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# A class of three dimensional almost coKähler manifolds

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**Abstract**. In this paper, we consider three-dimensional almost coKähler manifold $M^3$ satisfying $\nabla_\xi h = 0$. We prove that the Ricci tensor of $M^3$ is cyclic-parallel if and only if $M^3$ is conformally flat with $\xi$ an eigenvector field of the Ricci operator, and this is also equivalent to that $M^3$ is locally isometric to either the flat Euclidean space $\mathbb{R}^3$ or the Riemannian product $\mathbb{R} \times N^2(c)$, where $N^2(c)$ denotes a Kähler surface of constant curvature $c \neq 0$.

## 1 Introduction

As a special class of almost contact metric manifolds and an analogy of Kähler manifolds, the geometry of (almost) coKähler manifolds was introduced and studied in the last years by many authors (see for example Blair [1], Goldberg and Yano [7], Dacko et al. [4, 6] and Olszak [10, 11]). In the present paper, (almost) coKähler manifolds are just (almost) cosymplectic manifolds discussed in the above earlier literatures. The main reason why the new terminology recently was adopted widely lies in the fact that the coKähler manifolds are really the odd-dimensional analogy of Kähler manifolds (see [8]). In a recent survey [3], the authors collected some new results concerning (almost) coKähler manifolds both from geometrical and topological viewpoints. It is also worth pointing out that Perrone in [12, 13] obtained some classification results of three-dimensional almost coKähler manifolds which are homogeneous or the Reeb vector fields are minimal. Recently, three-dimensional almost coKähler manifolds were also studied by Wang [14].

In this paper, we investigate three-dimensional almost coKähler manifolds $M^3$ satisfying $\nabla_\xi h = 0$. An example satisfying this condition was provided in Section 3. We first give some classifications of $M^3$ for which the Ricci tensor is cyclic-parallel in Section 4. As stated in Dacko [5], it is difficult to give a complete classification of $M^3$ with a conformal flatness condition. In Section 5, we obtain that a three-dimensional conformally flat almost coKähler manifold with $\nabla_\xi h = 0$ and $\xi$ being an eigenvector field of the Ricci operator is locally isometric to either the flat Euclidean space $\mathbb{R}^3$ or the product space $\mathbb{R} \times N^2(c)$, where $N^2(c)$ denotes a Kähler surface of constant curvature $c \neq 0$.

## 2 Almost coKähler manifolds

On a smooth manifold $M^{2n+1}$ of dimensional $2n + 1$, if there exist a $(1, 1)$-type tensor field $\phi$, a global vector field $\xi$ and a 1-form $\eta$ such that

$$\phi^2 = -\mathrm{id} + \eta \otimes \xi, \quad \eta(\xi) = 1, \tag{2.1}$$

where id denotes the identity endomorphism, then we say that $M^{2n+1}$ admits an almost contact structure denoted by the triplet $(\phi, \xi, \eta)$, where $\xi$ is called the Reeb vector field. From (2.1) we have $\phi(\xi) = 0$, $\eta \circ \phi = 0$ and $\mathrm{rank}(\phi) = 2n$. We denote by $(M^{2n+1}, \phi, \xi, \eta)$ a smooth manifold $M^{2n+1}$ endowed with an almost contact structure, which is called an almost contact manifold.

On the product manifold $M^{2n+1}\times\mathbb{R}$ we define an almost complex structure $J$ by

$$J\left(X, f\frac{d}{dt}\right) = \left(\phi X - f\xi, \eta(X)\frac{d}{dt}\right),$$

where $X$ denotes the vector field tangent to $M^{2n+1}$, $t$ is the coordinate of $\mathbb{R}$ and $f$ is a smooth function defined on product $M^{2n+1}\times\mathbb{R}$.

An almost contact structure is said to be normal if the above almost complex structure $J$ is integrable, i.e., $J$ is a complex structure. According to Blair [2], the normality of an almost contact structure is expressed by $[\phi,\phi] = -2d\eta\otimes\xi$, where $[\phi,\phi]$ denotes the Nijenhuis tensor of $\phi$ defined by

$$[\phi,\phi](X,Y) = \phi^2[X,Y] + [\phi X,\phi Y] - \phi[\phi X,Y] - \phi[X,\phi Y]$$

for any vector fields $X, Y$ on $M^{2n+1}$. If on an almost contact manifold there exists a Riemannian metric $g$ satisfying

$$g(\phi X,\phi Y) = g(X,Y) - \eta(X)\eta(Y)$$

for any vector fields $X, Y$, then $g$ is said to be compatible with the associated almost contact structure. In general, an almost contact manifold furnished with a compatible Riemannian metric is said to be an almost contact metric manifold and is denoted by $(M^{2n+1},\phi,\xi,\eta,g)$. The fundamental 2-form $\Phi$ on an almost contact metric manifold $M^{2n+1}$ is defined by $\Phi(X,Y) = g(X,\phi Y)$ for any vector fields $X$ and $Y$.

In this paper, by an almost coKähler manifold, we mean an almost contact metric manifold such that both the 1-form $\eta$ and 2-form $\Phi$ are closed (see [3]). In particular, an almost coKähler manifold is said to be a coKähler manifold if the associated almost contact structure is normal, which is also equivalent to $\nabla\phi = 0$, or equivalently, $\nabla\Phi = 0$.

In this paper, we set $h = \frac{1}{2}\mathcal{L}_\xi\phi$ and $h' = h\circ\phi$ on an almost coKähler manifold $M^{2n+1}$. Note that both $h$ and $h'$ are symmetric operators. Then the following formulas can be found in Olszak [10] and Perrone [12]:

$$h\xi = 0,\ \ h\phi + \phi h = 0,\ \ \mathrm{tr}(h) = \mathrm{tr}(h') = 0, \tag{2.2}$$

$$\nabla_\xi\phi = 0,\ \ \nabla\xi = h',\ \ \mathrm{div}\xi = 0, \tag{2.3}$$

$$\nabla_\xi h = -h^2\phi - \phi l, \tag{2.4}$$

$$\phi l\phi - l = 2h^2, \tag{2.5}$$

where $l := R(\cdot,\xi)\xi$ is the Jacobi operator along the Reeb vector field and the Riemannian curvature tensor $R$ is defined by $R(X,Y)Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z$, and tr and div denote the trace and divergence operators, respectively.

## 3 Three dimensional almost coKähler manifolds with $\nabla_\xi h = 0$

Throughout this paper, we denote by $(M^3,\phi,\xi,\eta,g)$ an almost coKähler manifold of dimension 3. By using the second term of relation (2.3) we may obtain $(\mathcal{L}_\xi g)(X,Y) = 2g(h'X,Y)$, this means that $\xi$ is a Killing vector field if and only if $h = 0$. Then, from Goldberg and Yano [7, Proposition 3] we know that a three-dimensional almost coKähler manifold is coKähler if and only if $h$ is vanishing. However, the converse of this assertion is not necessarily true in case of dimension greater than three (for more details see [3]).

**Proposition 3.1.** *On any 3-dimension almost coKähler manifold, the following four conditions are equivalent.*

$$\nabla_\xi h = 0,\quad \nabla_\xi l = 0,\quad h^2 + l = 0,\quad \phi l = l\phi. \tag{3.1}$$

*Proof.* Firstly, the equivalence between $\nabla_\xi h = 0$ and $\nabla_\xi l = 0$ was already proved by Perrone [12, Lemma 3.1]. If $\nabla_\xi h = 0$, using it in (2.4) gives that $h^2 = \phi l \phi$ and using this in (2.5) yields the third term of relation (3.1). Conversely, using $h^2 = -l$ in (2.5) and (2.4) we obtain $\nabla_\xi h = 0$. By (2.5), the equivalence between the last two terms of relation (3.1) is easy to check. □

In what follows, we shall study 3-dimensional almost coKähler manifolds $M^3$ satisfying $\nabla_\xi h = 0$. Obviously, on any 3-dimensional coKähler manifold, such a condition holds trivially because of vanishing of $h$. Next, we present a non-trivial example as follows.

**Example 3.2.** Let $M^3$ be an almost coKähler manifold of dimension 3 with the Reeb vector field $\xi$ belonging to the $k$-nullity distribution (see [4]), that is,

$$R(X,Y)\xi = k(\eta(Y)X - \eta(X)Y) \tag{3.2}$$

for any vector fields $X, Y$, where $k$ is a non-zero constant. If $k$ in the above relation is a smooth function, Dacko in [4] proved that $k$ must be a constant. From relation (3.2) we have $l = -k\phi^2$. Using it in equation (2.5) we have $h^2 = k\phi^2$, hence $k$ is a negative constant and $M^3$ is non-coKähler. Using this in equation (2.4) gives that $\nabla_\xi h = 0$.

Let $\mathcal{U}_1$ be the open subset of $M^3$ on which $h \neq 0$ and $\mathcal{U}_2$ the open subset defined by $\mathcal{U}_2 = \{p \in M^3 : h = 0 \text{ in a neighborhood of } p\}$. Therefore, $\mathcal{U}_1 \cup \mathcal{U}_2$ is an open dense subset of $M^3$. For any point $p \in \mathcal{U}_1 \cup \mathcal{U}_2$, we may find a local orthonormal basis $\{\xi, e_1, e_2 = \phi e_1\}$ of three distinct unit eigenvector fields of $h$ in certain neighborhood of $p$. On $\mathcal{U}_1$ we may assume that $he_1 = \lambda e_1$ and hence $he_2 = -\lambda e_2$, where $\lambda$ is a positive function, continuous on $M^3$ and smooth on $\mathcal{U}_1 \cup \mathcal{U}_2$. Using $\nabla_\xi h = 0$, according to [13, Lemma 2.1] we have the following

**Lemma 3.3.** *On $\mathcal{U}_1$ we have*

$$\begin{aligned}
&\nabla_\xi e_1 = 0,\ \nabla_\xi e_2 = 0,\ \nabla_{e_1}\xi = -\lambda e_2,\ \nabla_{e_2}\xi = -\lambda e_1,\\
&\nabla_{e_1} e_1 = \frac{1}{2\lambda}\Big(e_2(\lambda) + \sigma(e_1)\Big)e_2,\ \nabla_{e_2} e_2 = \frac{1}{2\lambda}\Big(e_1(\lambda) + \sigma(e_2)\Big)e_1,\\
&\nabla_{e_2} e_1 = \lambda\xi - \frac{1}{2\lambda}\Big(e_1(\lambda) + \sigma(e_2)\Big)e_2,\ \nabla_{e_1} e_2 = \lambda\xi - \frac{1}{2\lambda}\Big(e_2(\lambda) + \sigma(e_1)\Big)e_1,
\end{aligned}$$

*where $\sigma$ is an $1$-form defined by $\sigma(\cdot) = S(\cdot\,,\xi)$ and $S$ denotes the Ricci tensor.*

Using Lemma 3.3, the Ricci operator $Q$ can be expressed (see [13]) on $\mathcal{U}_1$ by

$$\begin{cases}
Q\xi = -2\lambda^2\xi + \sigma(e_1)e_1 + \sigma(e_2)e_2,\\
Qe_1 = \sigma(e_1)\xi + \frac{1}{2}\Big(r + 2\lambda^2\Big)e_1,\\
Qe_2 = \sigma(e_2)\xi + \frac{1}{2}\Big(r + 2\lambda^2\Big)e_2,
\end{cases} \tag{3.3}$$

with respect to the local basis $\{\xi, e_1, e_2\}$, where $r$ denotes the scalar curvature.

## 4 Cyclic-parallel Ricci tensor

In this section, we shall classify three-dimensional almost coKähler manifolds whose Ricci tensor is cyclic-parallel, that is,

$$g((\nabla_X Q)Y, Z) + g((\nabla_Y Q)Z, X) + g((\nabla_Z Q)X, Y) = 0 \tag{4.1}$$

for any vector fields $X, Y$ and $Z$. Making use of the well-known formula $\mathrm{div}Q = \frac{1}{2}\mathrm{grad}(r)$ and the symmetry of the Ricci tensor in equation (4.1), we have

**Lemma 4.1.** *The scalar curvature of a Riemannian manifold with cyclic-parallel Ricci tensor is a constant.*

Before giving our main result, we first prove the following

**Proposition 4.2.** *A 3-dimensional coKähler manifold with cyclic-parallel Ricci tensor is locally isometric to either the flat Euclidean space $\mathbb{R}^3$ or the product space $\mathbb{R} \times N^2(c)$, where $N^2(c)$ denotes a Kähler surface of constant curvature $c \neq 0$.*

*Proof.* Recall that on any three-dimensional Riemannian manifold, the following relation

$$\begin{aligned} R(X,Y)Z =& g(Y,Z)QX - g(X,Z)QY + g(QY,Z)X \\ & - g(QX,Z)Y - \frac{r}{2}(g(Y,Z)X - g(X,Z)Y) \end{aligned} \tag{4.2}$$

holds for any vector fields $X, Y$ and $Z$. On a three-dimensional coKähler manifold, using $h = 0$ in the second term of relation (2.3) gives that $Q\xi = 0$ and also $l = 0$. Thus, putting $Y = Z = \xi$ in equation (4.2) gives that

$$Q = \frac{r}{2}\mathrm{id} - \frac{r}{2}\eta \otimes \xi.$$

Taking the covariant derivative of the above relation and using the second term of relation (2.3), we have

$$(\nabla_X Q)Y = \frac{1}{2}X(r)Y - \frac{1}{2}X(r)\eta(Y)\xi \tag{4.3}$$

for any vector fields $X, Y$. Applying Lemma 4.1 on equation (4.3), it follows that the Ricci tensor is symmetric and hence $M^3$ is locally symmetric. According to Perrone [12, Proposition 3.1] we know that any 3-dimensional locally symmetric almost coKähler manifold is locally isometric to either the flat Euclidean space $\mathbb{R}^3$ or the Riemannian product $\mathbb{R} \times N^2(c)$, where $N^2(c)$ denotes a Kähler surface of constant curvature $c \neq 0$. This completes the proof. □

Using Proposition 4.2 we obtain directly the following

**Corollary 4.3.** *Any 3-dimensional coKähler manifold with constant scalar curvature is locally isometric to either the flat Euclidean space $\mathbb{R}^3$ or the Riemannian product $\mathbb{R} \times N^2(c)$, where $N^2(c)$ denotes a Kähler surface of constant curvature $c \neq 0$.*

Applying the above results, we may present our main results as follows:

**Theorem 4.4.** *Let $M^3$ be a 3-dimensional almost coKähler manifold satisfying $\nabla_\xi h = 0$. Suppose that the Ricci tensor of $M^3$ is cyclic-parallel. Then, $M^3$ is locally isometric to either the flat Euclidean space $\mathbb{R}^3$ or the product $\mathbb{R} \times N^2(c)$, where $N^2(c)$ denotes a Kähler surface of constant curvature $c \neq 0$.*

*Proof.* If $\mathcal{U}_1$ is an empty subset, i.e., $M^3$ is a coKähler manifold, then the proof follows from Lemma 4.1 and Proposition 4.2. Next, we consider the case that $\mathcal{U}_1$ is a non-empty subset and $\lambda$ on it is a positive smooth function. Using $\nabla_\xi h = 0$, it follows that $\xi(\lambda) = 0$. Thus, on $\mathcal{U}_1$ by applying Lemma 3.3 and relation (3.3) we obtain the following relations.

$$(\nabla_\xi Q)\xi = \xi(\sigma(e_1))e_1 + \xi(\sigma(e_2))e_2. \tag{4.4}$$

$$(\nabla_\xi Q)e_1 = \xi(\sigma(e_1))\xi + \frac{1}{2}\xi(r)e_1. \tag{4.5}$$

$$(\nabla_\xi Q)e_2 = \xi(\sigma(e_2))\xi + \frac{1}{2}\xi(r)e_2. \tag{4.6}$$

$$\begin{aligned} (\nabla_{e_1} Q)e_1 =& \Big(e_1(\sigma(e_1)) - \frac{1}{2\lambda}\sigma(e_2)(e_2(\lambda) + \sigma(e_1))\Big)\xi \\ & + \Big(\frac{1}{2}e_1(r) + 2\lambda e_1(\lambda)\Big)e_1 - \lambda\sigma(e_1)e_2. \end{aligned} \tag{4.7}$$

$$\begin{aligned} (\nabla_{e_2} Q)e_2 =& \Big(e_2(\sigma(e_2)) - \frac{1}{2\lambda}\sigma(e_1)(e_1(\lambda) + \sigma(e_2))\Big)\xi \\ & - \lambda\sigma(e_2)e_1 + \Big(\frac{1}{2}e_2(r) + 2\lambda e_2(\lambda)\Big)e_2. \end{aligned} \tag{4.8}$$

$$
\begin{aligned}
(\nabla_{e_1}Q)e_2 =&\Big(e_1(\sigma(e_2))+\frac{1}{2}\lambda(r+6\lambda^2)+\frac{1}{2\lambda}\sigma(e_1)(e_2(\lambda)+\sigma(e_1))\Big)\xi\\
&-\lambda\sigma(e_1)e_1-\Big(2\lambda\sigma(e_2)-\frac{1}{2}e_1(r)-2\lambda e_1(\lambda)\Big)e_2.
\end{aligned}
\tag{4.9}
$$

$$
\begin{aligned}
(\nabla_{e_2}Q)e_1 =&\Big(e_2(\sigma(e_1))+\frac{1}{2}\lambda(r+6\lambda^2)+\frac{1}{2\lambda}\sigma(e_2)(e_1(\lambda)+\sigma(e_2))\Big)\xi\\
&-\Big(2\lambda\sigma(e_1)-\frac{1}{2}e_2(r)-2\lambda e_2(\lambda)\Big)e_1-\lambda\sigma(e_2)e_2.
\end{aligned}
\tag{4.10}
$$

$$
\begin{aligned}
&(\nabla_{e_1}Q)\xi\\
=&2\lambda\Big(\sigma(e_2)-2e_1(\lambda)\Big)\xi\\
&+\Big(e_1(\sigma(e_1))-\frac{1}{2\lambda}\sigma(e_2)(e_2(\lambda)+\sigma(e_1))\Big)e_1\\
&+\Big(2\lambda^3+e_1(\sigma(e_2))+\frac{1}{2}\lambda(r+2\lambda^2)+\frac{1}{2\lambda}\sigma(e_1)(e_2(\lambda)+\sigma(e_1))\Big)e_2.
\end{aligned}
\tag{4.11}
$$

$$
\begin{aligned}
&(\nabla_{e_2}Q)\xi\\
=&2\lambda\Big(\sigma(e_1)-2e_2(\lambda)\Big)\xi\\
&+\Big(e_2(\sigma(e_2))-\frac{1}{2\lambda}\sigma(e_1)(e_1(\lambda)+\sigma(e_2))\Big)e_2\\
&+\Big(2\lambda^3+e_2(\sigma(e_1))+\frac{1}{2}\lambda(r+2\lambda^2)+\frac{1}{2\lambda}\sigma(e_2)(e_1(\lambda)+\sigma(e_2))\Big)e_1.
\end{aligned}
\tag{4.12}
$$

Firstly, putting $X=Y=Z$ in equation (4.1) then it follows that

$$
g((\nabla_X Q)X,X)=0 \tag{4.13}
$$

for any vector field $X$. Therefore, applying Lemma 4.1 we obtain from relations (4.4) and (4.7)-(4.8) that $\lambda$ is a global constant, where we have used that $\lambda>0$ on $\mathcal{U}_1$ and $\lambda$ is continuous. Since the scalar curvature $r$ is a constant, by using (4.4) and (4.7)-(4.8) in the well-known formula $\mathrm{div}Q=\frac{1}{2}\mathrm{grad}(r)$ we have

$$
\begin{cases}
\xi(\sigma(e_1))-\lambda\sigma(e_2)=0,\\
\xi(\sigma(e_2))-\lambda\sigma(e_1)=0,\\
e_1(\sigma(e_1))+e_2(\sigma(e_2))=\frac{1}{\lambda}\sigma(e_1)\sigma(e_2).
\end{cases}
\tag{4.14}
$$

Next, putting $Y=Z$ into equation (4.1) and using the symmetry of Ricci tensor we have

$$
g((\nabla_X Q)Y+2(\nabla_Y Q)X,Y)=0 \tag{4.15}
$$

for any vector field $X,Y$. Then, putting $X=e_1, Y=\xi$ and $X=\xi, Y=e_1$ into equation (4.15), respectively, from relations (4.5) and (4.11) we have that

$$
\begin{cases}
\lambda\sigma(e_2)+\xi(\sigma(e_1))=0,\\
e_1(\sigma(e_1))-\frac{1}{2\lambda}\sigma(e_2)\sigma(e_1)=0.
\end{cases}
\tag{4.16}
$$

Similarly, putting $X=e_2, Y=\xi$ and $X=\xi, Y=e_2$ into equation (4.15), respectively, from relations (4.6) and (4.12) we have that

$$
\begin{cases}
\lambda\sigma(e_1)+\xi(\sigma(e_2))=0,\\
e_2(\sigma(e_2))-\frac{1}{2\lambda}\sigma(e_1)\sigma(e_2)=0.
\end{cases}
\tag{4.17}
$$

Due to $\lambda>0$ on $\mathcal{U}_1$, from the first terms of (4.14) and (4.16) we get $\sigma(e_2)=0$. Similarly, from the second term of (4.14) and the first term (4.17) we have that $\sigma(e_1)=0$. This means that

$\xi$ is an eigenvector field of the Ricci operator. In fact, such conclusion can also be deduced from using (4.9)-(4.10) in (4.15).

On the other hand, if we set $X = \xi$, $Y = e_1$ and $Z = e_2$ in equation (4.1), by using equations (4.5), (4.9) and (4.12) we have that

$$r + 6\lambda^2 = 0. \tag{4.18}$$

Using this in relation (3.3) we see that $M^3$ is an Einstein manifold whose Ricci operator is $Q = -2\lambda^2\mathrm{id}$. Thus, putting it into equation (4.2) gives that $M^3$ is of constant sectional curvature. According to Olszak [11, Theorem 3] we know that any three-dimensional almost coKähler manifold of constant sectional curvature is a locally flat coKähler manifold. It follows that $\lambda = 0$, a contradiction. □

**Theorem 4.5.** *On a three-dimensionalalmost coKähler manifold $M^3$ satisfying $\nabla_\xi h = 0$, the following conditions are equivalent.*

- *$M^3$ is locally symmetric.*
- *The Ricci tensor of $M^3$ is parallel*
- *The Ricci tensor of $M^3$ is cyclic-parallel.*
- *$M^3$ is locally isometric to either the flat Euclidean space $\mathbb{R}^3$ or the Riemannian product $\mathbb{R} \times N^2(c)$, where $N^2(c)$ denotes a Kähler surface of constant curvature $c \neq 0$.*

*Proof.* The proof follows from Theorem 4.5 and [12, Proposition 3.1]. □

**Corollary 4.6.** *Let $M^3$ be a 3-dimensional almost coKähler manifold such that $\xi$ belongs to the $k$-nullity distribution, where $k$ is a smooth function on $M^3$. If the Ricci tensor of $M^3$ is cyclic-parallel, then $M^3$ is locally isometric to either the flat Euclidean space $\mathbb{R}^3$ or the product space $\mathbb{R} \times N^2(c)$, where $N^2(c)$ denotes a Kähler surface of constant curvature $c \neq 0$.*

*Proof.* For the coKähler case, i.e., $k = 0$, the proof follows from Lemma 4.1 and Proposition 4.2. For the non-coKähler case, i.e., $k \neq 0$, the proof follows from Theorem 4.4 and Example 3.2. □

## 5 Conformal flatness

It is well-known that a three-dimensional Riemannian manifold $M$ is said to be conformally flat if and only if its Weyl-Schouten tensor is of Codazzi-type, or equivalently, its Ricci tensor satisfies

$$(\nabla_X Q)Y - (\nabla_Y Q)X = \frac{1}{4}\{X(r)Y - Y(r)X\} \tag{5.1}$$

for any vector fields $X, Y$ on $M$.

**Theorem 5.1.** *Any 3-dimensional conformally flat coKähler manifold is locally isometric to either the flat Euclidean space $\mathbb{R}^3$ or the product space $\mathbb{R} \times N^2(c)$, where $N^2(c)$ denotes a Kähler surface of constant curvature $c \neq 0$.*

*Proof.* Let $M^3$ be a 3-dimensional coKähler manifold, we observe that $h$ vanishes and equation (4.3) holds. Since $M^3$ is conformally flat, putting (4.3) into (5.1) yields that

$$X(r)Y - Y(r)X = X(r)\eta(Y)\xi - Y(r)\eta(X)\xi$$

for any vector fields $X$ and $Y$. Replacing $Y$ by $\phi X$, where $X$ is orthogonal to $\xi$, in the above relation gives that $r$ is invariant along the contact distribution $\ker\eta$. On the other hand, letting $Y = \xi$ and using the formula $\mathrm{div}Q = \frac{1}{2}\mathrm{grad}(r)$ in equation (4.3) gives that $\xi(r) = 0$. This means that $r$ is a constant and hence the following proof follows from Corollary 4.3. □

**Theorem 5.2.** *Let $M^3$ be a three-dimensional conformally flat almost coKähler manifold satisfying $\nabla_\xi h = 0$. Then $\xi$ is an eigenvector field of the Ricci operator if and only if $M^3$ is locally isometric to either the flat Euclidean space $\mathbb{R}^3$ or the product space $\mathbb{R} \times N^2(c)$, where $N^2(c)$ denotes a Kähler surface of constant curvature $c \neq 0$.*

*Proof.* Firstly, let us consider the non-coKähler case, i.e., $\mathcal{U}_1$ is an non-empty subset. As shown in Theorem 4.4, $\nabla_\xi h = 0$ implies that $\xi(\lambda) = 0$. Then, putting $X = e_1$ and $Y = \xi$ into (5.1) and using (4.5) and (4.11) gives that

$$\begin{cases} 2\lambda\sigma(e_2) - 4\lambda e_1(\lambda) - \xi(\sigma(e_1)) - \frac{1}{4}e_1(r) = 0, \\ e_1(\sigma(e_1)) - \frac{1}{2\lambda}\sigma(e_2)(e_2(\lambda) + \sigma(e_1)) - \frac{1}{4}\xi(r) = 0, \\ 2\lambda^3 + e_1(\sigma(e_2)) + \frac{1}{2}\lambda(r + 2\lambda^2) + \frac{1}{2\lambda}\sigma(e_1)(e_2(\lambda) + \sigma(e_1)) = 0. \end{cases} \tag{5.2}$$

Similarly, putting $X = e_2$ and $Y = \xi$ into equation (5.1), and making use of equations (4.6) and (4.12) gives that

$$\begin{cases} 2\lambda\sigma(e_1) - 4\lambda e_2(\lambda) - \xi(\sigma(e_2)) - \frac{1}{4}e_2(r) = 0, \\ e_2(\sigma(e_2)) - \frac{1}{2\lambda}\sigma(e_1)(e_1(\lambda) + \sigma(e_2)) - \frac{1}{4}\xi(r) = 0, \\ 2\lambda^3 + e_2(\sigma(e_1)) + \frac{1}{2}\lambda(r + 2\lambda^2) + \frac{1}{2\lambda}\sigma(e_2)(e_1(\lambda) + \sigma(e_2)) = 0. \end{cases} \tag{5.3}$$

Similarly, putting $X = e_2$ and $Y = e_1$ into equation (5.1), and making use of equations (4.9) and (4.10) gives that

$$\begin{cases} \lambda\sigma(e_1) - \frac{1}{4}e_2(r) - 2\lambda e_2(\lambda) = 0, \\ \lambda\sigma(e_2) - \frac{1}{4}e_1(r) - 2\lambda e_1(\lambda) = 0, \\ e_2(\sigma(e_1)) - e_1(\sigma(e_2)) \\ = \frac{1}{2\lambda}\Big(\sigma(e_1)(e_2(\lambda) + \sigma(e_1)) - \sigma(e_2)(e_1(\lambda) + \sigma(e_2))\Big) \end{cases} \tag{5.4}$$

The above three relations (5.2)-(5.4) are the necessary and sufficient condition for a 3-dimensional almost coKähler manifold with $\nabla_\xi h = 0$ to be conformally flat. Putting the second term of (5.4) into the first term of (5.2) gives that

$$\xi(\sigma(e_1)) = \lambda\sigma(e_2) - 2\lambda e_1(\lambda). \tag{5.5}$$

Similarly, putting the first term of (5.4) into the first term of (5.3) gives that

$$\xi(\sigma(e_2)) = \lambda\sigma(e_1) - 2\lambda e_2(\lambda). \tag{5.6}$$

Assuming that $\xi$ is an eigenvector field of the Ricci operator, it follows from equations (5.5) and (5.6) that $\lambda$ is a global positive constant, where we have used that $\lambda$ is positive and continuous. In this case, from the last terms of relations (5.2)-(5.3) we see that $r = -6\lambda^2$ being a negative constant. As shown in proof of Theorem 4.4, using $r = -6\lambda^2$ in (3.3) we see that $M^3$ is Einstein and hence by (4.2) we see that $M^3$ is of constant sectional curvature. According to Olszak [11, Theorem 3], we know that $M^3$ is a locally flat coKähler manifold, a contradiction. If $M^3$ is a coKähler manifold, the proof follows from Theorem 5.1. The converse is easy to check. □

From Example 3.1 and Theorem 5.2 we have the following

**Corollary 5.3.** *Any 3-dimensional almost coKähler manifold with $\xi$ belonging to the $k$-nullity distribution ($k$ a smooth function) is conformally flat if and only if it is locally isometric to either the flat Euclidean space $\mathbb{R}^3$ or the Riemannian product $\mathbb{R}\times N^2(c)$, where $N^2(c)$ denotes a Kähler surface of constant curvature $c \neq 0$.*

**Remark 5.4.** Any one condition in Theorem 4.5 is also equivalent to that $M^3$ is conformally flat and $\xi$ is an eigenvector of the Ricci operator.

**Remark 5.5.** Dacko and Olszak [6, Section 5] constructed a three-dimensional conformally flat almost coKähler manifold which is non-coKähler and not locally flat. Moreover, on such manifold $\xi$ is not an eigenvector field of the Ricci operator and $\nabla_\xi h \neq 0$.

**Remark 5.6.** Dacko in [5, Theorem 1] gave a necessary and sufficient condition for a special three-dimensional almost cosymplectic manifold to be conformally flat.

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