

On the modular relations and dissections for a continued fraction of order sixteen

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Abstract. We prove modular relations and 2-, 4-, 8-, 16-dissections for a continued fraction of order sixteen which are analogous to the Rogers–Ramanujan continued fraction $R(q)$. We also show that the sign of the coefficients in the power series expansion of $I_1^*(q) := q^{-1/2}I_1(q)$ and its reciprocal are periodic with period 16.

1 Introduction

Throughout this paper, we let $|q| < 1$ and we use the following notation

$$(a; q)_\infty = \prod_{n=1}^{\infty} (1 - aq^{n-1}),$$

$$(a_1, a_2, \dots, a_m; q)_\infty = (a_1; q)_\infty (a_2; q)_\infty \cdots (a_m; q)_\infty.$$

The Rogers-Ramanujan continued fraction is defined by [9]

$$R(q) := q^{1/5} \frac{f(-q, -q^4)}{f(-q^2, -q^3)} = \frac{q^{1/5}}{1} + \frac{q}{1} + \frac{q^2}{1} + \frac{q^3}{1} + \dots, \tag{1.1}$$

where

$$f(a, b) = \sum_{n=-\infty}^{\infty} a^{n(n+1)/2} b^{n(n-1)/2}, \quad |ab| < 1, \tag{1.2}$$

is Ramanujan’s general theta function.

Ramanujan eventually found several generalizations and ramifications of $R(q)$ which can be found in his notebooks [10] and “Lost Notebook” [11]. Ramanujan recorded many identities involving $R(q)$ namely,

$$\frac{1}{\sqrt{R(q)}} - \gamma\sqrt{R(q)} = \frac{1}{q^{1/10}} \sqrt{\frac{(q; q)_\infty}{(q^5; q^5)_\infty}} \prod_{n=1}^{\infty} \frac{1}{1 + \gamma q^{n/5} + q^{2n/5}}, \tag{1.3}$$

$$\frac{1}{\sqrt{R(q)}} - \delta\sqrt{R(q)} = \frac{1}{q^{1/10}} \sqrt{\frac{(q; q)_\infty}{(q^5; q^5)_\infty}} \prod_{n=1}^{\infty} \frac{1}{1 + \delta q^{n/5} + q^{2n/5}}, \tag{1.4}$$

$$\left(\frac{1}{\sqrt{R(q)}}\right)^5 - \left(\gamma\sqrt{R(q)}\right)^5 = \frac{1}{q^{1/2}} \sqrt{\frac{(q; q)_\infty}{(q^5; q^5)_\infty}} \prod_{n=1}^{\infty} \frac{1}{(1 + \gamma q^{n/5} + q^{2n/5})^5}, \tag{1.5}$$

$$\left(\frac{1}{\sqrt{R(q)}}\right)^5 - \left(\delta\sqrt{R(q)}\right)^5 = \frac{1}{q^{1/2}} \sqrt{\frac{(q; q)_\infty}{(q^5; q^5)_\infty}} \prod_{n=1}^{\infty} \frac{1}{(1 + \delta q^{n/5} + q^{2n/5})^5}, \tag{1.6}$$

where $\gamma = (1 - \sqrt{5})/2$ and $\delta = (1 + \sqrt{5})/2$.

Ramanujan was the first person to give dissections of q -series identities. Ramanujan [11, p. 50] gave the 2-dissections of the continued fraction $R^*(q)$ and its reciprocal.

$$R^*(q) = \frac{(q^4, q^4, q^{16}, q^{16}; q^{20})_\infty}{(q^2, q^{10}, q^{10}, q^{18}; q^{20})_\infty} - q \frac{(q^4, q^6, q^{14}, q^{16}; q^{20})_\infty}{(q^8, q^{10}, q^{10}, q^{12}; q^{20})_\infty}, \tag{1.7}$$

$$\frac{1}{R^*(q)} = \frac{(q^8, q^8, q^{12}, q^{12}; q^{20})_\infty}{(q^6, q^{10}, q^{10}, q^{14}; q^{20})_\infty} + q \frac{(q^2, q^8, q^{12}, q^{18}; q^{20})_\infty}{(q^4, q^{10}, q^{10}, q^{16}; q^{20})_\infty}, \tag{1.8}$$

where

$$R^*(q) := \frac{f(-q, -q^4)}{f(-q^2, -q^3)} = \frac{1}{1} + \frac{q}{1} + \frac{q^2}{1} + \frac{q^3}{1} + \dots$$

Ramanujan [11, p. 50] also gave 5-dissections of the continued fraction $R^*(q)$ and its reciprocal. These results were improved upon and proved by Hirschhorn [6]. Hirschhorn [6] presented a conjecture on the 4-dissections of the Rogers–Ramanujan continued fraction and its reciprocal. In [7], Lewis and Liu settled Hirschhorn’s conjecture. Hirschhorn was able to demonstrate the periodic behaviour of the sign of the coefficients in the series expansion of $R^*(q)$ and its reciprocal, first proved by Richmond and Szekeres [12]. In particular, if

$$R^*(q) = \sum_{n=0}^{\infty} a(n)q^n, \text{ they proved that there exists } N_0 \text{ such that for any } n \geq N_0,$$

$$a(5n), a(5n + 2) > 0 \text{ and } a(5n + 1), a(5n + 3), a(5n + 4) < 0.$$

Andrews [4], further showed that the above inequalities hold for all n except that $a(3)=a(8)=a(13)=a(23)=0$ by considering the formulas for $\sum_{n=0}^{\infty} a(5n + j)q^n, 0 \leq j \leq 4$, which were recorded in Ramanujan’s Lost Notebook [11]. In [4], Andrews also considered the 2-dissections of the Rogers-Ramanujan continued fraction and its reciprocal.

In his second notebook [1, p. 24], [10], Ramanujan recorded the following beautiful continued fraction identity:

$$\frac{(a^2q^3; q^4)_\infty (b^2q^3; q^4)_\infty}{(a^2q; q^4)_\infty (b^2q; q^4)_\infty} = \frac{1}{1 - ab} + \frac{(a - bq)(b - aq)}{(1 - ab)(q^2 + 1)} + \frac{(a - bq^3)(b - aq^3)}{(1 - ab)(q^4 + 1)} + \dots, \quad |ab| < 1. \tag{1.9}$$

For $|ab| > 1$, Lisa Jacobsen [8] has shown that,

$$\frac{1}{ab} \frac{(q^3/a^2; q^4)_\infty (q^3/b^2; q^4)_\infty}{(q/a^2; q^4)_\infty (q/b^2; q^4)_\infty} = \frac{1}{1 - ab} + \frac{(a - bq)(b - aq)}{(1 - ab)(q^2 + 1)} + \frac{(a - bq^3)(b - aq^3)}{(1 - ab)(q^4 + 1)} + \dots$$

Changing q to q^4 and then putting $a = q^{1/2}, b = q^{-9/2}$ in the above continued fraction, we obtain

$$\frac{1}{I_1(q)} := \frac{f(-q^5, -q^{11})}{q^{1/2} f(-q^3, -q^{13})} = \frac{-q^{-9/2}(1 - q^5)}{(1 - q^{-4})} + \frac{(q^{1/2} - q^{-1/2})(q^{-9/2} - q^{9/2})}{(1 - q^{-4})(1 + q^8)} + \frac{(q^{1/2} - q^{15/2})(q^{-9/2} - q^{25/2})}{(1 - q^{-4})(1 + q^{16})} + \dots \tag{1.10}$$

Similarly, changing q by q^4 and then putting $a = q^{3/2}$ and $b = q^{5/2}$ in (1.9), we obtain

$$I_2(q) := q^{3/2} \frac{f(-q, -q^{15})}{f(-q^7, -q^9)} = \frac{q^{3/2}(1 - q)}{(1 - q^4)} + \frac{q^4(1 - q^3)(1 - q^5)}{(1 - q^4)(1 + q^8)} + \frac{q^4(1 - q^{11})(1 - q^{13})}{(1 - q^4)(1 + q^{16})} + \dots \tag{1.11}$$

In Section 2 of this paper, we establish modular relations for $I_1(q)$ and $I_2(q)$ which are similar to (1.3) and (1.4). In Section 3, we prove the 2-, 4-, 8- and 16-dissections of $I_1^*(q)$ and its reciprocals, where

$$I_1^*(q) = \sum_{n=0}^{\infty} a_n q^n = \frac{f(-q^3, -q^{13})}{f(-q^5, -q^{11})}, \tag{1.12}$$

$$\frac{1}{I_1^*(q)} = \sum_{n=0}^{\infty} b_n q^n = \frac{f(-q^5, -q^{11})}{f(-q^3, -q^{13})}. \tag{1.13}$$

We also show that the sign of the coefficients in the power series expansion of $I_1^*(q)$ and its reciprocal are periodic with period 16.

2 Modular relations of $I_1(q)$

In [2], Adiga et al. studied new identities and properties of the Ramanujan’s continued fraction of order 12. In this section, we derive identities involving $I_1(q)$, which are similar to the identities (1.3) and (1.4).

Theorem 2.1. *We have*

$$\frac{1}{I_1(q)} - I_1(q) = \frac{f(-q, -q^7)f(q^4, q^4)}{q^{1/2}f(-q^3, -q^{13})f(-q^5, -q^{11})}. \tag{2.1}$$

Proof. From (1.10), we have

$$\frac{1}{\sqrt{I_1(q)}} - \sqrt{I_1(q)} = \frac{f(-q^{11}, -q^5) - q^{1/2}f(-q^{13}, -q^3)}{\sqrt{q^{1/2}f(-q^{11}, -q^5)f(-q^{13}, -q^3)}}. \tag{2.2}$$

From [1, Entry 30, p. 46], we have

$$f(a, b) = f(a^3b, ab^3) + af(b/a, a^5b^3). \tag{2.3}$$

Putting $a=-q^{1/2}$ and $b=q^{7/2}$ in (2.3), we get

$$f(-q^{1/2}, q^{7/2}) = f(-q^{11}, -q^5) - q^{1/2}f(-q^{13}, -q^3). \tag{2.4}$$

Employing (2.4) in (2.2), we obtain

$$\frac{1}{\sqrt{I_1(q)}} - \sqrt{I_1(q)} = \frac{f(-q^{1/2}, q^{7/2})}{\sqrt{q^{1/2}f(-q^{11}, -q^5)f(-q^{13}, -q^3)}}. \tag{2.5}$$

In a similar way, we deduce

$$\frac{1}{\sqrt{I_1(q)}} + \sqrt{I_1(q)} = \frac{f(q^{1/2}, -q^{7/2})}{\sqrt{q^{1/2}f(-q^{11}, -q^5)f(-q^{13}, -q^3)}}. \tag{2.6}$$

Multiplying (2.5) and (2.6), we deduce that

$$\frac{1}{I_1(q)} - I_1(q) = \frac{f(q^{1/2}, -q^{7/2})f(-q^{1/2}, q^{7/2})}{q^{1/2}f(-q^{11}, -q^5)f(-q^{13}, -q^3)}. \tag{2.7}$$

From [1, Entry 30, p. 46], we have

$$f(a, b)f(-a, -b) = f(-a^2, -b^2)f(-ab, -ab). \tag{2.8}$$

Putting $a = q^{1/2}$ and $b = -q^{7/2}$ in (2.8), we get

$$f(q^{1/2}, -q^{7/2})f(-q^{1/2}, q^{7/2}) = f(-q, -q^7)f(q^4, q^4). \tag{2.9}$$

Employing (2.9) in (2.7), we obtain (2.1).

Theorem 2.2. *We have*

$$\frac{1}{\sqrt{I_2(q)}} + \alpha\sqrt{I_2(q)} = \frac{1}{\sqrt{q^{3/2}f(-q^{15}, -q)f(-q^7, -q^9)}} \prod_{n=1}^{\infty} \left(1 - \gamma(-q^{1/4})^n + \gamma(-q^{1/4})^{2n} - (-q^{1/4})^{3n}\right) - q^{1/4}f(-q^5, -q^{11})\{\gamma - \beta I_1(q)\}, \tag{2.10}$$

where $\alpha = -2 + \sqrt{\frac{354(\sqrt{2} + 1)}{125\sqrt{2}}}$, $\beta = 1 - \sqrt{\frac{359(\sqrt{2} + 1)}{250\sqrt{2}}}$ and $\gamma = 1 - 2\sqrt{\frac{(\sqrt{2} + 1)}{2\sqrt{2}}}$.

Proof. We have

$$\frac{1}{\sqrt{I_2(q)}} + \alpha\sqrt{I_2(q)} = \frac{f(-q^7, -q^9) + \alpha q^{3/2}f(-q, -q^{15})}{\sqrt{q^{3/2}f(-q^7, -q^9)f(-q, -q^{15})}}. \tag{2.11}$$

From [1, p. 48], we have

$$f(U_1, V_1) = \sum_{r=0}^{k-1} U_r f\left(\frac{U_{k+r}}{U_r}, \frac{V_{k-r}}{U_r}\right), \tag{2.12}$$

where $U_n = a^{n(n+1)/2}b^{n(n-1)/2}$ and $V_n = a^{n(n-1)/2}b^{n(n+1)/2}$ for each integer n . Taking $k = 8$, $a = \xi$ and $b = \xi^7 q^{1/4}$ in (2.12), where $\xi = e^{\pi i/8}$, we obtain

$$\begin{aligned} f(\xi, \xi^7 q^{1/4}) &= f(-q^7, -q^9) + \xi f(-q^7, -q^9) + \xi^{10} q^{1/4} f(-q^5, -q^{11}) + \xi^{27} q^{3/4} \\ &\quad \times f(-q^3, -q^{13}) + \xi^{52} q^{3/2} f(-q, -q^{15}) + \xi^{85} q^{5/2} f(-q^{17}, -q^{-1}) \\ &\quad + \xi^{126} q^{15/4} f(-q^{19}, -q^{-3}) + \xi^{175} q^{21/4} f(-q^{21}, -q^{-5}), \end{aligned} \tag{2.13}$$

From [1], we have

$$f(a, b) = f(b, a)$$

and if n is an integer, then

$$f(a, b) = a^{n(n+1)/2}b^{n(n-1)/2}f(a(ab)^n, b(ab)^{-n}). \tag{2.14}$$

Using (2.14) in (2.13), we can deduce that

$$\begin{aligned} f(\xi, \xi^7 q^{1/4}) &= (1 + \xi)f(-q^7, -q^9) + q^{3/2}(\xi^{52} - \xi^{85})f(-q, -q^{15}) + q^{3/4} \\ &\quad \times (\xi^{27} - \xi^{126})f(-q^3, -q^{13}) + q^{1/4}(\xi^{10} - \xi^{175})f(-q^5, -q^{11}). \end{aligned} \tag{2.15}$$

Note that $\xi^{52} - \xi^{85} = \alpha(1 + \xi)$, $\xi^{27} - \xi^{126} = \beta(1 + \xi)$ and $\xi^{10} - \xi^{175} = \gamma(1 + \xi)$. It follows that

$$\frac{f(\xi, \xi^7 q^{1/4})}{1 + \xi} - q^{1/4}\{\gamma f(-q^5, -q^{11}) + \beta q^{1/2}f(-q^3, -q^{13})\} = f(-q^7, -q^9) + \alpha q^{3/2}f(-q, -q^{15}). \tag{2.16}$$

Substituting (2.16) in (2.11), we have

$$\frac{1}{\sqrt{I_2(q)}} + \alpha\sqrt{I_2(q)} = \frac{\frac{f(\xi, \xi^7 q^{1/4})}{1 + \xi} - q^{1/4}\{\gamma f(-q^5, -q^{11}) + \beta q^{1/2}f(-q^3, -q^{13})\}}{\sqrt{q^{3/2}f(-q^7, -q^9)f(-q, -q^{15})}}. \tag{2.17}$$

In Ramanujan’s notation Jacobi triple product identity [1] takes the form

$$f(a, b) = (-a; ab)_{\infty}(-b; ab)_{\infty}(ab; ab)_{\infty}. \tag{2.18}$$

Using (2.18), we have

$$\begin{aligned} \frac{f(\xi, \xi^7 q^{1/4})}{1 + \xi} &= \frac{(-\xi; -q^{1/4})_\infty (-\xi^7 q^{1/4}; -q^{1/4})_\infty (-q^{1/4}; -q^{1/4})_\infty}{1 + \xi} \\ &= (\xi q^{1/4}; -q^{1/4})_\infty (-\xi^7 q^{1/4}; -q^{1/4})_\infty (-q^{1/4}; -q^{1/4})_\infty \\ &= \prod_{n=1}^\infty (1 + \xi(-q^{1/4})^n)(1 - \xi^7(-q^{1/4})^n)(1 - (-q^{1/4})^n). \end{aligned}$$

Note that $\xi - \xi^7 - 1 = -1 + 2\sqrt{\frac{\sqrt{2} + 1}{2\sqrt{2}}}$ and $\xi^8 = -1$. Using these in above equation, we see that

$$\frac{f(\xi, \xi^7 q^{1/4})}{1 + \xi} = \prod_{n=1}^\infty (1 - \gamma(-q^{1/4})^n + \gamma(-q^{1/4})^{2n} - (q^{1/4})^{3n}). \tag{2.19}$$

Substituting (2.19) in (2.17), we get

$$\begin{aligned} \frac{1}{\sqrt{I_2(q)}} + \alpha\sqrt{I_2(q)} &= \frac{\prod_{n=1}^\infty (1 - \gamma(-q^{1/4})^n + \gamma(-q^{1/4})^{2n} - (q^{1/4})^{3n})}{\sqrt{q^{3/2}f(-q^7, -q^9)f(-q, -q^{15})}} \\ &\times -q^{1/4}f(-q^5, -q^{11}) \left\{ \gamma + \beta q^{1/2} \frac{f(-q^3, -q^{13})}{f(-q^5, -q^{11})} \right\} \\ &= \frac{\prod_{n=1}^\infty (1 - \gamma(-q^{1/4})^n + \gamma(-q^{1/4})^{2n} - (q^{1/4})^{3n}) - q^{1/4}f(-q^5, -q^{11}) \{ \gamma + \beta I_1(q) \}}{\sqrt{q^{3/2}f(-q^7, -q^9)f(-q, -q^{15})}}. \end{aligned} \tag{2.20}$$

3 Dissections of $I_1^*(q)$ and its reciprocal

In [5], Bernard L. S. Lin studied 2-, 3-, 4-, 6- and 12-dissections of a continued fraction of order twelve. In [3] Adiga et al. studied 2- and 4-dissection of Ramanujan’s continued fraction of order six. Motivated by these, in this section, we give 2-, 4-, 8- and 16-dissections of the continued fraction $I_1^*(q)$ and its reciprocal which are similar to (1.7)–(1.8).

Ramanujan recorded many identities involving $f(a, b)$ and its special cases $\varphi(q)$ and $\psi(q)$, which are defined by

$$\varphi(q) := f(q, q) = \sum_{n=-\infty}^\infty q^{n^2},$$

$$\psi(q) := f(q, q^3) = \sum_{n=0}^\infty q^{n(n+1)/2}.$$

We will require the following identity of Ramanujan [1, p. 45],

$$f(a, b)f(c, d) = f(ac, bd)f(ad, bc) + af(b/c, ac^2d)f(b/d, acd^2), \tag{3.1}$$

where $ab = cd$.

3.1 The 2-dissection of $I_1^*(q)$

Theorem 3.1. If $I_1^*(q) := \frac{f(-q^3, -q^{13})}{f(-q^5, -q^{11})} = \sum_{n=0}^{\infty} a_n q^n$, then

$$\sum_{n=0}^{\infty} a_{2n} q^n = \frac{\psi(-q^4) f(-q^7, -q^9)}{\varphi(-q^8) f(-q^5, -q^{11})}, \quad (3.2)$$

$$\sum_{n=0}^{\infty} a_{2n+1} q^n = -q \frac{\psi(-q^4) f(-q, -q^{15})}{\varphi(-q^8) f(-q^5, -q^{11})}. \quad (3.3)$$

Proof. We have

$$\begin{aligned} \sum_{n=0}^{\infty} a_n q^n &= \frac{f(-q^3, -q^{13})}{f(-q^5, -q^{11})} \\ &= \frac{f(-q^3, -q^{13}) f(q^5, q^{11})}{f(-q^5, -q^{11}) f(q^5, q^{11})}. \end{aligned} \quad (3.4)$$

Putting $a = -q^3$, $b = -q^{13}$, $c = q^5$ and $d = q^{11}$ in (3.1), we get

$$f(-q^3, -q^{13}) f(q^5, q^{11}) = f(-q^8, -q^{24}) f(-q^{14}, -q^{18}) - q^3 f(-q^8, -q^{24}) f(-q^2, -q^{30}). \quad (3.5)$$

Putting $a = q^5$ and $b = q^{11}$ in (2.8), we obtain

$$f(q^5, q^{11}) f(-q^5, -q^{11}) = f(-q^{10}, -q^{22}) \varphi(-q^{16}). \quad (3.6)$$

Employing (3.5) and (3.6) in (3.4), we deduce that

$$I_1^*(q) = \sum_{n=0}^{\infty} a_n q^n = \frac{f(-q^8, -q^{24}) f(-q^{14}, -q^{18}) - q^3 f(-q^8, -q^{24}) f(-q^2, -q^{30})}{f(-q^{10}, -q^{22}) \varphi(-q^{16})}.$$

Hence

$$I_1^*(q) + I_1^*(-q) = 2 \sum_{n=0}^{\infty} a_{2n} q^{2n} = 2 \frac{f(-q^8, -q^{24}) f(-q^{14}, -q^{18})}{f(-q^{10}, -q^{22}) \varphi(-q^{16})}$$

and

$$I_1^*(q) - I_1^*(-q) = 2 \sum_{n=0}^{\infty} a_{2n+1} q^{2n+1} = -2q^3 \frac{f(-q^8, -q^{24}) f(-q^2, -q^{30})}{f(-q^{10}, -q^{22}) \varphi(-q^{16})}.$$

Changing q to $q^{1/2}$ in the above equations, we obtain (3.2) and (3.3).

3.2 The 4-dissection of $I_1^*(q)$

Theorem 3.2. We have

$$\sum_{n=0}^{\infty} a_{4n} q^n = \frac{\psi(-q^2) f(-q^6, -q^{10}) f(-q^7, -q^9)}{\varphi(-q^4) \varphi(-q^8) f(-q^5, -q^{11})}, \quad (3.7)$$

$$\sum_{n=0}^{\infty} a_{4n+1} q^n = q \frac{\psi(-q^2) f(-q^2, -q^{14})}{\varphi(-q^4) \varphi(-q^8)}, \quad (3.8)$$

$$\sum_{n=0}^{\infty} a_{4n+2} q^n = q^2 \frac{\psi(-q^2) f(-q^2, -q^{14}) f(-q, -q^{15})}{\varphi(-q^4) \varphi(-q^8) f(-q^5, -q^{11})}, \quad (3.9)$$

$$\sum_{n=0}^{\infty} a_{4n+3} q^n = -\frac{\psi(-q^2) f(-q^6, -q^{10}) f(-q^3, -q^{13})}{\varphi(-q^4) \varphi(-q^8) f(-q^5, -q^{11})}. \quad (3.10)$$

Proof. From (3.2), we have

$$\sum_{n=0}^{\infty} a_{2n}q^n = \frac{\psi(-q^4)f(-q^7, -q^9) f(q^5, q^{11})}{\varphi(-q^8)f(-q^5, -q^{11}) f(q^5, q^{11})}. \tag{3.11}$$

Setting $a = q^5, b = q^{11}, c = -q^7$ and $d = -q^9$ in (3.1), we get

$$f(q^5, q^{11})f(-q^7, -q^9) = f(-q^{12}, -q^{20})f(-q^{14}, -q^{18}) + q^5 f(-q^4, -q^{28})f(-q^2, -q^{30}). \tag{3.12}$$

Employing (3.12) and (3.6) in (3.11), we find that

$$\sum_{n=0}^{\infty} a_{2n}q^n = \frac{\psi(-q^4)\{f(-q^{12}, -q^{20})f(-q^{14}, -q^{18}) + q^5 f(-q^4, -q^{28})f(-q^2, -q^{30})\}}{\varphi(-q^8)\varphi(-q^{16})f(-q^{10}, -q^{22})}.$$

This implies

$$\begin{aligned} \sum_{n=0}^{\infty} a_{4n}q^{2n} &= \frac{\psi(-q^4)f(-q^{12}, -q^{20})f(-q^{14}, -q^{18})}{\varphi(-q^8)\varphi(-q^{16})f(-q^{10}, -q^{22})}, \\ \sum_{n=0}^{\infty} a_{4n+2}q^{2n} &= q^4 \frac{\psi(-q^4)f(-q^4, -q^{28})f(-q^2, -q^{30})}{\varphi(-q^8)\varphi(-q^{16})f(-q^{10}, -q^{22})}. \end{aligned}$$

Changing q to $q^{1/2}$ in the above equations, we obtain (3.7) and (3.9).

Proofs of (3.8) and (3.10) are similar.

3.3 The 8-dissection of $I_1^*(q)$

Theorem 3.3. We have

$$\sum_{n=0}^{\infty} a_{8n}q^n = \frac{\psi(-q)f(-q^6, -q^{10})f(-q^5, -q^3)f(-q^7, -q^9)}{\varphi(-q^2)\varphi(-q^4)\varphi(-q^8)f(-q^5, -q^{11})}, \tag{3.13}$$

$$\sum_{n=0}^{\infty} a_{8n+2}q^n = q \frac{\psi(-q)f(-q^6, -q^{10})f(-q, -q^7)f(-q^3, -q^{13})}{\varphi(-q^2)\varphi(-q^4)\varphi(-q^8)f(-q^5, -q^{11})}, \tag{3.14}$$

$$\sum_{n=0}^{\infty} a_{8n+3}q^n = -\frac{\psi(-q)f(-q^4, -q^{12})f(-q^5, -q^3)f(-q^7, -q^9)}{\varphi(-q^2)\varphi(-q^4)\varphi(-q^8)f(-q^5, -q^{11})}, \tag{3.15}$$

$$\sum_{n=0}^{\infty} a_{8n+4}q^n = q^2 \frac{\psi(-q)f(-q^2, -q^{14})f(-q^5, -q^3)f(-q, -q^{15})}{\varphi(-q^2)\varphi(-q^4)\varphi(-q^8)f(-q^5, -q^{11})}, \tag{3.16}$$

$$\sum_{n=0}^{\infty} a_{8n+6}q^n = -q \frac{\psi(-q)f(-q^2, -q^{14})f(-q, -q^7)}{\varphi(-q^2)\varphi(-q^4)\varphi(-q^8)}, \tag{3.17}$$

$$\sum_{n=0}^{\infty} a_{8n+7}q^n = q \frac{\psi(-q)f(-q^4, -q^{12})f(-q^5, -q^3)f(-q, -q^{15})}{\varphi(-q^2)\varphi(-q^4)\varphi(-q^8)f(-q^5, -q^{11})}. \tag{3.18}$$

Proof. By (3.7), we have

$$\sum_{n=0}^{\infty} a_{4n}q^n = \frac{\psi(-q^2)f(-q^6, -q^{10})f(-q^7, -q^9)f(q^5, q^{11})}{\varphi(-q^4)\varphi(-q^8)f(-q^5, -q^{11})f(q^5, q^{11})}. \tag{3.19}$$

Employing (3.12) and (3.6) in (3.19), we find that

$$\begin{aligned} \sum_{n=0}^{\infty} a_{4n}q^n &= \frac{\psi(-q^2)f(-q^6, -q^{10})}{\varphi(-q^4)\varphi(-q^8)\varphi(-q^{16})f(-q^{10}, -q^{22})} \\ &\quad \times \{f(-q^{12}, -q^{20})f(-q^{14}, -q^{18}) + q^5 f(-q^4, -q^{28})f(-q^2, -q^{30})\}. \end{aligned}$$

It follows immediately that

$$\sum_{n=0}^{\infty} a_{8n}q^{2n} = \frac{\psi(-q^2)f(-q^6, -q^{10})f(-q^{12}, -q^{20})f(-q^{14}, -q^{18})}{\varphi(-q^4)\varphi(-q^8)\varphi(-q^{16})f(-q^{10}, -q^{22})},$$

$$\sum_{n=0}^{\infty} a_{8n+4}q^{2n} = q^4 \frac{\psi(-q^2)f(-q^6, -q^{10})f(-q^4, -q^{28})f(-q^2, -q^{30})}{\varphi(-q^4)\varphi(-q^8)\varphi(-q^{16})f(-q^{10}, -q^{22})}.$$

Changing q to $q^{1/2}$ in the above equations, we obtain (3.13) and (3.16).

Proofs of (3.14), (3.15) (3.17) and (3.18) are similar.

Theorem 3.4. *We have*

$$\sum_{n=0}^{\infty} a_{8n+1}q^{2n} \equiv 0 \pmod{4}, \tag{3.20}$$

$$\sum_{n=0}^{\infty} a_{8n+5}q^{2n} \equiv \frac{\psi(-q)f(-q, -q^7)}{\varphi(-q^2)\varphi(-q^4)\varphi^4(-q)} \pmod{4}. \tag{3.21}$$

Proof. From (3.8), we have

$$\begin{aligned} \sum_{n=0}^{\infty} a_{4n+1}q^{2n} &= q \frac{\psi(-q^2)f(-q^2, -q^{14})}{\varphi(-q^4)\varphi(-q^8)} \\ &= q \frac{\psi(-q^2)f(-q^2, -q^{14})}{\varphi(-q^4)\varphi(-q^8)} \frac{\varphi^2(q)}{\varphi^2(q)} \\ &= q \frac{\psi(-q^2)f(-q^2, -q^{14})\varphi^2(q)\varphi^2(-q)}{\varphi(-q^4)\varphi(-q^8)\varphi^4(-q^2)}. \end{aligned} \tag{3.22}$$

From (2.3) and $f(1, a) = 2f(a, a^3)$, we have

$$\varphi(q^4) - 2q\psi(q^8) = \varphi(-q). \tag{3.23}$$

Employing (3.23) in (3.22), we get

$$\sum_{n=0}^{\infty} a_{4n+1}q^{2n} = q \frac{\psi(-q^2)f(-q^2, -q^{14})\varphi^2(q)}{\varphi(-q^4)\varphi(-q^8)\varphi^4(-q^2)} \{ \varphi^2(q^4) - 4q\varphi(q^4)\psi(q^8) + 4q^2\psi^2(q^8) \}. \tag{3.24}$$

Note that $(\varphi(q))^{2^k} \equiv 1 \pmod{4}$ for $k \geq 1$. This is clear when one writes

$$(\varphi(q))^{2^k} = \left(1 + 2 \sum_{n \geq 1} q^{n^2} \right)^{2^k}$$

and then expands via the binomial theorem. Hence (3.24) becomes

$$\sum_{n=0}^{\infty} a_{4n+1}q^{2n} \equiv q \frac{\psi(-q^2)f(-q^2, -q^{14})}{\varphi(-q^4)\varphi(-q^8)\varphi^4(-q^2)} \pmod{4}. \tag{3.25}$$

Immediately it follows

$$\sum_{n=0}^{\infty} a_{8n+1}q^{2n} \equiv 0 \pmod{4}$$

and hence $a_{8n+1} \equiv 0 \pmod{4}$. From (3.25), we have also

$$\sum_{n=0}^{\infty} a_{8n+5}q^{2n} \equiv \frac{\psi(-q^2)f(-q^2, -q^{14})}{\varphi(-q^4)\varphi(-q^8)\varphi^4(-q^2)} \pmod{4}. \tag{3.26}$$

Changing q to $q^{1/2}$ in (3.26), we obtain (3.21).

3.4. The 16-dissection of $I_1^*(q)$

We define $A := f(-q, -q^{15}), A^* := f(q, q^{15}), B := f(-q^3, -q^{13}), B^* := f(q^3, q^{13}),$
 $C := f(-q^5, -q^{11}), C^* := f(q^5, q^{11}), D := f(-q^7, -q^9), D^* := f(q^7, q^9), P := f(-q^6, -q^{10}),$
 $Q := f(-q^2, -q^{14}), X := f(q^3, q^5), X^* := f(-q^3, -q^5), Y := f(q, q^7)$ and $Y^* := f(-q, -q^7),$
 $E := \frac{1}{C\varphi(-q)\varphi(-q^2)\varphi(-q^4)\varphi(-q^8)}, \quad F := \frac{1}{B\varphi(-q)\varphi(-q^2)\varphi(-q^4)\varphi(-q^8)}.$

Theorem 3.5. *We have*

$$\sum_{n=0}^{\infty} a_{16n}q^n = EX^*\{-AA^*QXq^4 - AD^*QYq^3 + A^*DPYq^2 + DD^*PX\}, \tag{3.27}$$

$$\sum_{n=0}^{\infty} a_{16n+2}q^n = EX^*\psi(-q^2)\{-AB^*Yq^3 - AC^*Xq^2 + (-B^*X - C^*Y)Dq\}, \tag{3.28}$$

$$\sum_{n=0}^{\infty} a_{16n+3}q^n = E\psi(-q^2)\{AA^*QXq^4 + AD^*QYq^3 - A^*DPYq^2 - DD^*PX\}, \tag{3.29}$$

$$\sum_{n=0}^{\infty} a_{16n+4}q^n = EY^*\{(A^*BPY + A^*CQX)q^3 + CD^*QYq^2 + BD^*PXq\}, \tag{3.30}$$

$$\sum_{n=0}^{\infty} a_{16n+6}q^n = \frac{Y^*(B^*X + C^*Y)q}{\varphi(-q)\varphi(-q^2)\varphi(-q^4)}, \tag{3.31}$$

$$\sum_{n=0}^{\infty} a_{16n+7}q^n = E\psi(-q^2)\{-A^*CQYq^3 - A^*BPXq^2 - BD^*PY - CD^*QX\}, \tag{3.32}$$

$$\sum_{n=0}^{\infty} a_{16n+8}q^n = EX^*\{AA^*QYq^4 + AD^*QXq^2 - A^*DPXq - DD^*PY\}, \tag{3.33}$$

$$\sum_{n=0}^{\infty} a_{16n+10}q^n = EX^*\psi(-q^2)\{(B^*X + C^*Y)Aq^2 + B^*DYq + C^*DX\}, \tag{3.34}$$

$$\sum_{n=0}^{\infty} a_{16n+11}q^n = E\psi(-q^2)\{-AA^*QYq^4 - AD^*QXq^2 + A^*DPXq + DD^*PY\}, \tag{3.35}$$

$$\sum_{n=0}^{\infty} a_{16n+12}q^n = EY^*\{-A^*CQYq^3 - A^*BPXq^2 - (BD^*PY + CD^*QX)q\}, \tag{3.36}$$

$$\sum_{n=0}^{\infty} a_{16n+14}q^n = -\frac{Y^*(B^*Yq + C^*X)}{\varphi(-q)\varphi(-q^2)\varphi(-q^4)}, \tag{3.37}$$

$$\sum_{n=0}^{\infty} a_{16n+15}q^n = E\psi(-q^2)\{(A^*BPY + A^*CQX)q^2 + CD^*QYq + XD^*PB\}. \tag{3.38}$$

Proof. From (3.13), we have

$$\sum_{n=0}^{\infty} a_{8n}q^n = \frac{\psi(-q)f(-q^6, -q^{10})f(-q^5, -q^3)f(-q^7, -q^9)}{\varphi(-q^2)\varphi(-q^4)\varphi(-q^8)f(-q^5, -q^{11})} \frac{f(q^5, q^{11})}{f(q^5, q^{11})} \tag{3.39}$$

Using (2.3), we obtain

$$\psi(-q) = f(q^6, q^{10}) - qf(q^2, q^{14}), \tag{3.40}$$

$$f(-q^3, -q^5) = f(q^{14}, q^{18}) - q^3f(q^2, q^{30}). \tag{3.41}$$

Employing (3.40), (3.41), (3.6) and (3.12) in (3.39), we deduce that

$$\begin{aligned} \sum_{n=0}^{\infty} a_{8n} q^n &= \frac{f(-q^6, -q^{10})}{\varphi(-q^2)\varphi(-q^4)\varphi(-q^8)\varphi(-q^{16})f(-q^{10}, -q^{22})} \\ &\quad \{(f(q^6, q^{10}) - qf(q^2, q^{14}))(f(q^{14}, q^{18}) - q^3f(q^2, q^{30})) \\ &\quad \times (f(-q^{12}, -q^{20})f(-q^{14}, -q^{18}) + q^5f(-q^4, -q^{28})f(-q^2, -q^{30}))\}, \\ &= \frac{f(-q^6, -q^{10})}{\varphi(-q^2)\varphi(-q^4)\varphi(-q^8)\varphi(-q^{16})f(-q^{10}, -q^{22})} \\ &\quad \{f(q^6, q^{10})f(q^{14}, q^{18})f(-q^{12}, -q^{20})f(-q^{14}, -q^{18}) \\ &\quad + q^5f(q^6, q^{10})f(q^{14}, q^{18})f(-q^4, -q^{28})f(-q^2, -q^{30}) \\ &\quad - q^3f(q^6, q^{10})f(q^2, q^{30})f(-q^{12}, -q^{20})f(-q^{14}, -q^{18}) \\ &\quad - q^8f(q^6, q^{10})f(q^2, q^{30})f(-q^4, -q^{28})f(-q^2, -q^{30}) \\ &\quad - qf(q^2, q^{14})f(q^{14}, q^{18})f(-q^{14}, -q^{18})f(-q^{12}, -q^{20}) \\ &\quad - q^6f(q^{14}, q^{18})f(q^2, q^{14})f(-q^4, -q^{28})f(-q^2, -q^{30}) \\ &\quad + q^4f(q^2, q^{14})f(q^2, q^{30})f(-q^{12}, -q^{20})f(-q^{14}, -q^{18}) \\ &\quad + q^9f(q^2, q^{14})f(q^2, q^{30})f(-q^4, -q^{28})f(-q^2, -q^{30})\}. \end{aligned}$$

Immediately it follows

$$\begin{aligned} \sum_{n=0}^{\infty} a_{16n} q^{2n} &= \frac{f(-q^6, -q^{10})}{\varphi(-q^2)\varphi(-q^4)\varphi(-q^8)\varphi(-q^{16})f(-q^{10}, -q^{22})} \\ &\quad \{f(q^6, q^{10})f(q^{14}, q^{18})f(-q^{12}, -q^{20})f(-q^{14}, -q^{18}) \\ &\quad - q^8f(q^6, q^{10})f(q^2, q^{30})f(-q^4, -q^{28})f(-q^2, -q^{30}) \\ &\quad - q^6f(q^{14}, q^{18})f(q^2, q^{14})f(-q^4, -q^{28})f(-q^2, -q^{30}) \\ &\quad + q^4f(q^2, q^{14})f(q^2, q^{30})f(-q^{12}, -q^{20})f(-q^{14}, -q^{18})\}, \\ \sum_{n=0}^{\infty} a_{16n+8} q^{2n+1} &= \frac{f(-q^6, -q^{10})}{\varphi(-q^2)\varphi(-q^4)\varphi(-q^8)\varphi(-q^{16})f(-q^{10}, -q^{22})} \\ &\quad \{q^9f(q^2, q^{14})f(q^2, q^{30})f(-q^4, -q^{28})f(-q^2, -q^{30}) \\ &\quad + q^5f(q^6, q^{10})f(q^{14}, q^{18})f(-q^4, -q^{28})f(-q^2, -q^{30}) \\ &\quad - q^3f(q^6, q^{10})f(q^2, q^{30})f(-q^{12}, -q^{20})f(-q^{14}, -q^{18}) \\ &\quad - qf(q^2, q^{14})f(q^{14}, q^{18})f(-q^{14}, -q^{18})f(-q^{12}, -q^{20})\}. \end{aligned}$$

Changing q to $q^{1/2}$ in above equations, we obtain (3.27) and (3.33), respectively. The remaining identities (3.28)–(3.32) and (3.34)–(3.38) follows similarly.

In a similar way, it is not hard to derive the following theorems.

4 Dissection of $\frac{1}{I_1^*(q)}$

Theorem 4.1. If $\frac{1}{I_1^*(q)} := \frac{f(-q^5, -q^{11})}{f(-q^3, -q^{13})} = \sum_{n=0}^{\infty} b_n q^n$, then

$$\sum_{n=0}^{\infty} b_{2n} q^n = \frac{\psi(-q^4)f(-q^7, -q^9)}{\varphi(-q^8)f(-q^3, -q^{13})}, \quad (4.1)$$

$$\sum_{n=0}^{\infty} b_{2n+1}q^n = q \frac{\psi(-q^4)f(-q, -q^{15})}{\varphi(-q^8)f(-q^3, -q^{13})}. \tag{4.2}$$

Theorem 4.2. *We have*

$$\sum_{n=0}^{\infty} b_{4n}q^n = \frac{\psi(-q^2)f(-q^6, -q^{10})f(-q^5, -q^{11})}{\varphi(-q^4)\varphi(-q^8)f(-q^3, -q^{13})}, \tag{4.3}$$

$$\sum_{n=0}^{\infty} b_{4n+1}q^n = -q \frac{\psi(-q^2)f(-q^6, -q^{10})f(-q, -q^{15})}{\varphi(-q^4)\varphi(-q^8)f(-q^3, -q^{13})}, \tag{4.4}$$

$$\sum_{n=0}^{\infty} b_{4n+2}q^n = q \frac{\psi(-q^2)f(-q^2, -q^{14})}{\varphi(-q^4)\varphi(-q^8)}, \tag{4.5}$$

$$\sum_{n=0}^{\infty} b_{4n+3}q^n = \frac{\psi(-q^2)f(-q^2, -q^{14})f(-q^7, -q^9)}{\varphi(-q^4)\varphi(-q^8)f(-q^3, -q^{13})}. \tag{4.6}$$

Theorem 4.3. *We have*

$$\sum_{n=0}^{\infty} b_{8n}q^n = \frac{\psi(-q)f(-q^4, -q^{12})f(-q^5, -q^3)f(-q^7, -q^9)}{\varphi(-q^2)\varphi(-q^4)\varphi(-q^8)f(-q^3, -q^{13})}, \tag{4.7}$$

$$\sum_{n=0}^{\infty} b_{8n+1}q^n = q \frac{\psi(-q)f(-q^6, -q^{10})f(-q^5, -q^3)f(-q, -q^{15})}{\varphi(-q^2)\varphi(-q^4)\varphi(-q^8)f(-q^3, -q^{13})}, \tag{4.8}$$

$$\sum_{n=0}^{\infty} b_{8n+3}q^n = \frac{\psi(-q)f(-q^6, -q^{10})f(-q, -q^7)f(-q^5, -q^{11})}{\varphi(-q^2)\varphi(-q^4)\varphi(-q^8)f(-q^3, -q^{13})}, \tag{4.9}$$

$$\sum_{n=0}^{\infty} b_{8n+4}q^n = q \frac{\psi(-q)f(-q^4, -q^{12})f(-q^5, -q^3)f(-q, -q^{15})}{\varphi(-q^2)\varphi(-q^4)\varphi(-q^8)f(-q^3, -q^{13})}, \tag{4.10}$$

$$\sum_{n=0}^{\infty} b_{8n+5}q^n = -\frac{\psi(-q)f(-q^2, -q^{14})f(-q^5, -q^3)f(-q^7, -q^9)}{\varphi(-q^2)\varphi(-q^4)\varphi(-q^8)f(-q^3, -q^{13})}, \tag{4.11}$$

$$\sum_{n=0}^{\infty} b_{8n+7}q^n = q \frac{\psi(-q)f(-q^2, -q^{14})f(-q, -q^7)}{\varphi(-q^2)\varphi(-q^4)\varphi(-q^8)}. \tag{4.12}$$

Theorem 4.4. *We have*

$$\sum_{n=0}^{\infty} b_{8n+2}q^n \equiv 0 \pmod{4}, \tag{4.13}$$

$$\sum_{n=0}^{\infty} b_{8n+6}q^n \equiv \frac{f(-q, -q^7)\psi^2(q)f^2(q^3, q^5)f^2(q, q^7)}{\varphi^5(-q^2)\varphi(-q^4)\psi^3(-q)} \{f^2(q^3, q^5) + 6q + q^2f^2(q, q^7)\} \pmod{4}. \tag{4.14}$$

Proof. From (4.5), we have

$$\begin{aligned} \sum_{n=0}^{\infty} b_{4n+2}q^n &= q \frac{\psi(-q^2)f(-q^2, -q^{14})}{\varphi(-q^4)\varphi(-q^8)} \frac{\varphi^2(q)}{\varphi^2(q)} \\ &= q \frac{\psi(-q^2)f(-q^2, -q^{14})\varphi^2(q)\psi^2(q^2)}{\varphi(-q^4)\varphi(-q^8)\psi^4(q)} \\ &\equiv q \frac{f(-q^2, -q^{14})\psi^2(q^2)\psi^4(-q)}{\varphi^5(-q^4)\varphi(-q^8)\psi^3(-q^2)} \pmod{4}. \end{aligned} \tag{4.15}$$

We have

$$\begin{aligned} \psi^4(-q) &= f^4(-q, -q^3) = (f(q^6, q^{10}) - qf(q^2, q^{14}))^4 \\ &= f^4(q^6, q^{10}) - 4qf^3(q^6, q^{10})f(q^2, q^{14}) + 6q^2f^2(q^6, q^{10})f^2(q^2, q^{14}) \\ &\quad - 4q^3f(q^6, q^{10})f^3(q^2, q^{14}) + q^4f^4(q^2, q^{14}). \end{aligned} \quad (4.16)$$

Employing (4.16) in (4.15), we obtain

$$\begin{aligned} \sum_{n=0}^{\infty} b_{4n+2}q^n &\equiv q \frac{f(-q^2, -q^{14})\psi^2(q^2)}{\varphi^5(-q^4)\varphi(-q^8)\psi^3(-q^2)} \{f^4(q^6, q^{10}) - 4qf^3(q^6, q^{10})f(q^2, q^{14}) \\ &\quad + 6q^2f^2(q^6, q^{10})f^2(q^2, q^{14}) - 4q^3f(q^6, q^{10})f^3(q^2, q^{14}) + q^4f^4(q^2, q^{14})\} \pmod{4}. \end{aligned} \quad (4.17)$$

Immediately, it follows that

$$\sum_{n=0}^{\infty} b_{8n+2}q^n \equiv 0 \pmod{4}, \quad (4.18)$$

$$\sum_{n=0}^{\infty} b_{8n+6}q^n \equiv \frac{f(-q, -q^7)\psi^2(q)f^2(q^3, q^5)f^2(q, q^7)}{\varphi^5(-q^2)\varphi(-q^4)\psi^3(-q)} \{f^2(q^3, q^5) + 6q + q^2f^2(q, q^7)\} \pmod{4}. \quad (4.19)$$

Theorem 4.5. *We have*

$$\sum_{n=0}^{\infty} b_{16n}q^n = F\psi(-q^2)\{-A^*BQXq^3 + (-BD^*QY + A^*CPY)q^2 + CD^*PX\} \quad (4.20)$$

$$\sum_{n=0}^{\infty} b_{16n+1}q^n = FX^*\{-AA^*PYq^3 - A^*DQXq^2 - (AD^*PX + DD^*QY)q\}, \quad (4.21)$$

$$\sum_{n=0}^{\infty} b_{16n+3}q^n = FX^*\psi(-q^2)\{(-B^*X - C^*Y)Aq^2 + B^*DYq + C^*DX\}, \quad (4.22)$$

$$\sum_{n=0}^{\infty} b_{16n+4}q^n = F\psi(-q^2)\{-AA^*PYq^3 - A^*DQXq^2 - (AD^*PX + DD^*QY)q\}, \quad (4.23)$$

$$\sum_{n=0}^{\infty} b_{16n+5}q^n = FY^*\{A^*BQXq^3 - (BD^*QY - A^*CPY)q^2 - CD^*PX\}, \quad (4.24)$$

$$\sum_{n=0}^{\infty} b_{16n+7}q^n = \frac{-Y^*(B^*X + C^*Y)q}{\varphi(-q)\varphi(-q^2)\varphi(-q^4)}, \quad (4.25)$$

$$\sum_{n=0}^{\infty} b_{16n+8}q^n = F\psi(-q^2)\{A^*BQYq^3 + (BD^*QX - A^*CPX)q - CD^*PY\}, \quad (4.26)$$

$$\sum_{n=0}^{\infty} b_{16n+9}q^n = FX^*\{(AA^*PX + A^*DQY)q^2 + AD^*PYq + DD^*QX\}, \quad (4.27)$$

$$\sum_{n=0}^{\infty} b_{16n+11}q^n = FX^*\psi(-q^2)\{AB^*Yq^2 + AC^*Xq + (-B^*X - C^*Y)D\}, \quad (4.28)$$

$$\sum_{n=0}^{\infty} b_{16n+12}q^n = F\psi(-q^2)\{(AA^*PX + A^*DQY)q^2 + AD^*PYq + DD^*QX\}, \quad (4.29)$$

$$\sum_{n=0}^{\infty} b_{16n+13}q^n = FY^*\{A^*BQYq^3 + (BD^*QX - A^*CPX)q - CD^*PY\}, \quad (4.30)$$

$$\sum_{n=0}^{\infty} b_{16n+15}q^n = \frac{Y^*(B^*Yq + C^*X)}{\varphi(-q)\varphi(-q^2)\varphi(-q^4)}. \tag{4.31}$$

Corollary 4.6. *We have*

$$\frac{\sum_{n=0}^{\infty} a_{4n+1}q^n}{\sum_{n=0}^{\infty} a_{4n+2}q^n} = -\frac{\sum_{n=0}^{\infty} b_{4n}q^n}{\sum_{n=0}^{\infty} b_{4n+1}q^n}, \quad q \frac{\sum_{n=0}^{\infty} a_{4n}q^n}{\sum_{n=0}^{\infty} a_{4n+3}q^n} = -\frac{\sum_{n=0}^{\infty} b_{4n+3}q^n}{\sum_{n=0}^{\infty} b_{4n+2}q^n}.$$

Proof. The proof the corollary follows from above Theorem.

Corollary 4.7. *We have*

$$\begin{aligned} \frac{\sum_{n=0}^{\infty} a_{8n}q^n}{\sum_{n=0}^{\infty} a_{8n+2}q^n} &= -\frac{\sum_{n=0}^{\infty} b_{8n+5}q^n}{\sum_{n=0}^{\infty} b_{8n+7}q^n}, & \frac{\sum_{n=0}^{\infty} a_{8n}q^n}{\sum_{n=0}^{\infty} a_{8n+3}q^n} &= -\frac{\sum_{n=0}^{\infty} b_{8n+1}q^n}{\sum_{n=0}^{\infty} b_{8n+4}q^n}, \\ \frac{\sum_{n=0}^{\infty} a_{8n+3}q^n}{\sum_{n=0}^{\infty} a_{8n+7}q^n} &= -\frac{\sum_{n=0}^{\infty} b_{8n}q^n}{\sum_{n=0}^{\infty} b_{8n+4}q^n}, & \frac{\sum_{n=0}^{\infty} a_{8n+4}q^n}{\sum_{n=0}^{\infty} a_{8n+6}q^n} &= -\frac{\sum_{n=0}^{\infty} b_{8n+1}q^n}{\sum_{n=0}^{\infty} b_{8n+3}q^n}, \\ & & q \frac{\sum_{n=0}^{\infty} a_{8n+4}q^n}{\sum_{n=0}^{\infty} a_{8n+7}q^n} &= -\frac{\sum_{n=0}^{\infty} b_{8n+5}q^n}{\sum_{n=0}^{\infty} b_{8n}q^n}. \end{aligned}$$

Proof. The proof the corollary follows from above Theorem.

Theorem 4.8. *We have $a_2 = a_4 = a_6 = a_7 = a_{12} = 0$. The remaining coefficients a_n satisfy the inequalities*

$$\begin{aligned} a_{16n}, a_{16n+4}, a_{16n+6}, a_{16n+10}, a_{16n+11}, a_{16n+15} &> 0, \\ a_{16n+2}, a_{16n+3}, a_{16n+7}, a_{16n+8}, a_{16n+12}, a_{16n+14} &< 0. \end{aligned}$$

Proof. Changing q to $-q$ in (3.13), we have

$$\begin{aligned} \sum_{n=0}^{\infty} a_{8n}(-q)^n &= \frac{\psi(q)f(-q^6, -q^{10})f(q^5, q^3)f(q^7, q^9)}{\varphi(-q^2)\varphi(-q^4)\varphi(-q^8)f(q^5, q^{11})} \\ &= \frac{(-q, -q^3, -q^3, -q^5, -q^7, -q^7, -q^9, -q^9, -q^{11}, -q^{13}, -q^{13}, -q^{15}; q^{16})_{\infty}}{(q^2, q^2, q^4, q^4, q^6, q^8, q^8, q^{10}, q^{12}, q^{12}, q^{14}, q^{14}; q^{16})_{\infty}}. \end{aligned}$$

From the above equality, we obtain the signs $a_{16n} > 0$ and $a_{16n+8} < 0$. Similarly, we can determine the signs of the remaining subsequences for a_n .

Theorem 4.9. *We have $b_1 = b_4 = b_7 = 0$. The remaining coefficients b_n satisfy the inequalities*

$$\begin{aligned} b_{16n}, b_{16n+3}, b_{16n+9}, b_{16n+12}, b_{16n+13}, b_{16n+15} &> 0, \\ b_{16n+1}, b_{16n+4}, b_{16n+5}, b_{16n+7}, b_{16n+8}, b_{16n+11} &< 0. \end{aligned}$$

Proof. Changing q to $-q$ in (4.7), we have

$$\begin{aligned} \sum_{n=0}^{\infty} b_{8n}(-q)^n &= \frac{\psi(q)f(-q^4, -q^{12})f(q^5, q^3)f(q^7, q^9)}{\varphi(-q^2)\varphi(-q^4)\varphi(-q^8)f(q^3, q^{13})} \\ &= \frac{(-q, -q^3, -q^5, -q^5, -q^7, -q^7, -q^9, -q^9, -q^{11}, -q^{11}, -q^{13}, -q^{15}; q^{16})_{\infty}}{(q^2, q^2, q^4, q^6, q^6, q^8, q^8, q^{10}, q^{10}, q^{12}, q^{14}, q^{14}; q^{16})_{\infty}}. \end{aligned}$$

From the above equality, we obtain the signs $b_{16n} > 0$ and $b_{16n+8} < 0$. Similarly, we can determine the signs of the remaining subsequences for b_n .

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