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ON SPECIAL NUMBER SEQUENCES VIA HESSENBERG MATRICES

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Abstract. There are many relations between number theory and matrix theory. In this paper, our aim is to obtain some relationships between the Padovan, Perrin and Tribonacci numbers and upper Hessenberg matrices.

1 Introduction

Determinants and permanents are two basic parameters for matrices. Let $A = [a_{ij}]$ be an $n \times n$ matrix and S_n be a symmetric group, which denotes the group of permutations over the set $\{1, 2, ..., n\}$. The determinant of matrix A is defined by [1]

$$\det A = \sum_{\sigma \epsilon S_n} sgn(\sigma) \prod_{i=1}^{n} a_{i\sigma(i)},$$

where the sum ranges over all the permutations of the integers $1, 2, ..., n$. It can be denoted by $sgn(\sigma) = \pm 1$ the signature of σ, equal to $+1$ if σ is the product an even number of transposition, and -1 otherwise. Similarly, the permanent of the matrix is defined by

$$perA = \sum_{\sigma \epsilon S_n} \prod_{i=1}^{n} a_{i\sigma(i)}.$$

In natural sciences, determinant and permanent calculations are very important issues especially in mathematics and physics. There are various methods to compute determinant and permanent in the literature. In this paper, we use the contraction method defined by Brualdi et al. [2].

Let $A = [a_{ij}]$ be an $m \times n$ matrix with row vectors $r_1, r_2, \ldots, r_m$. We call A *contractible* on column k, if column k contains exactly two non zero elements. Suppose that A is contractible on column k with $a_{ik} \neq 0, a_{jk} \neq 0$ and $i \neq j$. Then the $(m-1)\times(n-1)$ matrix $A_{ij:k}$ obtained from A replacing row i with $a_{jk}r_i + a_{ik}r_j$ and deleting row j and column k is called the contraction of A on column k relative to rows i and j. If A is contractible on row k with $a_{ki} \neq 0, a_{kj} \neq 0$ and $i \neq j$, then the matrix $A_{k:ij} = [A^T_{ij:k}]^T$ is called the contraction of A on row k relative to columns i and j. We know that if A is a nonnegative matrix and B is a contraction of A, then we have [2]

$$perA = perB. \tag{1.1}$$

Padovan, Tribonacci and Perrin sequences which are popular, are recursively defined as follow

$$\begin{aligned} P_n &= P_{n-2} + P_{n-3}, & P_0 &= P_1 = P_2 = 1, \\ T_n &= T_{n-1} + T_{n-2} + T_{n-3}, & T_0 &= T_1 = 0, T_2 = 1, \\ R_n &= R_{n-2} + R_{n-3}, & R_0 &= 3, R_1 = 0, R_2 = 2, \end{aligned}$$

for $n > 2$. The first few values of the sequences are shown below.

n	0	1	2	3	4	5	6	7	8	9
P_n	1	1	1	2	2	3	4	5	7	9
T_n	0	0	1	1	2	4	7	13	24	44
R_n	3	0	2	3	2	5	5	7	10	12

Previous studies pointed out that determinant and permanent of matrices and well-known number sequences have common relations. For example, the authors in [3] derived some relationships between the Fibonacci and Lucas numbers and determinants of matrices. The authors in [8] defined two Hessenberg matrices whose determinants are Pell and Perrin numbers. In [9], the authors defined two upper Hessenberg matrices and they showed that permanents of these matrices are Pell-Lucas and Jacobsthal numbers, respectively. In [5], Lee defined the matrix

$$E_n = \begin{bmatrix} 1 & 0 & 1 & 0 & \cdots & 0 \\ 1 & 1 & 1 & 0 & \cdots & 0 \\ 0 & 1 & 1 & 1 & & \vdots \\ 0 & 0 & 1 & 1 & \ddots & 0 \\ \vdots & \vdots & & \ddots & \ddots & 1 \\ 0 & 0 & \cdots & 0 & 1 & 1 \end{bmatrix}$$

and showed that

$$per(E_n) = L_{n-1},$$

where L_n is the nth Lucas number.

In [6], the authors found $(0, 1, -1)$ tridiagonal matrices whose determinants and permanents are negatively subscripted Fibonacci and Lucas numbers. Also, they give an $n \times n$ $(1, -1)$ matrix S, such that

$$perA = \det(A \circ S), \tag{1.2}$$

where $A \circ S$ denotes Hadamard product of A and S. Let S be a $(1, -1)$ matrix of order n, defined with

$$S = \begin{bmatrix} 1 & 1 & \ldots & 1 & 1 \\ -1 & 1 & \ldots & 1 & 1 \\ 1 & -1 & \ldots & 1 & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 1 & \ldots & -1 & 1 \end{bmatrix}. \tag{1.3}$$

In [4], the author investigated general tridiagonal matrix determinants and permanents. Also he showed that the permanent of tridiagonal matrix based on $\{a_i\}, \{b_i\}, \{c_i\}$ is equal to the determinant of matrix based on $\{-a_i\}, \{b_i\}, \{c_i\}$. In [7], the authors gave some determinantal and permanental representations of k-generalized Fibonacci and Lucas numbers.

2 Main Results

In this paper our aim is to make a contribution to the subject mentioned above concerning permanents. In the following part of this study, upper Hessenberg matrices are introduced and permanents of these matrices are Padovan, Tribonacci and Perrin numbers are shown, respectively. In the rest of the work, rth contraction of M_n is shown as $M_n^{(r)}$.

Let $W_n = [w_{ij}]_{n \times n}$ be an $n-$square Hessenberg matrix in which $w_{11} = 2$,$w_{24} = 1/2$ and $w_{(i,i+1)} = 1$ for $i = 1, 2, \ldots, n-1$ and $w_{(i+1,i)} = 1$ for $i = 1, 2, \ldots, n-1$ and $w_{(k,k+2)} = 1$ for $k = 3, 4, \ldots, n-2$ and otherwise 0. That is,

$$W_n=\begin{bmatrix}2&1&0&0&&&&&0\\1&0&1&\frac{1}{2}&&&&&\\0&1&0&1&1&&&&\\&&1&0&1&1&&&\\&&&\ddots&\ddots&\ddots&\ddots&&\\&&&&1&0&1&1&0\\&&&&&1&0&1&1\\&&&&&&1&0&1\\0&&&&&&0&1&0\end{bmatrix}. \tag{2.1}$$

Theorem 2.1. *Let W_n be an n-square matrix as in (2.1), then*

$$perW_n=perW_n^{(n-2)}=P_n,$$

where P_n is the nth Padovan number.

Proof. By definition of the matrix W_n, it can be contracted on first column If $r=1$, then

$$W_n^{(1)}=\begin{bmatrix}1&2&1&0&&&&&0\\1&0&1&1&&&&&\\0&1&0&1&1&&&&\\&&1&0&1&1&&&\\&&&\ddots&\ddots&\ddots&\ddots&&\\&&&&1&0&1&1&0\\&&&&&1&0&1&1\\&&&&&&1&0&1\\0&&&&&&0&1&0\end{bmatrix}.$$

Due to contractions of $W_n^{(1)}$ is performed based on the first column, it can be written

$$W_n^{(2)}=\begin{bmatrix}2&2&1&0&&&&&0\\1&0&1&1&&&&&\\0&1&0&1&1&&&&\\&&1&0&1&1&&&\\&&&\ddots&\ddots&\ddots&\ddots&&\\&&&&1&0&1&1&0\\&&&&&1&0&1&1\\&&&&&&1&0&1\\0&&&&&&0&1&0\end{bmatrix}.$$

If this method is applied continously to the rth step, the rth contraction is obtained by

$$W_n^{(r)}=\begin{bmatrix}P_{r+1}&P_{r+2}&P_r&0&&&&&0\\1&0&1&1&&&&&\\0&1&0&1&1&&&&\\&&1&0&1&1&&&\\&&&\ddots&\ddots&\ddots&\ddots&&\\&&&&1&0&1&1&0\\&&&&&1&0&1&1\\&&&&&&1&0&1\\0&&&&&&0&1&0\end{bmatrix},$$

where $1 \leq r \leq n-4$. Hence

$$W_n^{(n-3)} = \begin{bmatrix} P_{n-2} & P_{n-1} & P_{n-3} \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix},$$

which by contraction of $W_n^{(n-3)}$ on first column,

$$W_n^{(n-2)} = \begin{bmatrix} P_{n-1} & P_n \\ 1 & 0 \end{bmatrix}.$$

By (1.1), we have

$$perW_n = perW_n^{(n-2)} = P_n.$$

□

Let $U_n = [u_{ij}]$ be an n-square matrix with $u_{21} = 1$ and $u_{(i,i)} = 1$ for $i = 1, 2, \ldots, n$ and $u_{(i+1,i)} = 1$ for $i = 3, 4, \ldots, n-1$ and $u_{(i,i+1)} = 1$ for $i = 2, 3, ..., n-1$ and $u_{(i,i+2)} = 1$ for $i = 1, 2, ..., n-2$ and otherwise 0. Clearly

$$U_n = \begin{bmatrix} 1 & 0 & 1 & 0 & & & & & 0 \\ 1 & 1 & 1 & 1 & & & & & \\ 0 & 0 & 1 & 1 & 1 & & & & \\ & & 1 & 1 & 1 & 1 & & & \\ & & & \ddots & \ddots & \ddots & \ddots & & \\ & & & & 1 & 1 & 1 & 1 & 0 \\ & & & & & 1 & 1 & 1 & 1 \\ & & & & & & 1 & 1 & 1 \\ 0 & & & & & & 0 & 1 & 1 \end{bmatrix}. \tag{2.2}$$

Theorem 2.2. *If U_n is an n-square matrix as in (2.2), then we have that*

$$perU_n = perU_n^{(n-3)} = T_n,$$

where T_n is the nth Tribonacci number.

Proof. By definition of the matrix U_n, it can be contracted on last row. If $r = 1$, then

$$U_n^{(1)} = \begin{bmatrix} 1 & 0 & 1 & 0 & & & & & 0 \\ 1 & 1 & 1 & 1 & & & & & \\ 0 & 0 & 1 & 1 & 1 & & & & \\ & & 1 & 1 & 1 & 1 & & & \\ & & & \ddots & \ddots & \ddots & \ddots & & \\ & & & & 1 & 1 & 1 & 1 & 0 \\ & & & & & 1 & 1 & 1 & 1 \\ & & & & & & 1 & 1 & 2 \\ 0 & & & & & & 0 & 1 & 2 \end{bmatrix}.$$

$U_n^{(1)}$ also can be contracted according to the last row

$$U_n^{(2)} = \begin{bmatrix} 1 & 0 & 1 & 0 & & & & & 0 \\ 1 & 1 & 1 & 1 & & & & & \\ 0 & 0 & 1 & 1 & 1 & & & & \\ & & 1 & 1 & 1 & 1 & & & \\ & & & \ddots & \ddots & \ddots & \ddots & & \\ & & & & 1 & 1 & 1 & 1 & 0 \\ & & & & & 1 & 1 & 1 & 2 \\ & & & & & & 1 & 1 & 3 \\ 0 & & & & & & 0 & 1 & 4 \end{bmatrix}.$$

With applying the same process, we have

$$U_n^{(r)} = \begin{bmatrix} 1 & 0 & 1 & 0 & & & & & 0 \\ 1 & 1 & 1 & 1 & & & & & \\ 0 & 0 & 1 & 1 & 1 & & & & \\ & & 1 & 1 & 1 & 1 & & & \\ & & & \ddots & \ddots & \ddots & \ddots & & \\ & & & & 1 & 1 & 1 & 1 & 0 \\ & & & & & 1 & 1 & 1 & T_{r+2} \\ & & & & & & 1 & 1 & T_{r+1}+T_{r+2} \\ 0 & & & & & & 0 & 1 & T_{r+3} \end{bmatrix},$$

where $1 \le r \le n-4$. Hence

$$U_n^{(n-3)} = \begin{bmatrix} 1 & 0 & T_{n-1} \\ 1 & 1 & T_{n-2}+T_{n-1} \\ 0 & 0 & T_n \end{bmatrix}.$$

In this matrix if we consider the Laplace expansion according to third row, we obtain

$$perU_n = perU_n^{(n-3)} = per\begin{bmatrix} 1 & 0 & T_{n-1} \\ 1 & 1 & T_{n-2}+T_{n-1} \\ 0 & 0 & T_n \end{bmatrix} = T_n.$$

□

Let $V_n = [v_{ij}]$ be an n-square upper Hessenberg matrix with $v_{11} = v_{13} = 1$, $v_{21} = 2$ and $v_{(i,i+1)} = 1$ for $i = 1, 2, \ldots, n$ and $v_{(i+1,i)} = 1$ for $i = 2, 3, \ldots, n-1$ and $v_{(i,i+2)} = 1$ for $i = 3, 4, ..., n-2$ and otherwise 0. Clearly

$$V_n = \begin{bmatrix} 1 & 1 & 1 & 0 & & & & & 0 \\ 2 & 0 & 1 & 0 & & & & & \\ 0 & 1 & 0 & 1 & 1 & & & & \\ & & 1 & 0 & 1 & 1 & & & \\ & & & \ddots & \ddots & \ddots & \ddots & & \\ & & & & 1 & 0 & 1 & 1 & 0 \\ & & & & & 1 & 0 & 1 & 1 \\ & & & & & & 1 & 0 & 1 \\ 0 & & & & & & 0 & 1 & 0 \end{bmatrix}. \tag{2.3}$$

Theorem 2.3. *If V_n is an n-square matrix as in (2.3), then*

$$perV_n = perV_n^{(n-2)} = R_n,$$

where R_n is nth Perrin number.

Proof. By definition of the matrix V_n, it can be contracted on first column. That is,

$$V_n^{(1)} = \begin{bmatrix} 2 & 3 & 0 & 0 & & & & & 0 \\ 1 & 0 & 1 & 1 & & & & & \\ 0 & 1 & 0 & 1 & 1 & & & & \\ & & 1 & 0 & 1 & 1 & & & \\ & & & \ddots & \ddots & \ddots & \ddots & & \\ & & & & 1 & 0 & 1 & 1 & 0 \\ & & & & & 1 & 0 & 1 & 1 \\ & & & & & & 1 & 0 & 1 \\ 0 & & & & & & 0 & 1 & 0 \end{bmatrix}.$$

$V_n^{(1)}$also can be contracted on the first column. With applying the same process, in rth step, we obtain

$$V_n^{(r)} = \begin{bmatrix} R_{r+1} & R_{r+2} & R_r & 0 & & & & & 0 \\ 1 & 0 & 1 & 1 & & & & & \\ 0 & 1 & 0 & 1 & 1 & & & & \\ & & 1 & 0 & 1 & 1 & & & \\ & & & \ddots & \ddots & \ddots & \ddots & & \\ & & & & 1 & 0 & 1 & 1 & 0 \\ & & & & & 1 & 0 & 1 & 1 \\ & & & & & & 1 & 0 & 1 \\ 0 & & & & & & 0 & 1 & 0 \end{bmatrix}$$

for $1 \leq r \leq n-4$. Hence

$$V_n^{(n-3)} = \begin{bmatrix} R_{n-2} & R_{n-1} & R_{n-3} \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix},$$

which by contraction of $V_n^{(n-3)}$ on first column gives

$$V_n^{(n-2)} = \begin{bmatrix} R_{n-1} & R_n \\ 1 & 0 \end{bmatrix}.$$

By applying (1.1) we have $perV_n = perV_n^{(n-2)} = R_n$. □

Corollary 2.4. *For the matrices $A_n = W_n \circ S$, $B_n = U_n \circ S$ and $C_n = V_n \circ S$ we have*

$$\begin{aligned} \det A_n &= P_n, \\ \det B_n &= T_n, \\ \det C_n &= R_n. \end{aligned}$$

References

[1] D. Serre, *Matrices: Theory and Applications*, Springer, New York (2002).

[2] R. A. Brualdi and P. M. Gibson, Convex polyhedra of Doubly Stochastic matrices I; Application of the Permanent Functions, *Journal of Combinatorial Theory, Series A* **22**, 194–230 (1977).

[3] H. Minc, *Encylopedia of Mathematic and Its Applications, Permanents*, Vol.6, Addison-Wesley Publishing Company, London (1978).

[4] D. H. Lehmer, Fibonacci and relatied sequences in periodic tridiagonal matrices, *Fibonacci Quarterly*, **12**, 150–158 (1975).

[5] G. Y. Lee, k-Lucas numbers and associated bipartite graphs, *Linear Algebra and Its Applications*, **320**, 51–61 (2000).

[6] E. Kılıç and D. Taşçı, Negatively subscripted Fibonacci and Lucas numbers and their complex factorizations, *Ars Combinatoria*, **96**, 275–288 (2010).

[7] A. A. Öcal, N. Tuğlu, E. Altınışık, On the representation of k-generalized Fibonacci and Lucas Numbers, *Applied Mathematics and Computation*, **170**, 584–596 (2005).

[8] F. Yılmaz, D. Bozkurt, Hessenberg matrices and the Pell and Perrin numbers, *Journal of Number theory*, **131**, 1390–1396 (2011).

[9] İ. Aktaş, H. Köse, Hessenberg matrices and the Pell-Lucas and Jacobsthal numbers, *International Journal of Pure and Applied Mathematics*, V:101, **3**, 425–432 (2015).

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