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Trigonometric approximation of (signals) functions by Nörlund type means in the variable space $L^{p(x)}$

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Abstract. In this paper the results obtained in [7] are extended in three directions: they are extended for a wider class of numerical sequences, are obtained sharper degrees of approximation, and are used some recent new means.

1 Introduction

Let $p : \mathbb{R} \to [1, \infty)$ be a measurable 2π periodic function. Denote by $L^{p(x)} = L^{p(x)}([0, 2\pi])$ the set of all measurable 2π periodic functions f such that $m_p(\mu f) < \infty$ for $\mu = \mu(f) > 0$, where

$$m_p(f) := \int_0^{2\pi} |f(x)|^{p(x)} dx.$$

$L^{p(x)}$ becomes a Banach space with respect to the norm

$$\|f\|_{p(x)} := \inf \left\{ \mu > 0 : m_p \left(\frac{f}{\mu} \right) \leq 1 \right\}.$$

If the function $p(x) = p$ is a constant one ($1 \leq p < \infty$), then the space $L^{p(x)}$ is isometrically isomorphic to the Lebesgue space L^p.

Moreover, if the function p satisfies

$$1 < p_- := \text{ess} \inf_{x \in [0,2\pi]} p(x), \quad p_+ := \text{ess} \sup_{x \in [0,2\pi]} p(x) < \infty, \tag{1.1}$$

then the function

$$p'(x) := \frac{p(x)}{p(x) - 1}$$

is well defined and satisfies (1.1) itself.

The space $L^{p(x)}$ consists of all measurable 2π periodic functions f such that

$$\int_0^{2\pi} |f(x)g(x)| dx < \infty$$

for all measurable functions g with $m_{p'}(g) \leq 1$.

Denote by $M(f)$ the Hardy-Littlewood maximal operator, defined for $f \in L^1$ by

$$M(f)(x) = \sup_I \frac{1}{|I|} \int_I |f(t)| dt, \quad x \in [0, 2\pi],$$

where the supremum is taken over all intervals with $x \in I$.

It was proved in [6] that if the function $p(x)$ satisfies (1.1) and the condition

$$|p(x) - p(y)| \leq \frac{C}{-\ln |x - y|}, \quad 0 < |x - y| \leq \frac{1}{2}, \tag{1.2}$$

then the maximal operator $M(f)$ is bounded on $L^{p(x)}$, that is,

$$\|M(f)\|_{p(x)} \leq A\|f\|_{p(x)} \tag{1.3}$$

for all $f \in L^{p(x)}$, where A is a constant depending only on p.

The set of all measurable 2π periodic functions $p : \mathbb{R} \to [0, \infty)$ satisfies the conditions (1.1) and (1.2) will be denoted by $\mathcal{M}$.

Let $p \in \mathcal{M}$ and $f \in L^{p(x)}$. The modulus of continuity of the function f is defined by equality

$$\Omega_{p(x)}(f, \delta) = \sup_{|h| \leq \delta} \|T_h(f)\|_{p(x)}, \quad \delta > 0,$$

where

$$T_h(f; x) := \frac{1}{h} \int_0^h |f(x+t) - f(x)| dt.$$

The modulus of continuity $\Omega_{p(x)}(f, \delta)$ and the classical integral modulus of continuity $\omega_p(f, \delta)$ in the Lebesgue space L^p are equivalent (for details see [8]).

Let $f \in L$ has the Fourier series

$$f(x) \sim \frac{a_0}{2} + \sum_{k=1}^{\infty} (a_k \cos kx + b_k \sin kx) \tag{1.4}$$

with its $n-$th partial sums at the point x

$$S_n(f; x) = \sum_{k=0}^{n} U_k(f; x),$$

where

$$U_0(f; x) := \frac{a_0}{2}; \quad U_k(f; x) := a_k \cos kx + b_k \sin kx, \quad k = 1, 2, \dots .$$

Let $(p_n)_{n=0}^{\infty}$ be a sequence of positive real numbers. We consider the so-called Nörlund means of the sums $S_n(f; x)$ defined by

$$N_n(f; x) = \frac{1}{P_n} \sum_{m=0}^{n} p_{n-m} S_m(f; x),$$

where $P_n := \sum_{m=0}^{n} p_m$, $p_{-1} := P_{-1} := 0$. In the case $p_m = 1$ for all $m \geq 0$, the means $N_n(f; x)$ reduced to the Cesàro mean given by equality

$$\sigma_n(f; x) = \frac{1}{n+1} \sum_{m=0}^{n} S_m(f; x).$$

The approximation properties of the mean $\sigma_n(f; x)$ in classes $Lip(\alpha, p)$, $1 \leq p < +\infty$, $0 < \alpha \leq 1$ were established first by E. S. Quade [13]. His results are generalized by R. N. Mohapatra and D. C. Russell [11], P. Chandra [2]-[5] and L. Leindler [9].

Let $p \in \mathcal{M}$ and $0 < \alpha \leq 1$. Very recently, A. Guven and D. Israfilov [7] defined the Lipschitz class $Lip(\alpha, p(x))$ as

$$Lip(\alpha, p(x)) = \left\{ f \in L^{p(x)} : \Omega_{p(x)}(f, \delta) = \mathcal{O}(\delta^{\alpha}), \ \delta > 0 \right\},$$

and gave $L^{p(x)}$ counterparts of the results obtained by L. Leindler [9] and P. Chandra [5].

Before we write their results we need first to recall some known notions.

A sequence of positive real numbers $(p_n)_0^{\infty}$ is called almost monotone decreasing (increasing) if there exists a constant K, depending only on $(p_n)_0^{\infty}$ such that for all $n \geq m$ the inequality

$$p_n \leq K p_m \quad (p_n \geq K p_m)$$

holds. Such sequences will be denoted by $(p_n)_0^{\infty} \in AMDS$ ($(p_n)_0^{\infty} \in AMIS$).

Among others they have proved the following.

Theorem 1.1 ([7]). *Let $p \in \mathcal{M}$, $0 < \alpha < 1$, $f \in Lip(\alpha, p(x))$ and let $(p_n)_{n=0}^{\infty}$ be a sequence of positive real numbers. If*

$$(p_n)_{n=0}^{\infty} \in AMDS$$

or

$$(p_n)_{n=0}^{\infty} \in AMIS \quad \textit{and} \quad (n+1)p_n = \mathcal{O}(P_n),$$

then

$$\|f - N_n(f)\|_{p(x)} = \mathcal{O}(n^{-\alpha}).$$

holds.

Theorem 1.2 ([7]). *Let $p \in \mathcal{M}$, $f \in Lip(1, p(x))$ and let $(p_n)_{n=0}^{\infty}$ be a sequence of positive real numbers. If*

$$\sum_{k=1}^{n-1} k|\triangle p_k| = \mathcal{O}(P_n)$$

or

$$\sum_{k=1}^{n-1} |\triangle p_k| = \mathcal{O}\left(\frac{P_n}{n}\right),$$

then

$$\|f - N_n(f)\|_{p(x)} = \mathcal{O}(n^{-1}).$$

holds for $n = 1, 2, \ldots$.

Let $\mathbb{F}$ be an infinite subset of the set of natural numbers $\mathbb{N}$ and $\mathbb{F}$ as the range of strictly increasing sequence of positive integers, say $\mathbb{F} = (\lambda(n))_{n=1}^{\infty}$. The Cesàro submethod C_λ is defined by

$$(C_\lambda x)_n = \frac{1}{\lambda(n)} \sum_{k=1}^{\lambda(n)} x_k, \quad (n = 1, 2, \ldots),$$

where $(x_k)_{k=1}^{\infty}$ is a sequence of real or complex numbers.

The C_λ-method yields a subsequence of the Cesàro method C_1 and thus it is regular for any λ. A very important fact to point out here is that C_λ-method is obtained by deleting a set of rows from Cesàro matrix. An interested reader could find basic properties of C_λ-method in [1] and [12].

Next we shall consider trigonometric polynomials $N_n^\lambda(f;x)$ defined by (see [10])

$$N_n^\lambda(f;x) = \frac{1}{P_{\lambda(n)}} \sum_{m=0}^{\lambda(n)} p_{\lambda(n)-m} S_m(f;x),$$

where $P_{\lambda(n)} := \sum_{m=0}^{\lambda(n)} p_m$, $p_{-1} := P_{-1} := 0$. In the case $p_m = 1$ for all $m \geq 0$, the means $N_n^\lambda(f;x)$ reduced to the λ-Cesàro mean given by equality

$$\sigma_n^\lambda(f;x) = \frac{1}{\lambda(n)+1} \sum_{m=0}^{\lambda(n)} S_m(f;x),$$

where

$$S_m(f;x) = \frac{1}{\pi} \int_0^{2\pi} f(x+t) D_m(t) dt \quad \text{and} \quad D_m(t) = \sum_{j=1}^{m} \sin jx = \frac{\sin(m+1/2)t}{2\sin(t/2)}.$$

Motivated from [14] we introduce two new classes of numerical sequences.

Let $B_{\lambda(n),k} = \frac{1}{(k+1)P_{\lambda(n)}} \sum_{i=\lambda(n)-k}^{\lambda(n)} p_i$. If $(B_{\lambda(n),k}) \in AMDS$ ($(B_{\lambda(n),k}) \in AMIS$), then it is said that (p_k) is an λ-*almost monotone decreasing (increasing) upper mean sequence*, briefly $(p_k) \in \lambda - AMDUMS$ ($(p_k) \in \lambda - AMIUMS$).

Remark 1.3. Note that in particular cases for $\lambda(n) = n$, $n = 1, 2, \ldots$, we obtain classes $\lambda - AMDUMS \equiv AMDUMS$ and $\lambda - AMIUMS \equiv AMIUMS$ defined in [14]. So, the classes $\lambda - AMDUMS$ and $\lambda - AMIUMS$ are generalizations of the classes $AMDUMS$ and $AMIUMS$ respectively.

The main object of this paper is to prove the Theorems 1.1 and 1.2 using new means $N_n^\lambda(f;x)$ and considering new classes $\lambda - AMIUMS$ and $\lambda - AMIUMS$, which give better degrees of approximations than those means that are considered previously by others.

2 Helpful Lemmas

To achieve the aim, which we mentioned above, we need some helpful statements given below.

Lemma 2.1 ([7]). *Let $p \in \mathcal{M}$. Then the estimate*

$$\|\sigma_n(f) - S_n(f)\|_{p(x)} = \mathcal{O}\left(n^{-1}\right), \ n = 1, 2, \ldots,$$

holds for every $f \in Lip(1, p(x))$.

Lemma 2.2 ([7]). *Let $p \in \mathcal{M}$ and $0 < \alpha \leq 1$. Then the estimate*

$$\|f - S_n(f)\|_{p(x)} = \mathcal{O}\left(n^{-\alpha}\right), \ n = 1, 2, \ldots,$$

holds for every $f \in Lip(\alpha, p(x))$.

Lemma 2.3. *Let (p_n) be a positive sequence so that*

(i) $(p_n) \in \lambda - AMDUMS$ *or,*

(ii) $(p_n) \in \lambda - AMIUMS$, *and* $(\lambda(n) + 1)p_{\lambda(n)} = \mathcal{O}(P_{\lambda(n)})$

are satisfied. Then

$$\Lambda := \sum_{k=0}^{\lambda(n)} \frac{p_{\lambda(n)-k}}{(k+1)^\alpha} = \mathcal{O}_\alpha\left(\frac{P_{\lambda(n)}}{(\lambda(n)+1)^\alpha}\right)$$

holds for all $0 < \alpha < 1$.

Proof. Let $r = [\lambda(n)/2]$ be the integer part of $\lambda(n)/2$. Then under assumptions of the lemma, then applying the summation by parts and using the inequality

$$(j+1)^\beta - j^\beta \leq \beta j^{\beta-1}, \quad \text{for} \quad j \in \mathbb{N} \quad \text{and} \quad 0 < \beta < 1,$$

we have

$$\begin{aligned}
\Lambda &\leq \sum_{k=0}^{r} \frac{p_{\lambda(n)-k}}{(k+1)^\alpha} + \frac{1}{(r+1)^\alpha} \sum_{k=r+1}^{\lambda(n)} p_{\lambda(n)-k} \\
&= \sum_{k=0}^{r-1} \left[\frac{1}{(k+1)^\alpha} - \frac{1}{(k+2)^\alpha}\right] \sum_{i=0}^{k} p_{\lambda(n)-i} + \frac{1}{(r+1)^\alpha} \sum_{k=0}^{r} p_{\lambda(n)-k} + \frac{P_{\lambda(n)}}{(r+1)^\alpha} \\
&= P_{\lambda(n)} \sum_{k=0}^{r-1} \frac{(k+2)^\alpha - (k+1)^\alpha}{(k+1)^{\alpha-1}(k+2)^\alpha} B_{\lambda(n),k} + \frac{P_r}{(r+1)^\alpha} + \frac{P_{\lambda(n)}}{(r+1)^\alpha} \\
&\leq P_{\lambda(n)} \left[\sum_{k=0}^{r-1} \frac{\alpha B_{\lambda(n),k}}{(k+2)^\alpha} + \frac{2}{(r+1)^\alpha}\right].
\end{aligned}$$

If $(p_n) \in \lambda - AMDUMS$, then

$$\begin{aligned}\Lambda &\leq P_{\lambda(n)}\left[\alpha B_{\lambda(n),r}\sum_{k=0}^{r-1}\frac{1}{(k+2)^\alpha}+\frac{2}{(r+1)^\alpha}\right]\\ &\leq P_{\lambda(n)}\left[\frac{\alpha}{(r+1)P_{\lambda(n)}}\sum_{i=\lambda(n)-r}^{\lambda(n)}p_i\sum_{k=0}^{r-1}\frac{1}{(k+2)^\alpha}+\frac{2}{(r+1)^\alpha}\right]\\ &\leq P_{\lambda(n)}\left[\frac{\alpha}{(r+1)P_{\lambda(n)}}\sum_{i=0}^{\lambda(n)}p_i\cdot(r+1)^{1-\alpha}+\frac{2}{(r+1)^\alpha}\right]\\ &= \frac{C_\alpha P_{\lambda(n)}}{(r+1)^\alpha}\leq\frac{C_\alpha P_{\lambda(n)}}{(\lambda(n)+1)^\alpha},\end{aligned}$$

where C_α is a positive constant that depends only on α.

If $(p_n) \in \lambda - AMIUMS$ and $(\lambda(n)+1)p_{\lambda(n)} = \mathcal{O}(P_{\lambda(n)})$, we obtain

$$\begin{aligned}\Lambda &\leq P_{\lambda(n)}\left[\alpha B_{\lambda(n),0}\sum_{k=0}^{r-1}\frac{1}{(k+2)^\alpha}+\frac{2}{(r+1)^\alpha}\right]\\ &\leq P_{\lambda(n)}\left[\frac{\alpha p_{\lambda(n)}}{P_{\lambda(n)}}\sum_{k=0}^{r-1}\frac{1}{(k+2)^\alpha}+\frac{2}{(r+1)^\alpha}\right]\\ &\leq P_{\lambda(n)}\left[\frac{\alpha p_{\lambda(n)}}{P_{\lambda(n)}}(r+1)^{1-\alpha}+\frac{2}{(r+1)^\alpha}\right]\leq\frac{C_\alpha P_{\lambda(n)}}{(\lambda(n)+1)^\alpha}.\end{aligned}$$

The proof of the lemma is completed. □

Next section will be devoted to the main results.

3 Main Results

First we verify the following statement.

Theorem 3.1. *The following properties hold true:*

(i) If $(p_m) \in AMDS$, *then* $(p_m) \in \lambda - AMIUMS$,

(ii) If $(p_m) \in AMIS$, *then* $(p_m) \in \lambda - AMDUMS$,

(iii) If $\sum_{i=0}^{\lambda(n)-1}\left|\triangle\left(\frac{p_i}{P_{\lambda(n)}}\right)\right| = \mathcal{O}\left(\frac{1}{\lambda(n)}\right)$, *then* $\sum_{i=0}^{\lambda(n)-1}\left|\triangle\left(B_{\lambda(n),i}\right)\right| = \mathcal{O}\left(\frac{1}{\lambda(n)}\right)$,

(iv) If $\sum_{i=1}^{\lambda(n)-1} i\left|\triangle\left(\frac{p_i}{P_{\lambda(n)}}\right)\right| = \mathcal{O}(1)$, *then* $\sum_{i=0}^{\lambda(n)-2}\left|\triangle\left(B_{\lambda(n),i}\right)\right| = \mathcal{O}\left(\frac{1}{\lambda(n)}\right)$.

Proof. (i) If $(p_m) \in AMDS$, then $Kp_m \geq p_\ell$ for $m \leq \ell$. For $m = \ell$ the implication (i) is true. Let $m < \ell$. Then

$$\begin{aligned}(\ell+1)\sum_{i=\lambda(n)-m}^{\lambda(n)}p_i &= (m+1)\sum_{i=\lambda(n)-m}^{\lambda(n)}p_i+(\ell-m)\sum_{i=\lambda(n)-m}^{\lambda(n)}p_i\\ &\leq (m+1)\left[\sum_{i=\lambda(n)-m}^{\lambda(n)}p_i+K(\ell-m)p_{\lambda(n)-m}\right]\\ &\leq (m+1)\left[\sum_{i=\lambda(n)-m}^{\lambda(n)}p_i+K^2\sum_{i=\lambda(n)-\ell}^{\lambda(n)-m-1}p_i\right]\\ &\leq \max\left\{1,K^2\right\}(m+1)\sum_{i=\lambda(n)-\ell}^{\lambda(n)}p_i.\end{aligned}$$

Multiplying the above inequality by $P_{\lambda(n)}$ we clearly obtain

$$B_{\lambda(n),m} \leq \max\left\{1, K^2\right\} B_{\lambda(n),\ell}.$$

(ii) If $(p_m) \in AMIS$, then $p_m \leq K p_\ell$ for $m \leq \ell$. For $m = \ell$ the implication (ii) is true. Let $m < \ell$. Then

$$\begin{aligned}
(\ell+1) \sum_{i=\lambda(n)-m}^{\lambda(n)} p_i &= (m+1) \sum_{i=\lambda(n)-m}^{\lambda(n)} p_i + (\ell - m) \sum_{i=\lambda(n)-m}^{\lambda(n)} p_i \\
&\geq (m+1)\left[\sum_{i=\lambda(n)-m}^{\lambda(n)} p_i + \frac{1}{K}(\ell-m) p_{\lambda(n)-m}\right] \\
&\geq (m+1)\left[\sum_{i=\lambda(n)-m}^{\lambda(n)} p_i + \frac{1}{K^2} \sum_{i=\lambda(n)-\ell}^{\lambda(n)-m-1} p_i\right] \\
&\geq \min\left\{1, \frac{1}{K^2}\right\}(m+1) \sum_{i=\lambda(n)-\ell}^{\lambda(n)} p_i .
\end{aligned}$$

Multiplying the above inequality by $P_{\lambda(n)}$ we clearly obtain

$$\frac{1}{\min\left\{1, \frac{1}{K^2}\right\}} B_{\lambda(n),m} \geq B_{\lambda(n),\ell}.$$

(iii) After some calculations we have

$$\begin{aligned}
\sum_{i=0}^{\lambda(n)-1} \left|\triangle\left(B_{\lambda(n),i}\right)\right| &= \frac{1}{P_{\lambda(n)}} \sum_{i=0}^{\lambda(n)-1} \frac{1}{(i+1)(i+2)} \times \left|(i+2) \sum_{j=\lambda(n)-i}^{\lambda(n)} p_i - (i+1) \sum_{j=\lambda(n)-i-1}^{\lambda(n)} p_i\right| \\
&= \frac{1}{P_{\lambda(n)}} \sum_{i=0}^{\lambda(n)-1} \frac{1}{(i+1)(i+2)} \times \left|\sum_{j=\lambda(n)-i}^{\lambda(n)} p_i - (i+1) p_{\lambda(n)-i-1}\right| . \qquad (3.1)
\end{aligned}$$

On the other side we have

$$\sum_{j=\lambda(n)-i}^{\lambda(n)} p_j - (i+1) p_{\lambda(n)-i-1} = \sum_{j=0}^{i} (j+1)\left(p_{\lambda(n)-j} - p_{\lambda(n)-j-1}\right) \qquad (3.2)$$

for any $0 \leq k \leq n$.

Now using (3.1) and (3.2) we obtain

$$\begin{aligned}
\sum_{i=0}^{\lambda(n)-1} \left|\triangle\left(B_{\lambda(n),i}\right)\right| &= \frac{1}{P_{\lambda(n)}} \sum_{i=0}^{\lambda(n)-1} \frac{1}{(i+1)(i+2)} \times \left|\sum_{j=0}^{i} (j+1)\left(p_{\lambda(n)-j} - p_{\lambda(n)-j-1}\right)\right| \\
&\leq \frac{1}{P_{\lambda(n)}} \sum_{i=0}^{\lambda(n)-1} \frac{1}{(i+1)(i+2)} \times \sum_{j=0}^{i} (j+1)\left|p_{\lambda(n)-j} - p_{\lambda(n)-j-1}\right| \\
&\leq \frac{1}{P_{\lambda(n)}} \sum_{i=0}^{\lambda(n)-1} (i+1)\left|p_{\lambda(n)-i} - p_{\lambda(n)-i-1}\right| \times \sum_{j=i}^{\infty} \frac{1}{(j+1)(j+2)} \\
&= \frac{1}{P_{\lambda(n)}} \sum_{i=0}^{\lambda(n)-1} \left|p_{\lambda(n)-i} - p_{\lambda(n)-i-1}\right| = \frac{1}{P_{\lambda(n)}} \sum_{i=0}^{\lambda(n)-1} \left|p_{i+1} - p_i\right| .
\end{aligned}$$

Therefore, if $\sum_{i=0}^{\lambda(n)-1} \left|\triangle\left(\frac{p_i}{P_{\lambda(n)}}\right)\right| = \mathcal{O}\left(\frac{1}{\lambda(n)}\right)$, then we also have $\sum_{i=0}^{\lambda(n)-1} \left|\triangle\left(B_{\lambda(n),i}\right)\right| = \mathcal{O}\left(\frac{1}{\lambda(n)}\right)$.

(iv) Let $r=[\lambda(n)/2]$. Taking into consideration (3.2) we get

$$\begin{aligned}
\sum_{i=0}^{\lambda(n)-2}\left|\triangle\left(B_{\lambda(n),i}\right)\right| &\leq \frac{1}{P_{\lambda(n)}}\sum_{i=0}^{\lambda(n)-2}\frac{1}{(i+1)(i+2)}\sum_{j=0}^{i}(j+1)\left|p_{\lambda(n)-j}-p_{\lambda(n)-j-1}\right| \\
&= \frac{1}{P_{\lambda(n)}}\sum_{i=0}^{r-1}\frac{1}{(i+1)(i+2)}\sum_{j=0}^{i}(j+1)\left|p_{\lambda(n)-j}-p_{\lambda(n)-j-1}\right| \\
&+\frac{1}{P_{\lambda(n)}}\sum_{i=r}^{\lambda(n)-2}\frac{1}{(i+1)(i+2)}\sum_{j=0}^{i}(j+1)\left|p_{\lambda(n)-j}-p_{\lambda(n)-j-1}\right| \\
&= J_1+J_2.
\end{aligned}$$

Under assumption of the theorem we have

$$\begin{aligned}
J_1 &\leq \frac{1}{P_{\lambda(n)}}\sum_{i=0}^{r-1}(i+1)\left|p_{\lambda(n)-i}-p_{\lambda(n)-i-1}\right|\sum_{j=i}^{\infty}\frac{1}{(i+1)(i+2)} \\
&\leq \frac{1}{P_{\lambda(n)}}\sum_{i=0}^{r-1}\left|p_{\lambda(n)-i}-p_{\lambda(n)-i-1}\right| = \frac{1}{P_{\lambda(n)}}\sum_{i=\lambda(n)-r}^{\lambda(n)-1}|p_i-p_{i+1}| \\
&\leq \frac{2}{\lambda(n)P_{\lambda(n)}}\sum_{i=\lambda(n)-r}^{\lambda(n)-1} i\,|\triangle p_i| \leq \frac{2}{\lambda(n)P_{\lambda(n)}}\sum_{i=1}^{\lambda(n)-1} i\,|\triangle p_i| = \mathcal{O}\left(\frac{1}{\lambda(n)}\right).
\end{aligned}$$

Now we write

$$\begin{aligned}
J_2 &= \frac{1}{P_{\lambda(n)}}\sum_{i=r}^{\lambda(n)-2}\frac{1}{(i+1)(i+2)}\sum_{j=0}^{r-1}(j+1)\left|p_{\lambda(n)-j}-p_{\lambda(n)-j-1}\right| \\
&+\frac{1}{P_{\lambda(n)}}\sum_{i=r}^{\lambda(n)-2}\frac{1}{(i+1)(i+2)}\sum_{j=r}^{i}(j+1)\left|p_{\lambda(n)-j}-p_{\lambda(n)-j-1}\right| = J_2^{(1)}+J_2^{(2)}.
\end{aligned}$$

Then we have

$$\begin{aligned}
J_2^{(1)} &\leq \frac{1}{P_{\lambda(n)}}\sum_{i=r}^{\lambda(n)-2}\frac{1}{(i+1)\frac{\lambda(n)}{2}}\sum_{j=0}^{r-1}\frac{\lambda(n)}{2}\left|p_{\lambda(n)-j}-p_{\lambda(n)-j-1}\right| \\
&\leq \frac{1}{P_{\lambda(n)}}\frac{\lambda(n)-r-1}{r+1}\sum_{j=\lambda(n)-r}^{\lambda(n)-1}|p_j-p_{j+1}| \leq \frac{1}{P_{\lambda(n)}}\sum_{j=\lambda(n)-r}^{\lambda(n)-1}|\triangle p_j| \\
&= \frac{\lambda(n)}{\lambda(n)P_{\lambda(n)}}\sum_{j=\lambda(n)-r}^{\lambda(n)-1}|\triangle p_j| \leq \frac{2}{\lambda(n)P_{\lambda(n)}}\sum_{j=\lambda(n)-r}^{\lambda(n)-1} i\,|\triangle p_j| \\
&\leq \frac{2}{\lambda(n)P_{\lambda(n)}}\sum_{j=1}^{\lambda(n)-1} i\,|\triangle p_j| = \mathcal{O}\left(\frac{1}{\lambda(n)}\right)
\end{aligned}$$

and

$$\begin{aligned}
J_2^{(2)} &\leq \frac{1}{P_{\lambda(n)}} \sum_{i=r}^{\lambda(n)-2} \frac{1}{(i+1)\frac{\lambda(n)}{2}} \sum_{j=r}^{i} \frac{\lambda(n)}{2} \left|p_{\lambda(n)-j} - p_{\lambda(n)-j-1}\right| \\
&\leq \frac{1}{P_{\lambda(n)}(r+1)} \sum_{i=r}^{\lambda(n)-2} \sum_{j=r}^{i} \left|p_{\lambda(n)-j} - p_{\lambda(n)-j-1}\right| \\
&= \frac{1}{P_{\lambda(n)}(r+1)} \sum_{i=r}^{\lambda(n)-2} \sum_{j=\lambda(n)-i}^{\lambda(n)-r} |p_j - p_{j+1}| \\
&\leq \frac{1}{P_{\lambda(n)}(r+1)} \sum_{i=r}^{\lambda(n)-2} \sum_{j=1}^{\lambda(n)-r} |\triangle p_j| = \frac{\lambda(n)-r-1}{P_{\lambda(n)}(r+1)} \sum_{j=1}^{\lambda(n)-r} |\triangle p_j| \\
&\leq \frac{1}{P_{\lambda(n)}} \sum_{j=1}^{\lambda(n)-1} |\triangle p_j| \leq \frac{2}{\lambda(n) P_{\lambda(n)}} \sum_{j=0}^{\lambda(n)-1} i\,|\triangle p_j| = \mathcal{O}\left(\frac{1}{\lambda(n)}\right).
\end{aligned}$$

Inserting $J_2^{(1)}$ and $J_2^{(2)}$ into J_2 we obtain $J_2 = \mathcal{O}\left(\frac{1}{\lambda(n)}\right)$, which along with $J_1 = \mathcal{O}\left(\frac{1}{\lambda(n)}\right)$ we clearly find that $\sum_{i=0}^{\lambda(n)-2} \left|\triangle\left(B_{\lambda(n),i}\right)\right| = \mathcal{O}\left(\frac{1}{\lambda(n)}\right)$. The proof of the lemma is completed. □

Theorem 3.2. *Let* $p \in \mathcal{M}$, $f \in Lip(\alpha, p(x))$, $0 < \alpha < 1$, *and* $(p_n)_{n=0}^{\infty}$ *be a sequence of positive real numbers. Let*

$$(p_n)_{n=0}^{\infty} \in \lambda - AMDUMS \quad or$$

$$(p_n)_{n=0}^{\infty} \in \lambda - AMIUMS \quad and \quad (\lambda(n)+1)p_{\lambda(n)} = \mathcal{O}(P_{\lambda(n)}), \tag{3.3}$$

then

$$\|f - N_n^{\lambda}(f)\|_{p(x)} = \mathcal{O}\left(\frac{1}{(\lambda(n)+1)^{\alpha}}\right)$$

holds for all $n \in \mathbb{N} \cup \{0\}$.

Proof. Since

$$f(x) = \frac{1}{P_{\lambda(n)}} \sum_{m=0}^{\lambda(n)} p_{\lambda(n)-m} f(x),$$

then we can write

$$f(x) - N_n^{\lambda}(f;x) = \frac{1}{P_{\lambda(n)}} \sum_{m=0}^{\lambda(n)} p_{\lambda(n)-m}\{f(x) - S_m(f;x)\}.$$

Whence, using Lemma 2.2, Lemma 2.3, and conditions (3.3) we get

$$\begin{aligned}
\|f - N_n^{\lambda}(f)\|_{p(x)} &\leq \frac{1}{P_{\lambda(n)}} \sum_{m=0}^{\lambda(n)} p_{\lambda(n)-m} \|f(x) - S_m(f;x)\|_{p(x)} \\
&= \frac{1}{P_{\lambda(n)}} \mathcal{O}\left(\sum_{m=0}^{\lambda(n)} p_{\lambda(n)-m}(m+1)^{-\alpha}\right) \\
&= \frac{1}{P_{\lambda(n)}} \mathcal{O}\left(\frac{P_{\lambda(n)}}{(\lambda(n)+1)^{\alpha}}\right) = \mathcal{O}\left(\frac{1}{(\lambda(n)+1)^{\alpha}}\right).
\end{aligned}$$

□

Next theorem gives the same degree of approximation with different conditions from those of Theorem 3.1, considering the case $\alpha = 1$.

Theorem 3.3. *Let $p \in \mathcal{M}$, $f \in Lip(1, p(x))$ and let $(p_n)_{n=0}^{\infty}$ be a sequence of positive real numbers. If*

$$\sum_{m=0}^{\lambda(n)-2} \left| B_{\lambda(n),m} - B_{\lambda(n),m+1} \right| = \mathcal{O}\left(\frac{1}{\lambda(n)}\right),$$

then for $n = 1, 2, \ldots$ the estimate

$$\|f - N_n^{\lambda}(f)\|_{p(x)} = \mathcal{O}\left(\frac{1}{\lambda(n)}\right)$$

holds.

Proof. According to the definition of $N_n^{\lambda}(f;x)$ the following equality is true

$$E_n^{\lambda}(f;x) := N_n^{\lambda}(f;x) - f(x) = \frac{1}{P_{\lambda(n)}} \sum_{m=0}^{\lambda(n)} p_{\lambda(n)-m}\{S_m(f;x) - f(x)\}.$$

Applying the summation by parts twice we get

$$\begin{aligned} E_n^{\lambda}(f;x) &= \sum_{m=0}^{\lambda(n)-1} (S_m(f;x) - S_{m+1}(f;x)) \frac{1}{P_{\lambda(n)}} \sum_{i=0}^{m} p_{\lambda(n)-i} + S_{\lambda(n)}(f;x) - f(x) \\ &= -\sum_{m=0}^{\lambda(n)-1} (m+1)\, U_{m+1}(f;x) B_{\lambda(n),m} + S_{\lambda(n)}(f;x) - f(x) \\ &= -\sum_{m=0}^{\lambda(n)-2} \left(B_{\lambda(n),m} - B_{\lambda(n),m+1}\right) \sum_{j=0}^{m} (j+1) U_{j+1}(f;x) \\ &\quad - \frac{1}{\lambda(n) P_{\lambda(n)}} \sum_{j=1}^{\lambda(n)} p_i \sum_{j=0}^{\lambda(n)-1} (j+1) U_{j+1}(f;x) + S_{\lambda(n)}(f;x) - f(x). \end{aligned}$$

Thus,

$$\begin{aligned} \|E_n^{\lambda}(f)\|_{p(x)} &\leq \sum_{m=0}^{\lambda(n)-2} \left| B_{\lambda(n),m} - B_{\lambda(n),m+1} \right| \left\| \sum_{j=1}^{m+1} j U_j(f) \right\|_{p(x)} \\ &\quad + \frac{1}{\lambda(n)} \left\| \sum_{j=1}^{\lambda(n)} j U_j(f) \right\|_{p(x)} + \left\| S_{\lambda(n)}(f) - f \right\|_{p(x)}. \end{aligned}$$

Based on Lemma 2.1 and the equality

$$\sum_{j=1}^{\lambda(n)} j U_j(f;x) = (\lambda(n)+1)(S_{\lambda(n)}(f;x) - \sigma_{\lambda(n)}(f;x)),$$

we have

$$\left\| \sum_{j=1}^{\lambda(n)} j U_j(f) \right\|_{p(x)} = \mathcal{O}(1).$$

Hence, using Lemma 2.2 and the latter estimation we get

$$\|E_n(f)\|_{p(x)} = \mathcal{O}\left(\sum_{m=0}^{\lambda(n)-2} \left| B_{\lambda(n),m} - B_{\lambda(n),m+1} \right| \right) + \mathcal{O}\left(\frac{1}{\lambda(n)}\right).$$

Finally, if the condition $\sum_{m=0}^{\lambda(n)-2} \left|B_{\lambda(n),m} - B_{\lambda(n),m+1}\right| = \mathcal{O}\left(\frac{1}{\lambda(n)}\right)$ is satisfied, then we obtain

$$\|N_n^{\lambda}(f) - f(x)\|_{p(x)} = \mathcal{O}\left(\frac{1}{\lambda(n)}\right).$$

The proof of theorem is completed. □

Remark 3.4. If $\lambda(n) = n$, $n = 1, 2, \ldots$, then Theorems 3.2 and 3.3 reduce to Theorems 1.1 and 1.2, respectively. Moreover, for the same sequence Theorem 4 from [14] is an immediate result of Theorem 3.1.

Remark 3.5. Since $(\lambda(n))^{-\alpha} \leq n^{-\alpha}$ for $0 < \alpha \leq 1$, then Theorems 3.2 and 3.3 give shaper estimates than those of Theorems 1.1 and 1.2.

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