IMPLICIT ISHIKAWA TYPE ALGORITHM IN HYPERBOLIC SPACES

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Abstract. Strong convergence and △-convergence of an implicit Ishikawa type algorithm associated with two nonexpansive mappings on a hyperbolic metric space is established.

1 Introduction

Almost all disciplines of science deal with nonlinear problems. Therefore, finding nonlinear versions of results on linear domain is very much essential.

Iterative construction of fixed points is extremely important [1]. Implicit algorithms provide better approximation of fixed points than explicit algorithms [7, 17]. The number of steps of an algorithm also plays an important role in iterative methods. The case of two maps has a direct link with the minimization problem [21].

Let C be a nonempty subset of a metric space (X, d) and $T : C \to C$ be a mapping. Denote the set of fixed points of T by $F(T)$. T is nonexpansive if $d(Tx, Ty) \leq d(x, y)$ for $x, y \in C$.

The pioneering work of Xu and Ori [23] deals with weak convergence of one-step implicit algorithm for a finite family of nonexpansive mappings on a Hilbert space. They posed an open question about necessary and sufficient conditions required for strong convergence of the algorithm.

Kirk [11] proved a fixed point theorem for Browder's type implicit algorithm (i.e., $x_t = (1 - t)x + tT(x_t)$) in a complete $CAT(0)$ space.

The well-known Mann [17] and Ishikawa [7] iterative procedures are well-defined in a vector space through its in-built convexity. Several mathematicians have introduced notion of convexity in a metric space [18, 20].

It is worth mentioning that introducing and analyzing a general iterative algorithm in more general setup is a problem of interest in many aspects [1]. Khan et al. [10] proposed and analyzed a two-step implicit algorithm for two finite families of nonexpansive mappings on a hyperbolic space in the sense of Kohlenbach [14]. Recently, Khan [9] has introduced and studied an Ishikawa algorithm of two mappings on a hyperbolic space (via Menger convexity).

In this paper, we study strong convergence and △-convergence of an implicit Ishikawa type algorithm associated with a pair of nonexpansive mappings on a hyperbolic metric space, equipped with Menger convexity [18].

2 Menger convexity in metric spaces

Let (X, d) be a metric space. Assume that for any x and y in X, there exists a unique metric segment $[x, y]$, which is an isometric copy of the real line interval $[0, d(x, y)]$. Denote this family of metric segments in X by $\mathcal{F}$. If for any $\beta \in [0, 1]$, there exists a unique point $z \in [x, y]$ in $\mathcal{F}$ such that

$$d(x, z) = (1 - \beta)d(x, y), \quad \text{and} \quad d(z, y) = \beta d(x, y),$$

then we denote this point z by $\beta x \oplus (1 - \beta)y$. Metric spaces having this property are usually called *convex metric spaces* [18]. Moreover, if we have

$$d\Big(\alpha p \oplus (1 - \alpha)x, \alpha q \oplus (1 - \alpha)y\Big) \leq \alpha d(p, q) + (1 - \alpha)d(x, y),$$

for all p, q, x, y in X, and $\alpha \in [0, 1]$, then X is said to be a *hyperbolic space*. For $q = y$, the hyperbolic inequality reduces to the convex structure inequality of Takahashi [20].

An example of linear hyperbolic space is a normed space. Hadamard manifolds [2], the Hilbert open unit ball equipped with the hyperbolic metric [6], and CAT(0) spaces [12] are examples of nonlinear hyperbolic spaces which play a major role in metric fixed point theory. A subset C of a hyperbolic space X is said to be convex if $[x, y] \subset C$, whenever $x, y \in C$.

Lemma 2.1. *Let X be a hyperbolic space. Suppose that $\alpha, \beta \in [0, 1]$ and $x, y \in X$. If $z = \alpha x \oplus (1 - \alpha)y$ and $w = \beta x \oplus (1 - \beta)y$, then $d(z, w) = |\alpha - \beta|d(x, y)$.*

Proof. Without loss of generality, we assume $0 < \beta < \alpha < 1$ (otherwise the conclusion is trivial). As $z, w \in [x, y]$ and $d(x, z) < d(x, w)$, so $z \in [x, w]$.

Moreover, $d(z, w) = d(x, w) - d(x, z) = (1 - \beta)d(x, y) - (1 - \alpha)d(x, y) = (\alpha - \beta)d(x, y)$. □

Proposition 2.2. *Let X be a hyperbolic space, $\{x_n\}$, $\{y_n\}$ be sequences in X which converge, respectively, to x and y, and $\{\alpha_n\}$ be a sequence in $[0, 1]$ converging to α. Then $\alpha_n x_n \oplus (1 - \alpha_n)y_n$ converges to $\alpha x \oplus (1 - \alpha)y$.*

Proof. $d(\alpha_n x_n \oplus (1 - \alpha_n)y_n, \alpha x \oplus (1 - \alpha)y) \leq d(\alpha_n x_n \oplus (1 - \alpha_n)y_n, \alpha_n x \oplus (1 - \alpha_n)y) + d(\alpha_n x \oplus (1 - \alpha_n)y, \alpha x \oplus (1 - \alpha)y)$. Using Lemma 2.1 we get, $d(\alpha_n x_n \oplus (1 - \alpha_n)y_n, \alpha x \oplus (1 - \alpha)y) \leq \alpha_n d(x_n, x) + (1 - \alpha_n)d(y_n, y) + |\alpha_n - \alpha|d(x_n.y_n)$. Hence, $\lim_{n \to \infty} d(\alpha_n x_n \oplus (1 - \alpha_n)y_n, \alpha x \oplus (1 - \alpha)y) = 0$. □

Definition 2.3. Let (X, d) be a hyperbolic space. For any $r > 0$, $a \in X$ and $\varepsilon > 0$, set

$$\delta(r, \varepsilon) = \inf \left\{ 1 - \frac{1}{r}\, d\left(\frac{1}{2}x \oplus \frac{1}{2}y, a\right); d(x, a) \leq r, d(y, a) \leq r, d(x, y) \geq r\varepsilon \right\}.$$

If $\delta(r, \varepsilon) > 0$, then X is said to be uniformly convex.

Throughout this paper we assume that if X is a uniformly convex hyperbolic space, then for every $s \geq 0$ and $\varepsilon > 0$, there exists $\eta(s, \varepsilon) > 0$ such that

$$\delta(r, \varepsilon) > \eta(s, \varepsilon) > 0 \; for \; any \; r > s.$$

In a hyperbolic space X, an implicit Ishikawa type algorithm for nonexpansive mappings S and T is defined as:

$$\begin{aligned} x_n &= \alpha_n x_{n-1} \oplus (1 - \alpha_n)\, Sy_n, \\ y_n &= \beta_n x_{n-1} \oplus (1 - \beta_n)\, Tx_n, \end{aligned} \tag{2.1}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $[0, 1]$.

If $\beta_n = 0$ and $T = I$ (the identity mapping), then (2.1) reduces to an implicit Mann type algorithm:

$$x_n = \alpha_n x_{n-1} \oplus (1 - \alpha_n)\, Sx_n, \tag{2.2}$$

where $\{\alpha_n\}$ is a sequence in $[0, 1]$.

In order to establish that (2.1) exists, we define a mapping $G_1 : C \to C$ by: $G_1(x) = \alpha_1 x_0 \oplus (1 - \alpha_1)\, Sy$, where $y = \beta_1 x_0 \oplus (1 - \beta_1)\, Tx$. For a given $x_0 \in C$, the existence of x_1 is guaranteed if G_1 has a fixed point. Now for any $u, v \in C$, we have

$$d(G_1(u), G_1(\nu)) = d(\alpha_1 x_0 \oplus (1 - \alpha_1)S(\beta_1 x_0 \oplus (1 - \beta_1) Tu),$$
$$\alpha_1 x_0 \oplus (1 - \alpha_1)S(\beta_1 x_0 \oplus (1 - \beta_1) T\nu))$$
$$\leq (1 - \alpha_1) d\left(S\left(\beta_1 x_0 \oplus (1 - \beta_1) Tu\right), S\left(\beta_1 x_0 \oplus (1 - \beta_1) T\nu\right)\right)$$
$$\leq (1 - \alpha_1) d\left(\beta_1 x_0 \oplus (1 - \beta_1) Tu, \beta_1 x_0 \oplus (1 - \beta_1) T\nu\right)$$
$$\leq (1 - \alpha_1)(1 - \beta_1) d(Tu, T\nu)$$
$$\leq (1 - \alpha_1)(1 - \beta_1) d(u, \nu).$$

Since $(1 - \alpha_1)(1 - \beta_1) < 1$, therefore G_1 is a contraction. By Banach contraction principle, G_1 has a unique fixed point. Thus the existence of x_1 is established. Continuing in this way, we can establish the existence of x_2, x_3 and so on. Thus the implicit algorithm (2.1) is well-defined.

Remark 2.4. Let (X, d) be a hyperbolic space. Let C be a nonempty closed convex subset of X. Let $S, T : C \to C$ be nonexpansive mappings. We assume that $F = F(S) \cap F(T) \neq \phi$. Let $x_0 \in C$ and $p \in F$. Set $r = d(x_0, p)$. Then

$$C(x_0) = C \cap B(p, r) = \{x \in C; d(p, x) \leq r\}$$

is nonempty and invariant under both S and T. So in the sequel, we assume that C is bounded provided S and T have a common fixed point.

Lemma 2.5. *Let C be a nonempty closed convex subset of a hyperbolic space X. Let $S, T : C \to C$ be nonexpansive mappings. If $\{x_n\}$ is defined by (2.1), then $\lim\limits_{n \to \infty} d(x_n, p)$ exists for any $p \in F$.*

Proof.

$$d(x_n, p) = d(\alpha_n x_{n-1} \oplus (1 - \alpha_n) Sy_n, p)$$
$$\leq \alpha_n d(x_{n-1}, p) + (1 - \alpha_n) d(Sy_n, p)$$
$$\leq \alpha_n d(x_{n-1}, p) + (1 - \alpha_n) d(y_n, p)$$
$$= \alpha_n d(x_{n-1}, p) + (1 - \alpha_n) d(\beta_n x_{n-1} \oplus (1 - \beta_n) Tx_n, p) \qquad (2.3)$$
$$\leq \alpha_n d(x_{n-1}, p) + (1 - \alpha_n) [\beta_n d(x_{n-1}, p) + (1 - \beta_n) d(Tx_n, p)]$$
$$\leq \alpha_n d(x_{n-1}, p) + (1 - \alpha_n) [\beta_n d(x_{n-1}, p) + (1 - \beta_n) d(x_n, p)],$$

or,

$$[\alpha_n + (1 - \alpha_n) \beta_n] d(x_n, p) \leq [\alpha_n + (1 - \alpha_n) \beta_n] d(x_{n-1}, p)$$

hence,

$$d(x_n, p) \leq d(x_{n-1}, p). \qquad (2.4)$$

This proves that $\{d(x_n, p)\}$ is decreasing which implies that $\lim\limits_{n \to \infty} d(x_n, p)$ exists.

$\square$

Let $\{x_n\}$ be a bounded sequence in a metric space X and C be a nonempty subset of X. Define $r(., \{x_n\}) : C \to [0, \infty)$, by:

$$r(x, \{x_n\}) = \limsup_{n \to \infty} d(x, x_n).$$

The asymptotic radius ρ_C of $\{x_n\}$ with respect to C is given by

$$\rho_C = \inf\left\{r(x, \{x_n\}) : x \in C\right\}.$$

The asymptotic radius of $\{x_n\}$ with respect to X will be denoted by ρ. A point $\xi \in C$ is said to be an asymptotic center of $\{x_n\}$ with respect to C if $r(\xi, \{x_n\}) = r(C, \{x_n\}) = \min\{r(x, \{x_n\}) : x \in C\}$. We denote by $A(C, \{x_n\})$, the set of asymptotic centers of $\{x_n\}$ with respect to C.

When $C = X$, we call ξ an asymptotic center of $\{x_n\}$ and simply use the notation $A(\{x_n\})$. In general, the set $A(C, \{x_n\})$ of asymptotic centers of a bounded sequence $\{x_n\}$ may be empty or may even contain infinitely many points.

The $\Delta-$convergence, introduced several years ago independently by Kuczumow [15] and Lim [16], behaves in $CAT(0)$ spaces as weak convergence in Banach spaces.

Definition 2.6. A bounded sequence $\{x_n\}$ in X is said to $\Delta-$converge to $x \in X$ if x is the unique asymptotic center of every subsequence $\{u_n\}$ of $\{x_n\}$. Symbolically, $x_n \overset{\Delta}{\to} x$.

We study strong convergence of the algorithm (2.1) in strictly convex hyperbolic spaces and its $\Delta-$convergence in uniformly convex hyperbolic spaces. It is remarked that Takahashi and Tamura [22] have required the domain of mappings to be a Banach space satisfying Opial's condition or whose norm is Fréchet differentiable to get their weak convergence results. Incidently, neither these concepts are defined nor we need in our results on a nonlinear domain.
We need the following known results.

Lemma 2.7. *[4] Let C be a nonempty closed and convex subset of a complete uniformly convex space X. Then every bounded sequence $\{x_n\}$ in X has a unique asymptotic center with respect to C.*

Lemma 2.8. *[3] If $\{x_n\}$ is a bounded sequence in a complete uniformly convex space X with $A(\{x_n\}) = \{x\}$ and $\{u_n\}$ is a subsequence of $\{x_n\}$ with $A(\{u_n\}) = \{u\}$ and the sequence $\{d(x_n, u)\}$ converges, then $x = u$.*

Remark 2.9. If (X, d) is uniformly convex, then (X, d) is strictly convex, i.e., whenever

$$d\Big(\alpha x \oplus (1 - \alpha)y, a\Big) = d(x, a) = d(y, a)$$

for $\alpha \in (0, 1)$ and $x, y, a \in X$, then we must have $x = y$.

Lemma 2.10. *[8] Let (X, d) be a uniformly convex hyperbolic space. Let $R \in [0, +\infty)$ be such that $\limsup\limits_{n\to\infty} d(x_n, a) \leq R$, $\limsup\limits_{n\to\infty} d(y_n, a) \leq R$, and*

$$\lim_{n\to\infty} d\Big(a, \alpha_n x_n \oplus (1 - \alpha_n)y_n\Big) = R,$$

where $\alpha_n \in [a, b]$, with $0 < a \leq b < 1$. Then we have

$$\lim_{n\to\infty} d(x_n, y_n) = 0.$$

A subset C of a metric space X is Chebyshev if for every $x \in X$, there exists $c_0 \in C$ such that $d(c_0, x) < d(c, x)$ for all $c \in C$, $c \neq c_0$. In other words, for each point of the space, there is a well-defined nearest point of C. So we define the nearest point projection $P : X \to C$ by sending x to c_0.

Lemma 2.11. *[8] Let (X, d) be a complete uniformly convex hyperbolic space. Let C be nonempty convex and closed subset of X. Let $x \in X$ be such that $d(x, C) < \infty$. Then there exists a unique best approximant of x in C, i.e., there exists a unique $c_0 \in C$ such that*

$$d(x, c_0) = d(x, C) = \inf\{d(x, c); c \in C\},$$

i.e., C is Chebyshev.

3   Convergence in strictly convex hyperbolic spaces

In this section, X is a strictly convex hyperbolic space.

Theorem 3.1. *Let C be a nonempty bounded closed and convex subset of X. Let $S, T : C \to C$ be nonexpansive mappings. Assume that $F \neq \emptyset$. Let $x_0 \in C$ and $\{x_n\}$ be given by (2.1). Then the following holds:*

If α_n and $\beta_n \in [a, b]$, with $0 < a \leq b < 1$, then $x_{n_i} \to y$ implies $y \in F$. In this case, we have $x_n \to y$.

Proof. Assume that $x_{n_i} \to y$. Let $p \in F$. Without loss of generality, we may assume $\lim\limits_{n \to \infty} \alpha_{n_i} = \alpha$ and $\lim\limits_{n \to \infty} \beta_{n_i} = \beta$. By Lemma 2.5, $\lim_{n \to \infty} d(x_n, p)$ exists. Hence,

$$\lim_{n \to \infty} d(x_n, p) = \lim_{n \to \infty} d(x_{n_i}, p) = d(y, p).$$

The inequalities (2.3) and (2.4) and the conclusion of Proposition 2.1 imply:

$$
\begin{align}
d(y,p) = &\quad d(\alpha y \oplus (1-\alpha)S(\beta y \oplus (1-\beta)Ty), p) \tag{3.1} \\
= &\quad \alpha d(y,p) + (1-\alpha)d(S(\beta y \oplus (1-\beta)Ty), p) \tag{3.2} \\
= &\quad \alpha d(y,p) + (1-\alpha)d(\beta y \oplus (1-\beta)Ty, p) \tag{3.3} \\
= &\quad \alpha d(y,p) + (1-\alpha)[\beta d(y,p) + (1-\beta)d(Ty,p)]. \tag{3.4}
\end{align}
$$

Set $r = d(y,p)$. Without loss of generality, we may assume $r > 0$ (otherwise the conclusion is trivial).

From (3.4),

$$r = \alpha r + (1-\alpha)[\beta r + (1-\beta)d(Ty,p)],$$

hence,

$$d(Ty, p) = r. \tag{3.5}$$

From (3.3),

$$r = \alpha r + (1-\alpha)d(\beta y \oplus (1-\beta)Ty, p),$$

hence,

$$r = d(\beta y \oplus (1-\beta)Ty, p). \tag{3.6}$$

The strict convexity of X implies,

$$Ty = y. \tag{3.7}$$

Also, from (3.2),

$$r = \alpha r + (1-\alpha)d(Sy, p),$$

hence,

$$r = d(Sy, p). \tag{3.8}$$

Moreover, from (3.1),

$$r = d(\alpha y \oplus (1-\alpha)\text{S}y, p),$$

The strict convexity of X implies

$$Sy = y. \tag{3.9}$$

Therefore, (3.7) and (3.9) imply that $y \in F$.

Now by Lemma 2.2,

$$\lim_{n \to \infty} d(x_n, y) = \lim_{n \to \infty} d(x_{n_i}, y) = 0,$$

and so $x_n \to y$. □

Remark 3.2. Let C be a nonempty bounded closed and convex subset of X. Let $S : C \to C$ be a nonexpansive mapping. Assume that $F(S) \neq \emptyset$. Let $x_0 \in C$ and $\{x_n\}$ be given by (2.2). Then the following holds:

If $\alpha_n \in [a, b]$, with $0 < a \leq b < 1$, then $x_{n_i} \to y$ implies $y \in F(S)$. In this case, we have $x_n \to y$.

If we assume compactness, then Theorem 3.1 implies the following result.

Theorem 3.3. *Let C be a nonempty bounded closed and convex subset of X. Let $S, T : C \to C$ be nonexpansive mappings. Assume that $F \neq \emptyset$. Fix $x_0 \in C$. Assume that $\overline{co}\{\{x_0\} \cup S(C) \cup T(C)\}$ is a compact subset of C. Define $\{x_n\}$ as in (2.1) where α_n and $\beta_n \in [a, b]$, with $0 < a \leq b < 1$, and x_0 is the initial element of the sequence. Then $\{x_n\}$ converges strongly to a common fixed point of S and T.*

Proof. As $x_n \in \overline{co}\{\{x_0\} \cup S(C) \cup T(C)\}$ (compact), so $\{x_n\}$ has a convergent subsequence $\{x_{n_i}\}$, i.e., $x_{n_i} \to z$. By Theorem 3.2, we have $z \in F$ and $x_n \to z$.   □

4   Convergence in uniformly convex hyperbolic spaces

In this section, X is a uniformly convex hyperbolic space.

Lemma 4.1. *Let C be a nonempty closed convex subset of X and let T and S be nonexpansive selfmappings on C with $F \neq \phi$. If for the sequence $\{x_n\}$ in (2.1), α_n and $\beta_n \in [a, b]$, with $0 < a \leq b < 1$, then we have,*

$$\lim_{n \to \infty} d(x_n, Tx_n) = \lim_{n \to \infty} d(x_n, Sx_n) = 0.$$

i.e., $\{x_n\}$ is an approximate common fixed point sequence for T and S.

Proof. Let $p \in F$. Then by Lemma 2.5, $\lim_{n \to \infty} d(x_n, p)$ exists. Set $r = \lim_{n \to \infty} d(x_n, p)$. If $r = 0$, then the conclusion is trivial. Therefore, we assume that $r > 0$.

From (2.1),

$$\begin{aligned}
d(x_n, p) &= d(\alpha_n x_{n-1} \oplus (1 - \alpha_n) Sy_n, p) \\
&\leq \alpha_n d(x_{n-1}, p) + (1 - \alpha_n) d(Sy_n, p) \\
&\leq \alpha_n d(x_{n-1}, p) + (1 - \alpha_n) d(y_n, p).
\end{aligned}$$

Taking lim inf on both sides in the above estimate, we have

$$r \leq \liminf_{n \to \infty} d(y_n, p). \tag{4.1}$$

Moreover,

$$\begin{aligned}
d(y_n, p) &= d(\beta_n x_{n-1} \oplus (1 - \beta_n) Tx_n, p) \\
&\leq \beta_n d(x_{n-1}, p) + (1 - \beta_n) d(Tx_n, p) \\
&\leq \beta_n d(x_{n-1}, p) + (1 - \beta_n) d(x_n, p).
\end{aligned}$$

Taking lim sup on both sides in the above estimate, we have

$$\limsup_{n \to \infty} d(y_n, p) \leq r. \tag{4.2}$$

Therefore, from (4.1) and (4.2)

$$\lim_{n \to \infty} d(y_n, p) = r. \tag{4.3}$$

Next,

$$\lim_{n \to \infty} d(x_n, p) = \lim_{n \to \infty} d(\alpha_n x_{n-1} \oplus (1 - \alpha_n) Sy_n, p) = r.$$

So by Lemma 2.5 , we have

$$\lim_{n \to \infty} d(x_{n-1}, Sy_n) = 0. \tag{4.4}$$

Also,

$$\lim_{n \to \infty} d(y_n, p) = \lim_{n \to \infty} d(\beta_n x_{n-1} \oplus (1 - \beta_n) Tx_n, p) = r,$$

gives by Lemma 2.5,

$$\lim_{n\to\infty} d\left(x_{n-1}, Tx_n\right) = 0. \tag{4.5}$$

As,

$$\begin{aligned} d\left(x_n,\ x_{n-1}\right) &= d\left(\alpha_n x_{n-1} \oplus \left(1 - \alpha_n\right) Sy_n, x_{n-1}\right) \\ &\leq \left(1 - \alpha_n\right) d\left(Sy_n, x_{n-1}\right), \end{aligned}$$

so by (4.4),

$$\lim_{n\to\infty} d\left(x_n,\ x_{n-1}\right) = 0. \tag{4.6}$$

Moreover,

$$d\left(Tx_n, x_n\right) \leq d\left(Tx_n, x_{n-1}\right) + d\left(\ x_{n-1}, x_n\right),$$

hence, by (4.5) and (4.6),

$$\lim_{n\to\infty} d\left(Tx_n, x_n\right) = 0. \tag{4.7}$$

Finally,

$$d\left(Sx_n, x_n\right) \leq d\left(Sx_n, Sy_n\right) + d\left(Sy_n, x_{n-1}\right) + d\left(x_{n-1}, x_n\right),$$

gives by (4.4) and (4.6),

$$\begin{aligned} \lim_{n\to\infty} d\left(Sx_n, x_n\right) &\leq \lim_{n\to\infty} d\left(Sx_n, Sy_n\right) \\ &\leq \lim_{n\to\infty} d\left(\ x_n,\ y_n\right) \\ &= \lim_{n\to\infty} d\left(\ x_n,\ \beta_n x_{n-1} \oplus \left(1 - \beta_n\right) Tx_n\right) \\ &\leq \lim_{n\to\infty} \left(\beta_n d\left(\ x_n,\ x_{n-1}\right) + \left(1 - \beta_n\right) d\left(\ x_n,\ Tx_n\right)\right). \end{aligned}$$

This implies by (4.6) and (4.7),

$$\lim_{n\to\infty} d\left(Sx_n, x_n\right) = 0. \tag{4.8}$$

The result follows from (4.7) and (4.8).

$$\square$$

As a direct consequence of Lemma 4.1, we establish Δ-convergence of the algorithm (2.1).

Theorem 4.2. *If X is complete, C a nonempty closed convex subset of X, and T, S are non-expansive selfmappings on C with $F \neq \phi$. Then the sequence $\{x_n\}$ in (2.1), $\triangle-$converges to a common fixed point of T and S.*

Proof. As $\{x_n\}$ is bounded, so by Lemma 2.7, $\{x_n\}$ has a unique asymptotic center, that is, $A(\{x_n\}) = \{x\}$. Let $\{u_n\}$ be any subsequence of $\{x_n\}$ such that $A(\{u_n\}) = \{u\}$. Now by Lemma 4.1, we have $\lim_{n\to\infty} d(u_n, Tu_n) = 0 = \lim_{n\to\infty} d(u_n, Su_n)$.

We claim that u is a common fixed point of T and S.

Clearly,

$$d\left(Tu, u_n\right) \leq d\left(Tu, Tu_n\right) + d\left(Tu_n, u_n\right),$$

Taking limsup,

$$\limsup_{n\to\infty} d\left(Tu, u_n\right) \leq \limsup_{n\to\infty} d\left(u, u_n\right),$$

we get, $r(Tu, u_n) \leq r(u, u_n)$. i. e., $Tu \in A(u_n)$. Hence, $Tu = u$. Similarly, we can show that $Su = u$.

Therefore, u is the common fixed point of T and S.

Suppose $x \neq u$. By the uniqueness of asymptotic centers,

$$
\begin{aligned}
\limsup_{n \to \infty} d(u_n, u) \;&< \; \limsup_{n \to \infty} d(u_n, x) \\
&\leq \; \limsup_{n \to \infty} d(x_n, x) \\
&< \; \limsup_{n \to \infty} d(x_n, u).
\end{aligned}
$$

Since $\lim_{n \to \infty} d(x_n, u)$ exists. Therefore,

$$
\limsup_{n \to \infty} d(u_n, u) < \limsup_{n \to \infty} d(u_n, u),
$$

a contradiction. Hence $x = u$. Since $\{u_n\}$ is an arbitrary subsequence of $\{x_n\}$, therefore $A(\{u_n\}) = \{u\}$ for all subsequences $\{u_n\}$ of $\{x_n\}$. This proves that $\{x_n\}$ $\triangle-$converges to a common fixed point of T and S.

$\square$

Remark 4.3. *Let X and C be as in Theorem 4.1 and let S be a nonexpansive selfmapping on C with $F(S) \neq \phi$. Then the sequence $\{x_n\}$ in (2.2), $\triangle-$converges to a fixed point of S.*

Using the concept of near point projection, we establish the following amazing convergence result.

Theorem 4.4. *Let X, C, S, T, and F be as in Theorem 4.1. Let P be the nearest point projection of C onto F. For an initial value $x_0 \in C$, define $\{x_n\}$ as in (2.1), where α_n and $\beta_n \in [a, b]$, with $0 < a \leq b < 1$. Then $\{Px_n\}$ converges strongly to the asymptotic center of $\{x_n\}$.*

Proof. By calculations similar to those in the proof of Lemma 2.2 and mathematical induction, we get

$$d\left(Px_{n-1}, x_{n-1+m}\right) \leq d\left(Px_{n-1}, x_{n-1}\right), \text{ for } m \geq 1, n \geq 1. \tag{4.9}$$

We know by Theorem 4.2 that $\{x_n\}$ Δ-converges to $y \in F$ and $\{d(x_n, y)\}$ converges by Lemma 2.2. Now Lemmas 2.3 and 2.4 imply that $A(\{x_n\}) = \{y\}$.

Let us prove that $\{Px_n\}$ converges strongly to y. Assume not, i.e., there exist $\varepsilon > 0$ and a subsequence $\{Px_{n_i}\}$ such that $d(Px_{n_i}, y) \geq \varepsilon$, for any $n_i \geq 1$. We must have $R = d(x_0, y) > 0$, otherwise $\{x_n\}$ is a constant sequence. From

$$
\begin{cases}
d(x_{n_i}, y) & \leq \quad d(x_{n_i}, y) \\
d(x_{n_i}, Px_{n_i}) & \leq \quad d(x_{n_i}, y) \\
d(Px_{n_i}, y) & \geq \quad \varepsilon = d(x_{n_i}, y)\dfrac{\varepsilon}{d(x_{n_i}, y)} \geq d(x_{n_i}, y)\dfrac{\varepsilon}{R}
\end{cases}
$$

we get

$$
d\left(x_{n_i}, \frac{1}{2}Px_{n_i} \oplus \frac{1}{2}y\right) \leq d(x_{n_i}, y)\left(1 - \delta\left(d(x_{n_i}, y), \frac{\varepsilon}{R}\right)\right),
$$

for any $n_i \geq 1$. Using the properties of the modulus of uniform convexity, there exists $\eta > 0$ such that

$$
\delta\left(d(x_{n_i}, y), \frac{\varepsilon}{R}\right) \geq \eta,
$$

for any $n_i \geq 1$. Hence

$$
d\left(x_{n_i}, \frac{1}{2}Px_{n_i} \oplus \frac{1}{2}y\right) \leq d(x_{n_i}, y)\left(1 - \eta\right),
$$

for any $n_i \geq 1$. Using the definition of the nearest point projection P, we get

$$
d\left(x_{n_i}, Px_{n_i}\right) \leq d(x_{n_i}, y)\left(1 - \eta\right),
$$

for any $n_i \geq 1$. Using the inequality (4.9), we get

$$
d\left(x_{n_i+m}, Px_{n_i}\right) \leq d(x_{n_i}, y)\left(1 - \eta\right),
$$

for any $n_i \geq 1$, and $m \geq 1$. As $Px_{n_i} \in F$, so $\{d(x_n, Px_{n_i})$ is decreasing (in n and fixed n_i). Hence

$$\limsup_{m \to \infty} d\left(x_{n_i+m}, Px_{n_i}\right) = \lim_{n \to \infty} d\left(x_n, Px_{n_i}\right) \leq d\left(x_{n_i}, y\right)\left(1 - \eta\right),$$

for any $n_i \geq 1$. Since y is the asymptotic center of $\{x_n\}$, we get

$$\lim_{n \to \infty} d\left(x_n, y\right) \leq \lim_{n \to \infty} d\left(x_n, Px_{n_i}\right) \leq d\left(x_{n_i}, y\right)\left(1 - \eta\right),$$

for any $n_i \geq 1$. Finally, since $y \in F$, if we let $n_i \to \infty$, we get

$$\lim_{n \to \infty} d\left(x_n, y\right) \leq \lim_{n \to \infty} d\left(x_n, y\right)\left(1 - \eta\right).$$

From $\varepsilon \leq d(x_{n_i}, Px_{n_i}) \leq d(x_{n_i}, y)$, we conclude that $\varepsilon \leq \lim_{n \to \infty} d\left(x_n, y\right)$, a contradiction. Therefore $\{Px_n\}$ converges strongly to y. □

Remark 4.5. Let X, C, S, P, and F be as in Theorem 4.1. For an initial value $x_0 \in C$, define $\{x_n\}$ as in (2.2), where $\alpha_n \in [a, b]$, with $0 < a \leq b < 1$. Then $\{Px_n\}$ converges strongly to the asymptotic center of $\{x_n\}$.

Remark 4.6. (i) It is shown by Khamsi and Khan [8] that CAT(0) spaces are uniformly convex with

$$\delta(r, \varepsilon) = 1 - \sqrt{1 - \frac{\varepsilon^2}{4}}.$$

Hence, Theorem 4.1 sets analogue of Proposition 3.7 of Kirk and Panyanak [13] for two nonexpansive mappings on a uniformly convex hyperbolic space.

 (ii) Theorem 4.1, generalizes Theorem 3.3 of Plubtieng et al. [19] for two nonexpansive mappings on a uniformly convex hyperbolic space.

(iii) Theorems 3.1 and 4.2 extend the corresponding results of Takahashi and Tamura [22] to a nonlinear domain for the implicit algorithm (2.1).

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