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On the modular relations and dissections for a continued fraction of order sixteen

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Abstract. We prove modular relations and 2-, 4-, 8-, 16-dissections for a continued fraction of order sixteen which are analogous to the Rogers–Ramanujan continued fraction $R(q)$. We also show that the sign of the coefficients in the power series expansion of $I_1^*(q) := q^{-1/2}I_1(q)$ and its reciprocal are periodic with period 16.

1 Introduction

Throughout this paper, we let $|\ q\ |< 1$ and we use the following notation

$$(a;q)_\infty = \prod_{n=1}^{\infty}(1-aq^{n-1}),$$

$$(a_1,a_2,\cdots,a_m;q)_\infty = (a_1;q)_\infty(a_2;q)_\infty\cdots(a_m;q)_\infty.$$

The Rogers-Ramanujan continued fraction is defined by [9]

$$R(q) := q^{1/5}\frac{f(-q,-q^4)}{f(-q^2,-q^3)} = \frac{q^{1/5}}{1}\,{}_{+}\,\frac{q}{1}\,{}_{+}\,\frac{q^2}{1}\,{}_{+}\,\frac{q^3}{1}\,{}_{+}\,\cdots, \tag{1.1}$$

where

$$f(a,b) = \sum_{n=-\infty}^{\infty} a^{n(n+1)/2}\, b^{n(n-1)/2}, \quad |ab|<1, \tag{1.2}$$

is Ramanujan's general theta function.

Ramanujan eventually found several generalizations and ramifications of $R(q)$ which can be found in his notebooks [10] and “Lost Notebook" [11]. Ramanujan recorded many identities involving R(q) namely,

$$\frac{1}{\sqrt{R(q)}} - \gamma\sqrt{R(q)} = \frac{1}{q^{1/10}}\sqrt{\frac{(q;q)_\infty}{(q^5;q^5)_\infty}}\prod_{n=1}^{\infty}\frac{1}{1+\gamma q^{n/5}+q^{2n/5}}, \tag{1.3}$$

$$\frac{1}{\sqrt{R(q)}} - \delta\sqrt{R(q)} = \frac{1}{q^{1/10}}\sqrt{\frac{(q;q)_\infty}{(q^5;q^5)_\infty}}\prod_{n=1}^{\infty}\frac{1}{1+\delta q^{n/5}+q^{2n/5}}, \tag{1.4}$$

$$\left(\frac{1}{\sqrt{R(q)}}\right)^5 - \left(\gamma\sqrt{R(q)}\right)^5 = \frac{1}{q^{1/2}}\sqrt{\frac{(q;q)_\infty}{(q^5;q^5)_\infty}}\prod_{n=1}^{\infty}\frac{1}{(1+\gamma q^{n/5}+q^{2n/5})^5}, \tag{1.5}$$

$$\left(\frac{1}{\sqrt{R(q)}}\right)^5 - \left(\delta\sqrt{R(q)}\right)^5 = \frac{1}{q^{1/2}}\sqrt{\frac{(q;q)_\infty}{(q^5;q^5)_\infty}}\prod_{n=1}^{\infty}\frac{1}{(1+\delta q^{n/5}+q^{2n/5})^5}, \tag{1.6}$$

where $\gamma = (1-\sqrt{5})/2$ and $\delta = (1+\sqrt{5})/2$.

Ramanujan was the first person to give dissections of q-series identities. Ramanujan [11, p. 50] gave the 2-dissections of the continued fraction $R^*(q)$ and its reciprocal.

$$R^*(q) = \frac{(q^4,q^4,q^{16},q^{16};q^{20})_\infty}{(q^2,q^{10},q^{10},q^{18};q^{20})_\infty} - q\frac{(q^4,q^6,q^{14},q^{16};q^{20})_\infty}{(q^8,q^{10},q^{10},q^{12};q^{20})_\infty}, \tag{1.7}$$

$$\frac{1}{R^*(q)} = \frac{(q^8,q^8,q^{12},q^{12};q^{20})_\infty}{(q^6,q^{10},q^{10},q^{14};q^{20})_\infty} + q\frac{(q^2,q^8,q^{12},q^{18};q^{20})_\infty}{(q^4,q^{10},q^{10},q^{16};q^{20})_\infty}, \tag{1.8}$$

where

$$R^*(q) := \frac{f(-q,-q^4)}{f(-q^2,-q^3)} = \frac{1}{1}\,\frac{q}{+\,1}\,\frac{q^2}{+\,1}\,\frac{q^3}{+\,1}\,+\cdots.$$

Ramanujan [11, p. 50] also gave 5-dissections of the continued fraction $R^*(q)$ and its reciprocal. These results were improved upon and proved by Hirschhorn [6]. Hirschhorn [6] presented a conjecture on the 4-dissections of the Rogers–Ramanujan continued fraction and its reciprocal. In [7], Lewis and Liu settled Hirschhorn's conjecture. Hirschhorn was able to demonstrate the periodic behaviour of the sign of the coefficients in the series expansion of $R^*(q)$ and its reciprocal, first proved by Richmond and Szekeres [12]. In particular, if $R^*(q) = \sum_{n=0}^{\infty} a(n)q^n$, they proved that there exists N_0 such that for any $n \geq N_0$,

$$a(5n),\ a(5n+2) > 0 \ \text{ and } \ a(5n+1),\ a(5n+3),\ a(5n+4) < 0.$$

Andrews [4], further showed that the above inequalities hold for all n except that a(3)=a(8)=a(13)=a(23)=0 by considering the formulas for $\sum_{n=0}^{\infty} a(5n+j)q^n$, $0 \leq j \leq 4$, which were recorded in Ramanujan's Lost Notebook [11]. In [4], Andrews also considered the 2-dissections of the Rogers-Ramanujan continued fraction and its reciprocal.

In his second notebook [1, p. 24], [10], Ramanujan recorded the following beautiful continued fraction identity:

$$\frac{(a^2q^3;q^4)_\infty(b^2q^3;q^4)_\infty}{(a^2q;q^4)_\infty(b^2q;q^4)_\infty} = \frac{1}{1-ab}\,\frac{(a-bq)(b-aq)}{+\,(1-ab)(q^2+1)}\,\frac{(a-bq^3)(b-aq^3)}{+\,(1-ab)(q^4+1)}\,+\cdots, \quad |\,ab\,| < 1. \tag{1.9}$$

For $|\,ab\,| > 1$, Lisa Jacobsen [8] has shown that,

$$-\frac{1}{ab}\frac{(q^3/a^2;q^4)_\infty(q^3/b^2;q^4)_\infty}{(q/a^2;q^4)_\infty(q/b^2;q^4)_\infty} = \frac{1}{1-ab}\,\frac{(a-bq)(b-aq)}{+\,(1-ab)(q^2+1)}\,\frac{(a-bq^3)(b-aq^3)}{+\,(1-ab)(q^4+1)}\,+\cdots.$$

Changing q to q^4 and then putting $a = q^{1/2}$, $b = q^{-9/2}$ in the above continued fraction, we obtain

$$\frac{1}{I_1(q)} := \frac{f(-q^5,-q^{11})}{q^{1/2}f(-q^3,-q^{13})} = \frac{-q^{-9/2}(1-q^5)}{(1-q^{-4})}\,\frac{(q^{1/2}-q^{-1/2})(q^{-9/2}-q^{9/2})}{+\,(1-q^{-4})(1+q^8)}\,\frac{(q^{1/2}-q^{15/2})(q^{-9/2}-q^{25/2})}{+\,(1-q^{-4})(1+q^{16})}\,+\cdots. \tag{1.10}$$

Similarly, changing q by q^4 and then putting $a = q^{3/2}$ and $b = q^{5/2}$ in (1.9), we obtain

$$I_2(q) := q^{3/2}\frac{f(-q,-q^{15})}{f(-q^7,-q^9)} = \frac{q^{3/2}(1-q)}{(1-q^4)}\,\frac{q^4(1-q^3)(1-q^5)}{+\,(1-q^4)(1+q^8)}\,\frac{q^4(1-q^{11})(1-q^{13})}{+\,(1-q^4)(1+q^{16})}\,+\cdots. \tag{1.11}$$

In Section 2 of this paper, we establish modular relations for $I_1(q)$ and $I_2(q)$ which are similar to (1.3) and (1.4). In Section 3, we prove the 2-, 4-, 8- and 16-dissections of $I_1^*(q)$ and its reciprocals, where

$$I_1^*(q) = \sum_{n=0}^{\infty} a_n q^n = \frac{f(-q^3,-q^{13})}{f(-q^5,-q^{11})}, \tag{1.12}$$

$$\frac{1}{I_1^*(q)} = \sum_{n=0}^{\infty} b_n q^n = \frac{f(-q^5,-q^{11})}{f(-q^3,-q^{13})}. \tag{1.13}$$

We also show that the sign of the coefficients in the power series expansion of $I_1^*(q)$ and its reciprocal are periodic with period 16.

2 Modular relations of $I_1(q)$

In [2], Adiga et al. studied new identities and properties of the Ramanujan's continued fraction of order 12. In this section, we derive identities involving $I_1(q)$, which are similar to the identities (1.3) and (1.4).

Theorem 2.1. *We have*

$$\frac{1}{I_1(q)} - I_1(q) = \frac{f(-q,-q^7)f(q^4,q^4)}{q^{1/2}f(-q^3,-q^{13})f(-q^5,-q^{11})}. \tag{2.1}$$

Proof. From (1.10), we have

$$\frac{1}{\sqrt{I_1(q)}} - \sqrt{I_1(q)} = \frac{f(-q^{11},-q^5) - q^{1/2}f(-q^{13},-q^3)}{\sqrt{q^{1/2}f(-q^{11},-q^5)f(-q^{13},-q^3)}}. \tag{2.2}$$

From [1, Entry 30, p. 46], we have

$$f(a,b) = f(a^3b,ab^3) + af(b/a,a^5b^3). \tag{2.3}$$

Putting a=$-q^{1/2}$ and b=$q^{7/2}$ in (2.3), we get

$$f(-q^{1/2},q^{7/2}) = f(-q^{11},-q^5) - q^{1/2}f(-q^{13},-q^3). \tag{2.4}$$

Employing (2.4) in (2.2), we obtain

$$\frac{1}{\sqrt{I_1(q)}} - \sqrt{I_1(q)} = \frac{f(-q^{1/2},q^{7/2})}{\sqrt{q^{1/2}f(-q^{11},-q^5)f(-q^{13},-q^3)}}. \tag{2.5}$$

In a similar way, we deduce

$$\frac{1}{\sqrt{I_1(q)}} + \sqrt{I_1(q)} = \frac{f(q^{1/2},-q^{7/2})}{\sqrt{q^{1/2}f(-q^{11},-q^5)f(-q^{13},-q^3)}}. \tag{2.6}$$

Multiplying (2.5) and (2.6), we deduce that

$$\frac{1}{I_1(q)} - I_1(q) = \frac{f(q^{1/2},-q^{7/2})f(-q^{1/2},q^{7/2})}{q^{1/2}f(-q^{11},-q^5)f(-q^{13},-q^3)}. \tag{2.7}$$

From [1, Entry 30, p. 46], we have

$$f(a,b)f(-a,-b) = f(-a^2,-b^2)f(-ab,-ab). \tag{2.8}$$

Putting $a = q^{1/2}$ and $b = -q^{7/2}$ in (2.8), we get

$$f(q^{1/2},-q^{7/2})f(-q^{1/2},q^{7/2}) = f(-q,-q^7)f(q^4,q^4). \tag{2.9}$$

Employing (2.9) in (2.7), we obtain (2.1).

Theorem 2.2. *We have*

$$\frac{1}{\sqrt{I_2(q)}}+\alpha\sqrt{I_2(q)}=\frac{1}{\sqrt{q^{3/2}f(-q^{15},-q)f(-q^7,-q^9)}}\prod_{n=1}^{\infty}\Big(1-\gamma(-q^{1/4})^n$$
$$+\gamma(-q^{1/4})^{2n}-(-q^{1/4})^{3n}\Big)-q^{1/4}f(-q^5,-q^{11})\{\gamma-\beta I_1(q)\}, \tag{2.10}$$

where $\alpha=-2+\sqrt{\frac{354(\sqrt{2}+1)}{125\sqrt{2}}}$, $\beta=1-\sqrt{\frac{359(\sqrt{2}+1)}{250\sqrt{2}}}$ *and* $\gamma=1-2\sqrt{\frac{(\sqrt{2}+1)}{2\sqrt{2}}}$.

Proof. We have

$$\frac{1}{\sqrt{I_2(q)}}+\alpha\sqrt{I_2(q)}=\frac{f(-q^7,-q^9)+\alpha q^{3/2}f(-q,-q^{15})}{\sqrt{q^{3/2}f(-q^7,-q^9)f(-q,-q^{15})}}. \tag{2.11}$$

From [1, p. 48], we have

$$f(U_1,V_1)=\sum_{r=0}^{k-1}U_rf\left(\frac{U_{k+r}}{U_r},\frac{V_{k-r}}{U_r}\right), \tag{2.12}$$

where $U_n=a^{n(n+1)/2}b^{n(n-1)/2}$ and $V_n=a^{n(n-1)/2}b^{n(n+1)/2}$ for each integer n. Taking $k=8$, $a=\xi$ and $b=\xi^7q^{1/4}$ in (2.12), where $\xi=e^{\pi i/8}$, we obtain

$$f(\xi,\xi^7q^{1/4})=f(-q^7,-q^9)+\xi f(-q^7,-q^9)+\xi^{10}q^{1/4}f(-q^5,-q^{11})+\xi^{27}q^{3/4}$$
$$\times f(-q^3,-q^{13})+\xi^{52}q^{3/2}f(-q,-q^{15})+\xi^{85}q^{5/2}f(-q^{17},-q^{-1})$$
$$+\xi^{126}q^{15/4}f(-q^{19},-q^{-3})+\xi^{175}q^{21/4}f(-q^{21},-q^{-5}), \tag{2.13}$$

From [1], we have

$$f(a,b)\ =\ f(b,a)$$

and if n is an integer, then

$$f(a,b)=a^{n(n+1)/2}b^{n(n-1)/2}f(a(ab)^n,b(ab)^{-n}). \tag{2.14}$$

Using (2.14) in (2.13), we can deduce that

$$f(\xi,\xi^7q^{1/4})=(1+\xi)f(-q^7,-q^9)+q^{3/2}(\xi^{52}-\xi^{85})f(-q,-q^{15})+q^{3/4}$$
$$\times(\xi^{27}-\xi^{126})f(-q^3,-q^{13})+q^{1/4}(\xi^{10}-\xi^{175})f(-q^5,-q^{11}). \tag{2.15}$$

Note that $\xi^{52}-\xi^{85}=\alpha(1+\xi)$, $\xi^{27}-\xi^{126}=\beta(1+\xi)$ and $\xi^{10}-\xi^{175}=\gamma(1+\xi)$. It follows that

$$\frac{f(\xi,\xi^7q^{1/4})}{1+\xi}-q^{1/4}\{\gamma f(-q^5,-q^{11})+\beta q^{1/2}f(-q^3,-q^{13})\}=f(-q^7,-q^9)+\alpha q^{3/2}f(-q,-q^{15}). \tag{2.16}$$

Substituting (2.16) in (2.11), we have

$$\frac{1}{\sqrt{I_2(q)}}+\alpha\sqrt{I_2(q)}=\frac{\frac{f(\xi,\xi^7q^{1/4})}{1+\xi}-q^{1/4}\left\{\gamma f(-q^5,-q^{11})+\beta q^{1/2}f(-q^3,-q^{13})\right\}}{\sqrt{q^{3/2}f(-q^7,-q^9)f(-q,-q^{15})}}. \tag{2.17}$$

In Ramanujan's notation Jacobi triple product identity [1] takes the form

$$f(a,b)=(-a;ab)_\infty(-b;ab)_\infty(ab;ab)_\infty. \tag{2.18}$$

Using (2.18), we have

$$\begin{aligned}\frac{f(\xi,\xi^7q^{1/4})}{1+\xi} &= \frac{(-\xi;-q^{1/4})_\infty(-\xi^7q^{1/4};-q^{1/4})_\infty(-q^{1/4};-q^{1/4})_\infty}{1+\xi}\\ &= (\xi q^{1/4};-q^{1/4})_\infty(-\xi^7q^{1/4};-q^{1/4})_\infty(-q^{1/4};-q^{1/4})_\infty\\ &= \prod_{n=1}^{\infty}(1+\xi(-q^{1/4})^n)(1-\xi^7(-q^{1/4})^n)(1-(-q^{1/4})^n).\end{aligned}$$

Note that $\xi-\xi^7-1=-1+2\sqrt{\dfrac{\sqrt{2}+1}{2\sqrt{2}}}$ and $\xi^8=-1$. Using these in above equation, we see that

$$\frac{f(\xi,\xi^7q^{1/4})}{1+\xi}=\prod_{n=1}^{\infty}(1-\gamma(-q^{1/4})^n+\gamma(-q^{1/4})^{2n}-(q^{1/4})^{3n}). \tag{2.19}$$

Substituting (2.19) in (2.17), we get

$$\begin{aligned}&\frac{1}{\sqrt{I_2(q)}}+\alpha\sqrt{I_2(q)}=\frac{\prod_{n=1}^{\infty}(1-\gamma(-q^{1/4})^n+\gamma(-q^{1/4})^{2n}-(q^{1/4})^{3n})}{\sqrt{q^{3/2}f(-q^7,-q^9)f(-q,-q^{15})}}\\ &\times -q^{1/4}f(-q^5,-q^{11})\left\{\gamma+\beta q^{1/2}\frac{f(-q^3,-q^{13})}{f(-q^5,-q^{11})}\right\}\\ &=\frac{\prod_{n=1}^{\infty}(1-\gamma(-q^{1/4})^n+\gamma(-q^{1/4})^{2n}-(q^{1/4})^{3n})-q^{1/4}f(-q^5,-q^{11})\{\gamma+\beta I_1(q)\}}{\sqrt{q^{3/2}f(-q^7,-q^9)f(-q,-q^{15})}}.\end{aligned} \tag{2.20}$$

3 Dissections of $I_1^*(q)$ and its reciprocal

In [5], Bernard L. S. Lin studied 2-, 3-, 4-, 6- and 12-dissections of a continued fraction of order twelve. In [3] Adiga et al. studied 2- and 4-dissection of Ramanujan's continued fraction of order six. Motivated by these, in this section, we give 2-, 4-, 8- and 16-dissections of the continued fraction $I_1^*(q)$ and its reciprocal which are similar to (1.7)–(1.8).

Ramanujan recorded many identities involving $f(a,b)$ and its special cases $\varphi(q)$ and $\psi(q)$, which are defined by

$$\varphi(q):=f(q,q)=\sum_{n=-\infty}^{\infty}q^{n^2},$$

$$\psi(q):=f(q,q^3)=\sum_{n=0}^{\infty}q^{n(n+1)/2}.$$

We will require the following identity of Ramanujan [1, p. 45],

$$f(a,b)f(c,d)=f(ac,bd)f(ad,bc)+af(b/c,ac^2d)f(b/d,acd^2), \tag{3.1}$$

where $ab=cd$.

3.1 The 2-dissection of $I_1^*(q)$

Theorem 3.1. *If* $I_1^*(q) := \dfrac{f(-q^3,-q^{13})}{f(-q^5,-q^{11})} = \sum_{n=0}^{\infty} a_n q^n$, *then*

$$\sum_{n=0}^{\infty} a_{2n} q^n = \frac{\psi(-q^4) f(-q^7,-q^9)}{\varphi(-q^8) f(-q^5,-q^{11})}, \tag{3.2}$$

$$\sum_{n=0}^{\infty} a_{2n+1} q^n = -q \frac{\psi(-q^4) f(-q,-q^{15})}{\varphi(-q^8) f(-q^5,-q^{11})}. \tag{3.3}$$

Proof. We have

$$\begin{aligned}\sum_{n=0}^{\infty} a_n q^n &= \frac{f(-q^3,-q^{13})}{f(-q^5,-q^{11})} \\ &= \frac{f(-q^3,-q^{13})}{f(-q^5,-q^{11})} \frac{f(q^5,q^{11})}{f(q^5,q^{11})}. \end{aligned}\tag{3.4}$$

Putting $a=-q^3$, $b=-q^{13}$, $c=q^5$ and $d=q^{11}$ in (3.1), we get

$$f(-q^3,-q^{13}) f(q^5,q^{11}) = f(-q^8,-q^{24}) f(-q^{14},-q^{18}) - q^3 f(-q^8,-q^{24}) f(-q^2,-q^{30}). \tag{3.5}$$

Putting $a=q^5$ and $b=q^{11}$ in (2.8), we obtain

$$f(q^5,q^{11}) f(-q^5,-q^{11}) = f(-q^{10},-q^{22}) \varphi(-q^{16}). \tag{3.6}$$

Employing (3.5) and (3.6) in (3.4), we deduce that

$$I_1^*(q) = \sum_{n=0}^{\infty} a_n q^n = \frac{f(-q^8,-q^{24}) f(-q^{14},-q^{18}) - q^3 f(-q^8,-q^{24}) f(-q^2,-q^{30})}{f(-q^{10},-q^{22}) \varphi(-q^{16})}.$$

Hence

$$I_1^*(q) + I_1^*(-q) = 2\sum_{n=0}^{\infty} a_{2n} q^{2n} = 2\frac{f(-q^8,-q^{24}) f(-q^{14},-q^{18})}{f(-q^{10},-q^{22}) \varphi(-q^{16})}$$

and

$$I_1^*(q) - I_1^*(-q) = 2\sum_{n=0}^{\infty} a_{2n+1} q^{2n+1} = -2q^3 \frac{f(-q^8,-q^{24}) f(-q^2,-q^{30})}{f(-q^{10},-q^{22}) \varphi(-q^{16})}.$$

Changing q to $q^{1/2}$ in the above equations, we obtain (3.2) and (3.3).

3.2 The 4-dissection of $I_1^*(q)$

Theorem 3.2. *We have*

$$\sum_{n=0}^{\infty} a_{4n} q^n = \frac{\psi(-q^2) f(-q^6,-q^{10}) f(-q^7,-q^9)}{\varphi(-q^4)\varphi(-q^8) f(-q^5,-q^{11})}, \tag{3.7}$$

$$\sum_{n=0}^{\infty} a_{4n+1} q^n = q \frac{\psi(-q^2) f(-q^2,-q^{14})}{\varphi(-q^4)\varphi(-q^8)}, \tag{3.8}$$

$$\sum_{n=0}^{\infty} a_{4n+2} q^n = q^2 \frac{\psi(-q^2) f(-q^2,-q^{14}) f(-q,-q^{15})}{\varphi(-q^4)\varphi(-q^8) f(-q^5,-q^{11})}, \tag{3.9}$$

$$\sum_{n=0}^{\infty} a_{4n+3} q^n = -\frac{\psi(-q^2) f(-q^6,-q^{10}) f(-q^3,-q^{13})}{\varphi(-q^4)\varphi(-q^8) f(-q^5,-q^{11})}. \tag{3.10}$$

Proof. From (3.2), we have

$$\sum_{n=0}^{\infty} a_{2n}q^n = \frac{\psi(-q^4)f(-q^7,-q^9)}{\varphi(-q^8)f(-q^5,-q^{11})}\frac{f(q^5,q^{11})}{f(q^5,q^{11})}. \tag{3.11}$$

Setting $a=q^5$, $b=q^{11}$, $c=-q^7$ and $d=-q^9$ in (3.1), we get

$$f(q^5,q^{11})f(-q^7,-q^9) = f(-q^{12},-q^{20})f(-q^{14},-q^{18}) + q^5 f(-q^4,-q^{28})f(-q^2,-q^{30}). \tag{3.12}$$

Employing (3.12) and (3.6) in (3.11), we find that

$$\sum_{n=0}^{\infty} a_{2n}q^n = \frac{\psi(-q^4)\{f(-q^{12},-q^{20})f(-q^{14},-q^{18}) + q^5 f(-q^4,-q^{28})f(-q^2,-q^{30})\}}{\varphi(-q^8)\varphi(-q^{16})f(-q^{10},-q^{22})}.$$

This implies

$$\sum_{n=0}^{\infty} a_{4n}q^{2n} = \frac{\psi(-q^4)f(-q^{12},-q^{20})f(-q^{14},-q^{18})}{\varphi(-q^8)\varphi(-q^{16})f(-q^{10},-q^{22})},$$

$$\sum_{n=0}^{\infty} a_{4n+2}q^{2n} = q^4\frac{\psi(-q^4)f(-q^4,-q^{28})f(-q^2,-q^{30})\}}{\varphi(-q^8)\varphi(-q^{16})f(-q^{10},-q^{22})}.$$

Changing q to $q^{1/2}$ in the above equations, we obtain (3.7) and (3.9).
Proofs of (3.8) and (3.10) are similar.

3.3 The 8-dissection of $I_1^*(q)$

Theorem 3.3. *We have*

$$\sum_{n=0}^{\infty} a_{8n}q^n = \frac{\psi(-q)f(-q^6,-q^{10})f(-q^5,-q^3)f(-q^7,-q^9)}{\varphi(-q^2)\varphi(-q^4)\varphi(-q^8)f(-q^5,-q^{11})}, \tag{3.13}$$

$$\sum_{n=0}^{\infty} a_{8n+2}q^n = q\frac{\psi(-q)f(-q^6,-q^{10})f(-q,-q^7)f(-q^3,-q^{13})}{\varphi(-q^2)\varphi(-q^4)\varphi(-q^8)f(-q^5,-q^{11})}, \tag{3.14}$$

$$\sum_{n=0}^{\infty} a_{8n+3}q^n = -\frac{\psi(-q)f(-q^4,-q^{12})f(-q^5,-q^3)f(-q^7,-q^9)}{\varphi(-q^2)\varphi(-q^4)\varphi(-q^8)f(-q^5,-q^{11})}, \tag{3.15}$$

$$\sum_{n=0}^{\infty} a_{8n+4}q^n = q^2\frac{\psi(-q)f(-q^2,-q^{14})f(-q^5,-q^3)f(-q,-q^{15})}{\varphi(-q^2)\varphi(-q^4)\varphi(-q^8)f(-q^5,-q^{11})}, \tag{3.16}$$

$$\sum_{n=0}^{\infty} a_{8n+6}q^n = -q\frac{\psi(-q)f(-q^2,-q^{14})f(-q,-q^7)}{\varphi(-q^2)\varphi(-q^4)\varphi(-q^8)}, \tag{3.17}$$

$$\sum_{n=0}^{\infty} a_{8n+7}q^n = q\frac{\psi(-q)f(-q^4,-q^{12})f(-q^5,-q^3)f(-q,-q^{15})}{\varphi(-q^2)\varphi(-q^4)\varphi(-q^8)f(-q^5,-q^{11})}. \tag{3.18}$$

Proof. By (3.7), we have

$$\sum_{n=0}^{\infty} a_{4n}q^n = \frac{\psi(-q^2)f(-q^6,-q^{10})f(-q^7,-q^9)f(q^5,q^{11})}{\varphi(-q^4)\varphi(-q^8)f(-q^5,-q^{11})f(q^5,q^{11})}. \tag{3.19}$$

Employing (3.12) and (3.6) in (3.19), we find that

$$\sum_{n=0}^{\infty} a_{4n}q^n = \frac{\psi(-q^2)f(-q^6,-q^{10})}{\varphi(-q^4)\varphi(-q^8)\varphi(-q^{16})f(-q^{10},-q^{22})}$$
$$\times\{f(-q^{12},-q^{20})f(-q^{14},-q^{18}) + q^5 f(-q^4,-q^{28})f(-q^2,-q^{30})\}.$$

It follows immediately that

$$\sum_{n=0}^{\infty} a_{8n} q^{2n} = \frac{\psi(-q^2) f(-q^6,-q^{10}) f(-q^{12},-q^{20}) f(-q^{14},-q^{18})}{\varphi(-q^4)\varphi(-q^8)\varphi(-q^{16}) f(-q^{10},-q^{22})},$$

$$\sum_{n=0}^{\infty} a_{8n+4} q^{2n} = q^4 \frac{\psi(-q^2) f(-q^6,-q^{10}) f(-q^4,-q^{28}) f(-q^2,-q^{30})}{\varphi(-q^4)\varphi(-q^8)\varphi(-q^{16}) f(-q^{10},-q^{22})}.$$

Changing q to $q^{1/2}$ in the above equations, we obtain (3.13) and (3.16).
Proofs of (3.14), (3.15) (3.17) and (3.18) are similar.

Theorem 3.4. *We have*

$$\sum_{n=0}^{\infty} a_{8n+1} q^{2n} \equiv 0 \pmod 4, \tag{3.20}$$

$$\sum_{n=0}^{\infty} a_{8n+5} q^{n} \equiv \frac{\psi(-q) f(-q,-q^7)}{\varphi(-q^2)\varphi(-q^4)\varphi^4(-q)} \pmod 4. \tag{3.21}$$

Proof. From (3.8), we have

$$\begin{aligned}\sum_{n=0}^{\infty} a_{4n+1} q^{2n} &= q\frac{\psi(-q^2) f(-q^2,-q^{14})}{\varphi(-q^4)\varphi(-q^8)} \\ &= q\frac{\psi(-q^2) f(-q^2,-q^{14})}{\varphi(-q^4)\varphi(-q^8)} \frac{\varphi^2(q)}{\varphi^2(q)} \\ &= q\frac{\psi(-q^2) f(-q^2,-q^{14})\varphi^2(q)\varphi^2(-q)}{\varphi(-q^4)\varphi(-q^8)\varphi^4(-q^2)}.\end{aligned} \tag{3.22}$$

From (2.3) and $f(1,a) = 2f(a,a^3)$, we have

$$\varphi(q^4) - 2q\psi(q^8) = \varphi(-q). \tag{3.23}$$

Employing (3.23) in (3.22), we get

$$\sum_{n=0}^{\infty} a_{4n+1} q^{n} = q\frac{\psi(-q^2) f(-q^2,-q^{14})\varphi^2(q)}{\varphi(-q^4)\varphi(-q^8)\varphi^4(-q^2)} \left\{\varphi^2(q^4) - 4q\varphi(q^4)\psi(q^8) + 4q^2\psi^2(q^8)\right\}. \tag{3.24}$$

Note that $(\varphi(q))^{2^k} \equiv 1 \pmod 4$ for $k \geq 1$. This is clear when one writes

$$(\varphi(q))^{2^k} = \left(1 + 2\sum_{n\geq 1} q^{n^2}\right)^{2^k}$$

and then expands via the binomial theorem. Hence (3.24) becomes

$$\sum_{n=0}^{\infty} a_{4n+1} q^{n} \equiv q\frac{\psi(-q^2) f(-q^2,-q^{14})}{\varphi(-q^4)\varphi(-q^8)\varphi^4(-q^2)} \pmod 4. \tag{3.25}$$

Immediately it follows

$$\sum_{n=0}^{\infty} a_{8n+1} q^{2n} \equiv 0 \pmod 4$$

and hence $a_{8n+1} \equiv 0 \pmod 4$. From (3.25), we have also

$$\sum_{n=0}^{\infty} a_{8n+5} q^{2n} \equiv \frac{\psi(-q^2) f(-q^2,-q^{14})}{\varphi(-q^4)\varphi(-q^8)\varphi^4(-q^2)} \pmod 4. \tag{3.26}$$

Changing q to $q^{1/2}$ in (3.26), we obtain (3.21).

3.4. **The 16-dissection of $I_1^*(q)$**

We define $A := f(-q,-q^{15})$, $A^* := f(q,q^{15})$, $B := f(-q^3,-q^{13})$, $B^* := f(q^3,q^{13})$, $C := f(-q^5,-q^{11})$, $C^* := f(q^5,q^{11})$ $D := f(-q^7,-q^9)$, $D^* := f(q^7,q^9)$, $P := f(-q^6,-q^{10})$, $Q := f(-q^2,-q^{14})$, $X := f(q^3,q^5)$, $X^* := f(-q^3,-q^5)$, $Y := f(q,q^7)$ and $Y^* := f(-q,-q^7)$, $E := \dfrac{1}{C\varphi(-q)\varphi(-q^2)\varphi(-q^4)\varphi(-q^8)}$, $F := \dfrac{1}{B\varphi(-q)\varphi(-q^2)\varphi(-q^4)\varphi(-q^8)}$.

Theorem 3.5. *We have*

$$\sum_{n=0}^{\infty} a_{16n}q^n = EX^*\{-AA^*QXq^4 - AD^*QYq^3 + A^*DPYq^2 + DD^*PX\}, \tag{3.27}$$

$$\sum_{n=0}^{\infty} a_{16n+2}q^n = EX^*\psi(-q^2)\{-AB^*Yq^3 - AC^*Xq^2 + (-B^*X - C^*Y)Dq\}, \tag{3.28}$$

$$\sum_{n=0}^{\infty} a_{16n+3}q^n = E\psi(-q^2)\{AA^*QXq^4 + AD^*QYq^3 - A^*DPYq^2 - DD^*PX\}, \tag{3.29}$$

$$\sum_{n=0}^{\infty} a_{16n+4}q^n = EY^*\{(A^*BPY + A^*CQX)q^3 + CD^*QYq^2 + BD^*PXq\}, \tag{3.30}$$

$$\sum_{n=0}^{\infty} a_{16n+6}q^n = \frac{Y^*(B^*X + C^*Y)q}{\varphi(-q)\varphi(-q^2)\varphi(-q^4)}, \tag{3.31}$$

$$\sum_{n=0}^{\infty} a_{16n+7}q^n = E\psi(-q^2)\{-A^*CQYq^3 - A^*BPXq^2 - BD^*PY - CD^*QX\}, \tag{3.32}$$

$$\sum_{n=0}^{\infty} a_{16n+8}q^n = EX^*\{AA^*QYq^4 + AD^*QXq^2 - A^*DPXq - DD^*PY\}, \tag{3.33}$$

$$\sum_{n=0}^{\infty} a_{16n+10}q^n = EX^*\psi(-q^2)\{(B^*X + C^*Y)Aq^2 + B^*DYq + C^*DX\}, \tag{3.34}$$

$$\sum_{n=0}^{\infty} a_{16n+11}q^n = E\psi(-q^2)\{-AA^*QYq^4 - AD^*QXq^2 + A^*DPXq + DD^*PY\}, \tag{3.35}$$

$$\sum_{n=0}^{\infty} a_{16n+12}q^n = EY^*\{-A^*CQYq^3 - A^*BPXq^2 - (BD^*PY + CD^*QX)q\}, \tag{3.36}$$

$$\sum_{n=0}^{\infty} a_{16n+14}q^n = -\frac{Y^*(B^*Yq + C^*X)}{\varphi(-q)\varphi(-q^2)\varphi(-q^4)}, \tag{3.37}$$

$$\sum_{n=0}^{\infty} a_{16n+15}q^n = E\psi(-q^2)\{(A^*BPY + A^*CQX)q^2 + CD^*QYq + XD^*PB\}. \tag{3.38}$$

Proof. From (3.13), we have

$$\sum_{n=0}^{\infty} a_{8n}q^n = \frac{\psi(-q)f(-q^6,-q^{10})f(-q^5,-q^3)f(-q^7,-q^9)}{\varphi(-q^2)\varphi(-q^4)\varphi(-q^8)f(-q^5,-q^{11})}\,\frac{f(q^5,q^{11})}{f(q^5,q^{11})} \tag{3.39}$$

Using (2.3), we obtain

$$\psi(-q) = f(q^6,q^{10}) - qf(q^2,q^{14}), \tag{3.40}$$

$$f(-q^3,-q^5) = f(q^{14},q^{18}) - q^3 f(q^2,q^{30}). \tag{3.41}$$

Employing (3.40), (3.41), (3.6) and (3.12) in (3.39), we deduce that

$$\sum_{n=0}^{\infty} a_{8n}q^n = \frac{f(-q^6,-q^{10})}{\varphi(-q^2)\varphi(-q^4)\varphi(-q^8)\varphi(-q^{16})f(-q^{10},-q^{22})}$$
$$\{(f(q^6,q^{10}) - qf(q^2,q^{14}))(f(q^{14},q^{18}) - q^3f(q^2,q^{30}))$$
$$\times(f(-q^{12},-q^{20})f(-q^{14},-q^{18}) + q^5f(-q^4,-q^{28})f(-q^2,-q^{30}))\},$$
$$= \frac{f(-q^6,-q^{10})}{\varphi(-q^2)\varphi(-q^4)\varphi(-q^8)\varphi(-q^{16})f(-q^{10},-q^{22})}$$
$$\{f(q^6,q^{10})f(q^{14},q^{18})f(-q^{12},-q^{20})f(-q^{14},-q^{18})$$
$$+ q^5f(q^6,q^{10})f(q^{14},q^{18})f(-q^4,-q^{28})f(-q^2,-q^{30})$$
$$- q^3f(q^6,q^{10})f(q^2,q^{30})f(-q^{12},-q^{20})f(-q^{14},-q^{18})$$
$$- q^8f(q^6,q^{10})f(q^2,q^{30})f(-q^4,-q^{28})f(-q^2,-q^{30})$$
$$- qf(q^2,q^{14})f(q^{14},q^{18})f(-q^{14},-q^{18})f(-q^{12},-q^{20})$$
$$- q^6f(q^{14},q^{18})f(q^2,q^{14})f(-q^4,-q^{28})f(-q^2,-q^{30})$$
$$+ q^4f(q^2,q^{14})f(q^2,q^{30})f(-q^{12},-q^{20})f(-q^{14},-q^{18})$$
$$+ q^9f(q^2,q^{14})f(q^2,q^{30})f(-q^4,-q^{28})f(-q^2,-q^{30})\}.$$

Immediately it follows

$$\sum_{n=0}^{\infty} a_{16n}q^{2n} = \frac{f(-q^6,-q^{10})}{\varphi(-q^2)\varphi(-q^4)\varphi(-q^8)\varphi(-q^{16})f(-q^{10},-q^{22})}$$
$$\{f(q^6,q^{10})f(q^{14},q^{18})f(-q^{12},-q^{20})f(-q^{14},-q^{18})$$
$$- q^8f(q^6,q^{10})f(q^2,q^{30})f(-q^4,-q^{28})f(-q^2,-q^{30})$$
$$- q^6f(q^{14},q^{18})f(q^2,q^{14})f(-q^4,-q^{28})f(-q^2,-q^{30})$$
$$+ q^4f(q^2,q^{14})f(q^2,q^{30})f(-q^{12},-q^{20})f(-q^{14},-q^{18})\},$$

$$\sum_{n=0}^{\infty} a_{16n+8}q^{2n+1} = \frac{f(-q^6,-q^{10})}{\varphi(-q^2)\varphi(-q^4)\varphi(-q^8)\varphi(-q^{16})f(-q^{10},-q^{22})}$$
$$\{q^9f(q^2,q^{14})f(q^2,q^{30})f(-q^4,-q^{28})f(-q^2,-q^{30})$$
$$+ q^5f(q^6,q^{10})f(q^{14},q^{18})f(-q^4,-q^{28})f(-q^2,-q^{30})$$
$$- q^3f(q^6,q^{10})f(q^2,q^{30})f(-q^{12},-q^{20})f(-q^{14},-q^{18})$$
$$- qf(q^2,q^{14})f(q^{14},q^{18})f(-q^{14},-q^{18})f(-q^{12},-q^{20})\}.$$

Changing q to $q^{1/2}$ in above equations, we obtain (3.27) and (3.33), respectively. The remaining identities (3.28)–(3.32) and (3.34)–(3.38) follows similarly.

In a similar way, it is not hard to derive the following theorems.

4 Dissection of $\dfrac{1}{I_1^*(q)}$

Theorem 4.1. *If* $\dfrac{1}{I_1^*(q)} := \dfrac{f(-q^5,-q^{11})}{f(-q^3,-q^{13})} = \sum_{n=0}^{\infty} b_nq^n$, *then*

$$\sum_{n=0}^{\infty} b_{2n}q^n = \frac{\psi(-q^4)f(-q^7,-q^9)}{\varphi(-q^8)f(-q^3,-q^{13})}, \tag{4.1}$$

$$\sum_{n=0}^{\infty} b_{2n+1}q^n = q\frac{\psi(-q^4)f(-q,-q^{15})}{\varphi(-q^8)f(-q^3,-q^{13})}. \tag{4.2}$$

Theorem 4.2. *We have*

$$\sum_{n=0}^{\infty} b_{4n}q^n = \frac{\psi(-q^2)f(-q^6,-q^{10})f(-q^5,-q^{11})}{\varphi(-q^4)\varphi(-q^8)f(-q^3,-q^{13})}, \tag{4.3}$$

$$\sum_{n=0}^{\infty} b_{4n+1}q^n = -q\frac{\psi(-q^2)f(-q^6,-q^{10})f(-q,-q^{15})}{\varphi(-q^4)\varphi(-q^8)f(-q^3,-q^{13})}, \tag{4.4}$$

$$\sum_{n=0}^{\infty} b_{4n+2}q^n = q\frac{\psi(-q^2)f(-q^2,-q^{14})}{\varphi(-q^4)\varphi(-q^8)}, \tag{4.5}$$

$$\sum_{n=0}^{\infty} b_{4n+3}q^n = \frac{\psi(-q^2)f(-q^2,-q^{14})f(-q^7,-q^9)}{\varphi(-q^4)\varphi(-q^8)f(-q^3,-q^{13})}. \tag{4.6}$$

Theorem 4.3. *We have*

$$\sum_{n=0}^{\infty} b_{8n}q^n = \frac{\psi(-q)f(-q^4,-q^{12})f(-q^5,-q^3)f(-q^7,-q^9)}{\varphi(-q^2)\varphi(-q^4)\varphi(-q^8)f(-q^3,-q^{13})}, \tag{4.7}$$

$$\sum_{n=0}^{\infty} b_{8n+1}q^n = q\frac{\psi(-q)f(-q^6,-q^{10})f(-q^5,-q^3)f(-q,-q^{15})}{\varphi(-q^2)\varphi(-q^4)\varphi(-q^8)f(-q^3,-q^{13})}, \tag{4.8}$$

$$\sum_{n=0}^{\infty} b_{8n+3}q^n = \frac{\psi(-q)f(-q^6,-q^{10})f(-q,-q^7)f(-q^5,-q^{11})}{\varphi(-q^2)\varphi(-q^4)\varphi(-q^8)f(-q^3,-q^{13})}, \tag{4.9}$$

$$\sum_{n=0}^{\infty} b_{8n+4}q^n = q\frac{\psi(-q)f(-q^4,-q^{12})f(-q^5,-q^3)f(-q,-q^{15})}{\varphi(-q^2)\varphi(-q^4)\varphi(-q^8)f(-q^3,-q^{13})}, \tag{4.10}$$

$$\sum_{n=0}^{\infty} b_{8n+5}q^n = -\frac{\psi(-q)f(-q^2,-q^{14})f(-q^5,-q^3)f(-q^7,-q^9)}{\varphi(-q^2)\varphi(-q^4)\varphi(-q^8)f(-q^3,-q^{13})}, \tag{4.11}$$

$$\sum_{n=0}^{\infty} b_{8n+7}q^n = q\frac{\psi(-q)f(-q^2,-q^{14})f(-q,-q^7)}{\varphi(-q^2)\varphi(-q^4)\varphi(-q^8)}. \tag{4.12}$$

Theorem 4.4. *We have*

$$\sum_{n=0}^{\infty} b_{8n+2}q^n \equiv 0 \pmod 4, \tag{4.13}$$

$$\sum_{n=0}^{\infty} b_{8n+6}q^n \equiv \frac{f(-q,-q^7)\psi^2(q)f^2(q^3,q^5)f^2(q,q^7)}{\varphi^5(-q^2)\varphi(-q^4)\psi^3(-q)}\left\{f^2(q^3,q^5)+6q+q^2f^2(q,q^7)\right\} (mod\ 4). \tag{4.14}$$

Proof. From (4.5), we have

$$\begin{aligned}\sum_{n=0}^{\infty} b_{4n+2}q^n &= q\frac{\psi(-q^2)f(-q^2,-q^{14})}{\varphi(-q^4)\varphi(-q^8)}\frac{\varphi^2(q)}{\varphi^2(q)}\\ &= q\frac{\psi(-q^2)f(-q^2,-q^{14})\varphi^2(q)\psi^2(q^2)}{\varphi(-q^4)\varphi(-q^8)\psi^4(q)}\\ &\equiv q\frac{f(-q^2,-q^{14})\psi^2(q^2)\psi^4(-q)}{\varphi^5(-q^4)\varphi(-q^8)\psi^3(-q^2)} \pmod 4. \end{aligned} \tag{4.15}$$

We have

$$\psi^4(-q) = f^4(-q,-q^3) = (f(q^6,q^{10}) - qf(q^2,q^{14}))^4$$
$$= f^4\left(q^6,q^{10}\right) - 4q\, f^3\left(q^6,q^{10}\right) f\left(q^2,q^{14}\right) + 6\,q^2 f^2\left(q^6,q^{10}\right) f^2\left(q^2,q^{14}\right)$$
$$- 4q^3 f\left(q^6,q^{10}\right) f^3\left(q^2,q^{14}\right) + q^4 f^4\left(q^2,q^{14}\right). \tag{4.16}$$

Employing (4.16) in (4.15), we obtain

$$\sum_{n=0}^{\infty} b_{4n+2}q^n \equiv q\frac{f(-q^2,-q^{14})\psi^2(q^2)}{\varphi^5(-q^4)\varphi(-q^8)\psi^3(-q^2)}\{f^4\left(q^6,q^{10}\right) - 4q\, f^3\left(q^6,q^{10}\right) f\left(q^2,q^{14}\right)$$
$$+ 6\,q^2 f^2\left(q^6,q^{10}\right) f^2\left(q^2,q^{14}\right) - 4q^3 f\left(q^6,q^{10}\right) f^3\left(q^2,q^{14}\right) + q^4 f^4\left(q^2,q^{14}\right)\} \pmod 4. \tag{4.17}$$

Immediately, it follows that

$$\sum_{n=0}^{\infty} b_{8n+2}q^n \equiv 0 \pmod 4, \tag{4.18}$$

$$\sum_{n=0}^{\infty} b_{8n+6}q^n \equiv \frac{f(-q,-q^7)\psi^2(q)f^2(q^3,q^5)f^2(q,q^7)}{\varphi^5(-q^2)\varphi(-q^4)\psi^3(-q)}\left\{f^2(q^3,q^5) + 6q + q^2 f^2(q,q^7)\right\} \pmod 4. \tag{4.19}$$

Theorem 4.5. *We have*

$$\sum_{n=0}^{\infty} b_{16n}q^n = F\psi(-q^2)\{-A^*BQXq^3 + (-BD^*QY + A^*CPY)q^2 + CD^*PX\} \tag{4.20}$$

$$\sum_{n=0}^{\infty} b_{16n+1}q^n = FX^*\{-AA^*PYq^3 - A^*DQXq^2 - (AD^*PX + DD^*QY)q\}, \tag{4.21}$$

$$\sum_{n=0}^{\infty} b_{16n+3}q^n = FX^*\psi(-q^2)\{(-B^*X - C^*Y)Aq^2 + B^*DYq + C^*DX\}, \tag{4.22}$$

$$\sum_{n=0}^{\infty} b_{16n+4}q^n = F\psi(-q^2)\{-AA^*PYq^3 - A^*DQXq^2 - (AD^*PX + DD^*QY)q\}, \tag{4.23}$$

$$\sum_{n=0}^{\infty} b_{16n+5}q^n = FY^*\{A^*BQXq^3 - (BD^*QY - A^*CPY)q^2 - CD^*PX\}, \tag{4.24}$$

$$\sum_{n=0}^{\infty} b_{16n+7}q^n = \frac{-Y^*(B^*X + C^*Y)q}{\varphi(-q)\varphi(-q^2)\varphi(-q^4)}, \tag{4.25}$$

$$\sum_{n=0}^{\infty} b_{16n+8}q^n = F\psi(-q^2)\{A^*BQYq^3 + (BD^*QX - A^*CPX)q - CD^*PY\}, \tag{4.26}$$

$$\sum_{n=0}^{\infty} b_{16n+9}q^n = FX^*\{(AA^*PX + A^*DQY)q^2 + AD^*PYq + DD^*QX\}, \tag{4.27}$$

$$\sum_{n=0}^{\infty} b_{16n+11}q^n = FX^*\psi(-q^2)\{AB^*Yq^2 + AC^*Xq + (-B^*X - C^*Y)D\}, \tag{4.28}$$

$$\sum_{n=0}^{\infty} b_{16n+12}q^n = F\psi(-q^2)\{(AA^*PX + A^*DQY)q^2 + AD^*PYq + DD^*QX\}, \tag{4.29}$$

$$\sum_{n=0}^{\infty} b_{16n+13}q^n = FY^*\{A^*BQYq^3 + (BD^*QX - A^*CPX)q - CD^*PY\}, \tag{4.30}$$

$$\sum_{n=0}^{\infty} b_{16n+15}q^n = \frac{Y^*(B^*Yq + C^*X)}{\varphi(-q)\varphi(-q^2)\varphi(-q^4)}. \tag{4.31}$$

Corollary 4.6. *We have*

$$\frac{\sum_{n=0}^{\infty} a_{4n+1}q^n}{\sum_{n=0}^{\infty} a_{4n+2}q^n} = -\frac{\sum_{n=0}^{\infty} b_{4n}q^n}{\sum_{n=0}^{\infty} b_{4n+1}q^n}, \qquad q\frac{\sum_{n=0}^{\infty} a_{4n}q^n}{\sum_{n=0}^{\infty} a_{4n+3}q^n} = -\frac{\sum_{n=0}^{\infty} b_{4n+3}q^n}{\sum_{n=0}^{\infty} b_{4n+2}q^n}.$$

Proof. The proof the corollary follows from above Theorem.

Corollary 4.7. *We have*

$$\frac{\sum_{n=0}^{\infty} a_{8n}q^n}{\sum_{n=0}^{\infty} a_{8n+2}q^n} = -\frac{\sum_{n=0}^{\infty} b_{8n+5}q^n}{\sum_{n=0}^{\infty} b_{8n+7}q^n}, \qquad \frac{\sum_{n=0}^{\infty} a_{8n}q^n}{\sum_{n=0}^{\infty} a_{8n+3}q^n} = -\frac{\sum_{n=0}^{\infty} b_{8n+1}q^n}{\sum_{n=0}^{\infty} b_{8n+4}q^n},$$

$$\frac{\sum_{n=0}^{\infty} a_{8n+3}q^n}{\sum_{n=0}^{\infty} a_{8n+7}q^n} = -\frac{\sum_{n=0}^{\infty} b_{8n}q^n}{\sum_{n=0}^{\infty} b_{8n+4}q^n}, \qquad \frac{\sum_{n=0}^{\infty} a_{8n+4}q^n}{\sum_{n=0}^{\infty} a_{8n+6}q^n} = -\frac{\sum_{n=0}^{\infty} b_{8n+1}q^n}{\sum_{n=0}^{\infty} b_{8n+3}q^n},$$

$$q\frac{\sum_{n=0}^{\infty} a_{8n+4}q^n}{\sum_{n=0}^{\infty} a_{8n+7}q^n} = -\frac{\sum_{n=0}^{\infty} b_{8n+5}q^n}{\sum_{n=0}^{\infty} b_{8n}q^n}.$$

Proof. The proof the corollary follows from above Theorem.

Theorem 4.8. *We have $a_2 = a_4 = a_6 = a_7 = a_{12} = 0$. The remaining coefficients a_n satisfy the inequalities*

$$a_{16n},\ a_{16n+4},\ a_{16n+6},\ a_{16n+10},\ a_{16n+11},\ a_{16n+15}\ >\ 0,$$

$$a_{16n+2},\ a_{16n+3},\ a_{16n+7},\ a_{16n+8},\ \ a_{16n+12},\ a_{16n+14}\ <\ 0.$$

Proof. Changing q to $-q$ in (3.13), we have

$$\sum_{n=0}^{\infty} a_{8n}(-q)^n = \frac{\psi(q)f(-q^6,-q^{10})f(q^5,q^3)f(q^7,q^9)}{\varphi(-q^2)\varphi(-q^4)\varphi(-q^8)f(q^5,q^{11})}$$
$$= \frac{(-q,-q^3,-q^3,-q^5,-q^7,-q^7,-q^9,-q^9,-q^{11},-q^{13},-q^{13},-q^{15};q^{16})_\infty}{(q^2,q^2,q^4,q^4,q^6,q^8,q^8,q^{10},q^{12},q^{12},q^{14},q^{14};q^{16})_\infty}.$$

From the above equality, we obtain the signs $a_{16n}\ >\ 0$ and $a_{16n+8}\ <\ 0$.
Similarly, we can determine the signs of the remaining subsequences for a_n.

Theorem 4.9. *We have $b_1 = b_4 = b_7 = 0$. The remaining coefficients b_n satisfy the inequalities*

$$b_{16n},\ b_{16n+3},\ b_{16n+9},\ b_{16n+12},\ b_{16n+13},\ b_{16n+15}\ >\ 0,$$

$$b_{16n+1},\ b_{16n+4},\ b_{16n+5},\ b_{16n+7},\ b_{16n+8},\ ,\ b_{16n+11}\ <\ 0.$$

Proof. Changing q to $-q$ in (4.7), we have

$$\sum_{n=0}^{\infty} b_{8n}(-q)^n = \frac{\psi(q)f(-q^4,-q^{12})f(q^5,q^3)f(q^7,q^9)}{\varphi(-q^2)\varphi(-q^4)\varphi(-q^8)f(q^3,q^{13})}$$
$$= \frac{(-q,-q^3,-q^5,-q^5,-q^7,-q^7,-q^9,-q^9,-q^{11},-q^{11},-q^{13},-q^{15};q^{16})_\infty}{(q^2,q^2,q^4,q^6,q^6,q^8,q^8,q^{10},q^{10},q^{12},q^{14},q^{14};q^{16})_\infty}.$$

From the above equality, we obtain the signs $b_{16n} > 0$ and $b_{16n+8} < 0$.
Similarly, we can determine the signs of the remaining subsequences for b_n.

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