,U) = g(QX,U) = S(X,U). \tag{0.29}$$

This leads the following,

Theorem 6: *In a $G(SPRS)_n$ (which is not a $(SPRS)_n$)) the scalar curvature r is an eigen value of the Ricci tensor corresponding to the eigen vector $U = V - W$.*

3.Ricci tensor and torse forming vector field

Now let the scalar curvature of $G(SPRS)_n$ is zero. Then the vector field U satisfies the following,

$$S(X,U) = 0 \;\; \forall X \in \chi(M). \tag{0.30}$$

Now we know that,

$$(\nabla_X S)(Y,Z) = \nabla_X S(Y,Z) - S(\nabla_X Y,Z) - S(Y,\nabla_X Z). \tag{0.31}$$

In virtue of (7) and (30), the above equation reduces to

$$B(U)S(X,Y) + S(Y,\nabla_X U) = 0. \tag{0.32}$$

Let us now suppose that U is a torse forming vector field given by (10).
Using (10) and (32) we have,

$$a + B(U) = 0. \tag{0.33}$$

Similarly we can show,

$$a + A(U) = 0, \tag{0.34}$$

which implies using (33) that,

$$\delta(U) = 0. \tag{0.35}$$

Hence we can state that,

Theorem 7: *If the scalar curvature of a $G(SPRS)_n$ is zero and U is a torse forming vector field given by (10), then $\delta(U)$ is equal to zero and the scalar a associated to U is equal to -$A(U)$.*

Let

$$f = \frac{1}{2} g(U,U) \tag{0.36}$$

be the energy of the torse forming vector field U, given by (10) and let

$$g(\xi,Y) = \omega(Y), \tag{0.37}$$

$\forall Y \in \chi(M)$.
From (36) and (37), we get

$$df(Y) = g(aU + 2f\xi, Y) \tag{0.38}$$

and hence,

$$gradf = aU + 2f\xi = -A(U)U + A(U)\xi. \tag{0.39}$$

Now if f is constant, then from (37) we get,

$$A(U)[\xi - U] = 0. \tag{0.40}$$

Now since $A(U) \neq 0$, then $U = \xi$,
hence

$$\omega(X) = A(X), \tag{0.41}$$

$\forall X \in \chi(M)$.
Then A is closed implies ω is closed. Then U is concircular.
Thus we can state the following,

Theorem 8: *If in a $G(SPRS)_n$ the 1-form A is closed and scalar curvature is zero, then the torse forming vector field U of which energy is constant, is concircular.*

4.$G(SPRS)_n$ with Ricci tensor of Codazzi type

Let $e_i : 1 \leq i \leq n$ be the orthonormal basis of the tangent space at each point of the manifold. Then by the definition of S we have,

$$S(e_i, e_i) = \sum_{k=1}^{n} g(R(e_k, e_i)e_i, e_k) = \sum_{k=1}^{n} \prod_{ik} \tag{0.42}$$

and

$$S(e_i, e_j) = 0 \quad if \quad i \neq j \tag{0.43}$$

where $\prod_{ij}$ is the sectional curvature of a plane spanned by the vectors e_i and e_j.
Now let the Ricci tensor of $G(SPRS)_n$ is of Codazzi type. Then we have from (11),

$$(\nabla_X S)(Y, Z) - (\nabla_Y S)(X, Z) = 0. \tag{0.44}$$

Putting $X, Z = e_i, Y = e_j$ in above we can obtain,

$$A(e_j)S(e_i, e_i) = 0. \tag{0.45}$$

Since $A(e_j) \neq 0$,

$$S(e_i, e_i) = 0. \tag{0.46}$$

Then

$$S(X, Y) = 0 \tag{0.47}$$

$\forall$ vector field X, Y, which is not admissible for $G(SPRS)_n$.
Thus we can state,

Theorem 9: *The Ricci tensor of $G(SPRS)_n$ is never of Codazzi type.*

5.Conformally flat $G(SPRS)_n$

From the paper[2] we have already known that $(SPRS)_n$ can not be Einstein manifold. In this section we assume the manifold $G(SPRS)_n$ (which is not $(SPRS)_n$) is conformally flat. Hence we have,

$$(\nabla_X S)(Y, Z) - (\nabla_Y S)(X, Z) = \frac{1}{2(n-1}[g(Y, Z)dr(X) - g(X, Z)dr(Y)]. \tag{0.48}$$

Using $X, Y = e_i, Z = e_j$ in (35) and then taking summation over $i = 1$ to n on both side of (35) we have,

$$S(e_i, e_i) = \frac{r}{n}. \tag{0.49}$$

Thus we can state the following:

Theorem 10: *A conformally flat $G(SPRS)_n$ (which is not $(SPRS)_n$) is Einstein manifold.*

6. Example of $G(SPRS)_n$

Let us consider M^3 be an open subsets of R^3 endowed with the metric g defined by

$$ds^2 = g_{ij}dx^i dx^j = (x^3)^2 x^1 (dx^1)^2 + 2dx^1 dx^2 + (dx^3)^2 \tag{0.50}$$

$i, j = 1, 2, 3$

Then

$$S_{11} = x^1, S_{11,1} = 1 \tag{0.51}$$

and all others vanish. where $(,)$ denotes the covariant differentiation with respect to x^1.
Now we define

$$A_i(x) = \frac{1}{x^1}, \quad i = 1 \tag{0.52}$$

$= 0$ otherwise

$$B_i(x) = \frac{2}{x^1}, \quad i = 1 \tag{0.53}$$

$= 0$ otherwise
for any point $x \in M$, Then

$$S_{11} = A_1 S_{11} + B_1 S_{11}, \tag{0.54}$$

and all other forms vanish identically.
The relation(54) implies that the above Riemannian manifold(M^3, g) is a $G(SPRS)_3$

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Received: October 22, 2015.

Accepted: March 1, 2016.