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# On the Poisson Transform on the bounded domain of type IV.

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**Abstract**. Let $\mathcal{D}$ be the Lie ball in $\mathbb{C}^2$ and let $A'(S)$ be the space of all hyperfunctions over the Shilov boundary $S$ of $\mathcal{D}$.
The aim of this paper is to give a necessary and sufficient condition on the Poisson transform $P_\lambda f$ of an element $f$ in the space $A'(S)$ for $f$ to be in $L^2(S)$. More precisely, we establish for any $\lambda \in \mathbb{R}\backslash\{0\}$ that:
(i) Let $F = P_\lambda f, f \in L^2(S)$. Then we have

$$||F||_*^2 = \sup_{t>0} \frac{1}{t^2} \int_0^t \Big( \int_0^{r_1} \int_{S(O(2)\times O(2))} |F(ka_R.0)|^2 \sinh(r_1 - r_2) \sinh(r_1 + r_2) dk dr_2 \Big) dr_1 < \infty.$$

(ii) Let $f$ be a hyperfunction on $S$ such that its image $F = P_\lambda f$ satisfies the growth condition $||F||_* < \infty$, then necessarily such $f$ is in $L^2(S)$.

## 1 Introduction

Let $\mathcal{D} = G/K$ be a Riemannian symmetric space of the non-compact type with Furstenberg boundary (maximal boundary)$S_F$. Each eigenfunction of all invariant differential operators on $\mathcal{D}$ can be represented by the Poisson integral of a hyperfunction on $S_F$. This was conjectured by S. Helgason and proved by Kashiwara *et al.*
If $\mathcal{D} = G/K$ is an irreducible bounded hermitian symmetric domain, it is well known that the Hua operator associated to $\mathcal{D}$ characterizes the Poisson transform on the Shilov boundary (minimal boundary) $S$ of $\mathcal{D}$.
More precisely, the Poisson transform $P_\lambda$ is a $G$-isomorphism from the space $A^{'}(S)$ of all hyperfunctions on $S$ onto an eigenspace of the Hua operator for $\lambda$ varying in a subset of $\mathbb{C}$.
Hence it becomes natural to look a characterization of the range of the Poisson transform on classical spaces of $S$. This problem was handled by the authors in a series of papers for the case of the classical $L^p$-space on $S$.
More precisely, they showed that for $\lambda$ ranges a subset of $\lambda \in \mathbb{C}\backslash\mathbb{R}$ the eigenfunctions of The Hua operator that are Poisson integral of $L^p$-functions on the Shilov boundary $S$ are characterized by an $H^p$-condition.
In order to prove those result they established a Fatou-type theorem for the eigenfunctions of the Hua operator. More precisely, they gave the $L^p$-boundedness properties of the Poisson transform $P_\lambda$ associated to the space $\mathcal{D}$. And they established the asymptotic behaviour of the generalized spherical function $\Phi_{\lambda,m}$

$$\Phi_{\lambda,m}(z) = \int_S P_\lambda(z,u)\phi_m(u)du$$

where $\phi_m$ is the zonal spherical function.
Thus, we address the question of the characterization of the $L^p$-range of the Poisson transform $P_\lambda$ for $\lambda \in \mathbb{R}$. In [1], [5] the authors gave an answer of this question in the case of rank one symmetric space with $p = 2$.
The aim of this paper is to give a necessary and sufficient condition on the Poisson transform $P_\lambda f, \lambda \in \mathbb{R}\backslash\{0\}$ of an element $f$ in the space $A'(S)$ for $f$ to be in $L^2(S)$ in the case of Lie ball $\mathcal{D} = SO(2,2)/SO(2) \times SO(2)$.We are led:
First to establish the following lemma on the asymptotic behaviour of the generalized spherical function $\phi_m$

**Lemma 1.1.** *There exists a constant $\gamma_1 > 0$ such that we have:*

$$\lim_{t\longrightarrow\infty}\frac{1}{t^2}\int_0^t\Big(\int_0^{r_1}\Big|\Phi_{\lambda,m}(a_R.0)\Big|^2\sinh(r_1-r_2)\sinh(r_1+r_2)dr_2\Big)dr_1=\gamma_1^2\Big|\frac{\Gamma^2(i\lambda)}{\Gamma^4(\frac{i\lambda+1}{2})}\Big|^2$$

*for every $\lambda\in\mathbb{R}\backslash\{0\}$ and for every $m=(m_1,m_2)\in\wedge$.*

Second to investigate $L^2$-boundedness properties of the Poisson transform associated to $\mathcal{D}$. More precisely, we give the following lemma

**Lemma 1.2.** *Let $\lambda$ be a non zero real number. Then, there exists a positive constant $\gamma_2(\lambda)$ such that*

$$\sup_{t>0}\frac{1}{t^2}\int_0^t\Big(\int_0^{r_1}\Big|\Phi_{\lambda,m}(a_R.0)\Big|^2\sinh(r_1-r_2)\sinh(r_1+r_2)dr_2\Big)dr_1<\gamma_2^2(\lambda).$$

The paper is organized as follows. In Section 2, we recall some preliminaries of harmonic analysis on the Lie ball in $\mathbb{C}^2$ and we state the main results of this paper. In Section 3, we give the precise action of the Poisson transform $L^2(S)$. Section 4 is devoted to the proof of Theorem 2.1 and Theorem 2.2. We conclude with an appendix in which we give the proof of Lemma 1.1.

## 2 Notation and statement of the main results

First, we recall some well known results of harmonic analysis in the Lie ball (see [3], [4]). For any matrix we denote by $a^t$ and $\bar{a}$ the transpose and conjugate of $a$ respectively.
Let

$$\mathcal{D}=\{z=(z_1,z_2)\in\mathbb{C}^2\ /\ 1-2\overline{z}z^t+|zz^t|^2>0\ \text{ and }\ |zz^t|<1\},$$

be the Lie ball, where $|w|^2=\bar{w}^tw$ for any $w\in\mathbb{C}^2$. The Shilov boundary $S$ of $\mathcal{D}$ is given by

$$S=\{u=e^{i\theta}x\in\mathbb{C}^2;\quad 0\le\theta<2\pi,\quad x\in S^1\},$$

with

$$S^1=\{(x_1,x_2)\in\mathbb{R}^2;\quad x_1^2+x_2^2=1\}.$$

Let $G=SO(2,2)$ be the group of all matrices $g$ in $SL(4,\mathbb{R})$ such that $g^tJg=J$, where $J=\begin{pmatrix} I_2 & 0\\ 0 & -I_2\end{pmatrix}$. Then, the group $G=SO(2,2)$ acts transitivity on $\mathcal{D}$ by:

$$\begin{aligned} g:z\longmapsto g.z &= \Big\{\Big[\Big((\frac{zz^t+1}{2}),i(\frac{zz^t-1}{2})\Big)A^t+zB^t\Big]\begin{pmatrix}1\\ i\end{pmatrix}\Big\}^{-1}\\ &\times\ \Big\{\Big((\frac{zz^t+1}{2}),i(\frac{zz^t-1}{2})\Big)C^t+zD^t\Big\}.\end{aligned}$$

for $g=\begin{pmatrix} A & B\\ C & D\end{pmatrix}\in G=SO(2,2)$. Thus as homogeneous space, we have the identification $\mathcal{D}=G/K$ where $K$ is the stabilizer in $G$ of 0 given by $K=S(O(2)\times O(2))$. The action of G extends naturally to $\overline{\mathcal{D}}$ and under this action the group $K$ acts transitively on the Shilov boundary $S$ and we have $S=K/\{I_4\}$.
Finally recall that every $z$ in $\mathcal{D}$ can be written as $z=ka_R.0$, with respect to the Cartan decomposition of $G$ is given by $SO(2,2)=KAK$. Here

$$a_R\in A=\Big\{\begin{pmatrix} diag(\cosh r_1,\cosh r_2) & diag(\sinh r_1,\sinh r_2)\\ diag(\sinh r_1,\sinh r_2) & diag(\cosh r_1,\cosh r_2)\end{pmatrix};\qquad R=(r_1,r_2)\in\mathbb{R}_+^2\Big\}.$$

Let $L^2(S)$ be the space of all square integrable $\mathbb{C}$-valued functions on $S$ with respect to the measure $du$. Then the group $K$ acts on $L^2(S)$ by:

$$f\longrightarrow\pi(k)f=f\circ k^{-1},\quad k\in K,$$

and under this action the space $L^2(S)$ has the following Peter-weyl decomposition (see [3]) :

$$L^2(S) = \bigoplus_{m\in\wedge} V_m,$$

where $\wedge$ is the set of all two-tuple $m = (m_1, m_2) \in \mathbb{Z}^2$ with $m_1 \geq m_2$. The K-irreducible component $V_m$ is the finite linear span $\{\phi_m \circ k, k \in K\}$. where $\phi_m \in V_m$ is the zonal spherical function given by

$$\phi_m(u) = (u_1 - iu_2)^{m_1-m_2}(u_1^2 + u_2^2)^{m_2}, \quad u = (u_1, u_2) \in S, \quad m = (m_1, m_2).$$

Let P(z,u) be the Poisson kernel of the Lie ball $\mathcal{D}$ with respect to the Shilov boundary S of $\mathcal{D}$, given by (see [4])

$$P(z, u) = \frac{1 - 2\bar{z}z^t + |zz^t|^2}{|(z-u)(z-u)^t|^2}.$$

Let $\lambda \in \mathbb{C}$ the Poisson transform $P_\lambda$ is defined for $f \in A'(S)$ by:

$$[P_\lambda f](z) = \int_S P_\lambda(z, u) f(u) du, \tag{2.1}$$

where

$$P_\lambda(z, u) = (P(z, u))^{\frac{i\lambda+1}{2}}.$$

The main result of this paper is the following theorems.

**Theorem 2.1.** *Let $\lambda \in \mathbb{R}\backslash\{0\}$ . Then, we have:*
*(1) Let $F = P_\lambda f, f \in L^2(S)$. Then*

$$||F||_*^2 = \sup_{t>0} \frac{1}{t^2} \int_0^t \Big( \int_0^{r_1} \int_{S(O(2)\times O(2))} |F(ka_R.0)|^2 \sinh(r_1 - r_2) \sinh(r_1 + r_2) dk dr_2 \Big) dr_1 < \infty.$$

*(2) Let $f \in A'(S)$. If $F = P_\lambda f$ satisfies $||F||_* < \infty$, then $f \in L^2(S)$. Moreover, there exists a positive constants $\gamma_1$ and $\gamma_2(\lambda)$ such that for every function $f \in L^2(S)$ we have:*

$$\gamma_1 |C(\lambda)| ||f||_{L^2(S)} \leq ||P_\lambda f||_* \leq \gamma_2(\lambda) ||f||_{L^2(S)} \tag{2.2}$$

*where* $C(\lambda) = \dfrac{\Gamma^2(i\lambda)}{\Gamma^4(\frac{i\lambda+1}{2})}$.

As a second result of this paper, we give an $L^2$-type inversion formula for the Poisson transform.

**Theorem 2.2.** *Let $F = P_\lambda f$, $f \in L^2(S)$. Then its $L^2$-boundary value $f$ is given by the following inversion formula*

$$\begin{aligned} &\lim_{t\longrightarrow\infty} \frac{1}{t^2} \int_0^t \Big( \int_0^{r_1} \int_K F(ka_R.0)\overline{P_\lambda(ha_R.0, ke)} \sinh(r_1 - r_2) \sinh(r_1 + r_2) dk dr_2 \Big) dr_1 \\ = \;& \gamma_1^2 |C(\lambda)|^2 f(h.e), \quad \textit{in } L^2(S). \end{aligned}$$

The difficult part in proving our result is to show that every $F = P_\lambda f$, $f \in A'(S)$ such that $||F||_* < \infty$ is the poisson transform of an $L^2$-function on the Shilov boundary $S$. Indeed, expanding $F$ into a $C^\infty$ series (see corollary below)

$$F(ka_R.0) = \sum_{m\in\wedge} a_m \Phi_{\lambda,m}(a_R.0) f_m(k.e)$$

next, applying the Lemma 1.1 of asymptotic behaviour of integral type of $\Phi_{\lambda,m}(a_R.0)$.

## 3 The Poisson transform.

In this section, we give the precise action of the Poisson transform $P_\lambda$ on $L^2(S)$.
For $\lambda \in \mathbb{C}$ and for $k \in \mathbb{Z}^+$, let $\varphi_{\lambda,k}(r)$ denote the following $\mathbb{C}$-valued function on $r \in [0,1[$

$$\varphi_{\lambda,k}(r) = (1-r^2)^{\frac{i\lambda+1}{2}} r^k \frac{(\frac{i\lambda+1}{2})_k}{(1)_k} F(\frac{i\lambda+1}{2}, \frac{i\lambda+1}{2}+k, 1+k; r^2),$$

where $(a)_k = a(a+1)(a+2)...(a+k-1)$ is the Pochhammer's symbol and $F(a,b,c;x)$ is the classical Gauss hypergeometric function.

**Proposition 3.1.** [2] *Let $m=(m_1,m_2) \in \wedge$ and let $f \in V_m$. Then, we have*

$$(P_\lambda f)(ka_R.0) = \Phi_{\lambda,m}(a_R.0) f(k.e),$$

*where the generalized spherical function $\Phi_{\lambda,m}$ is given by*

$$\Phi_{\lambda,m}(a_R.0) = 4\pi^2 \Big[\varphi_{\lambda,|m_1|}(\tanh(\frac{r_1-r_2}{2})\varphi_{\lambda,|m_2|}(\tanh(\frac{r_1+r_2}{2})\Big]$$

**Corollary 3.2.** *Let $F = P_\lambda f, f \in A'(S)$. Then, there exists a sequence of spherical harmonic functions $(f_m)_{m\in\wedge}$ such that for every $z = ka_R.0 \in \mathcal{D}$, $k \in K$ $a_R \in A$, $F$ may be written in the form as follows*

$$F(z) = \sum_{m\in\wedge} \Phi_{\lambda,m}(a_R.0) f_m(k.e), \qquad f_m \in V_m.$$

**Proof.** For $f$ in $A'(S)$. Let $f = \sum_{m\in\wedge} f_m$ it's K-type decomposition. Then using Proposition 3.1 we get

$$F(ka_R.0) = \sum_{m\in\wedge} \Phi_{\lambda,m}(a_R.0) f_m(k.e), \qquad f_m \in V_m.$$

## 4 Proof of main results

### 4.1 Proof of Theorem 2.1

For the proof Theorem 2.1, we will need the Lemma 1.2, which we recall below
**Lemma 1.2** Let $\lambda$ be a non zero real number. Then, there exists a positive constant $\gamma_2(\lambda)$ such that

$$\sup_{t>0} \frac{1}{t^2} \int_0^t \Big( \int_0^{r_1} \Big|\Phi_{\lambda,m}(a_R.0)\Big|^2 \sinh(r_1-r_2)\sinh(r_1+r_2) dr_2 \Big) dr_1 < \gamma_2^2(\lambda)$$

**Proof.** In order to get the proof for this lemma, we introduce the following lemma

**Lemma** (see [1]): Let $\lambda$ be a non zero real number. Then, there exists a positive constant $A(\lambda)$ such that for every $t>0$, we have

$$\sup_{k\in\mathbb{Z}^+} \Big|\varphi_{\lambda,k}(\tanh t)\Big| \le A(\lambda)\cosh^{-1} t.$$

For fixed $t>0$, we have

$$\begin{aligned}
& \frac{1}{t^2} \int_0^t \Big( \int_0^{r_1} \Big|\Phi_{\lambda,m}(a_R.0)\Big|^2 \sinh(r_1-r_2)\sinh(r_1+r_2) dr_2 \Big) dr_1 \\
= & \frac{4}{t^2} \int_0^t \Big( \int_0^{r_1} \coth^2(\frac{r_1-r_2}{2})\coth^2(\frac{r_1+r_2}{2}) \Big|\Phi_{\lambda,m}(a_R.0)\Big|^2 \tanh(\frac{r_1-r_2}{2})\tanh(\frac{r_1+r_2}{2}) dr_2 \Big) dr_1 \\
\le & \frac{4}{t^2} \int_0^t \Big( \int_0^{r_1} \coth^2(\frac{r_1-r_2}{2})\coth^2(\frac{r_1+r_2}{2}) \Big|\Phi_{\lambda,m}(a_R.0)\Big|^2 dr_2 \Big) dr_1.
\end{aligned}$$

Then, we deduce from the above lemma that there exists a positive constant $\gamma_2(\lambda)$ such that

$$\sup_{t>0} \frac{1}{t^2} \int_0^t \Big( \int_0^{r_1} \Big|\Phi_{\lambda,m}(a_R.0)\Big|^2 \sinh(r_1 - r_2) \sinh(r_1 + r_2) dr_2 \Big) dr_1 \leq \frac{1}{4\pi^4} A^2(\lambda) = \gamma_2^2(\lambda).$$

Then, for the necessary condition let $f \in L^2(S)$ and let $f = \sum_{m\in\wedge} f_m$ be its K-type decomposition. By Proposition 3.1, with $\sum_{m\in\wedge} |\Phi_{\lambda,m}(a_R.0)|^2 ||f_m||^2_{L^2(S)} < \infty$, for every $R = (r_1, r_2) \in \mathbb{R}^2_+$, we have

$$(P_\lambda f)(ka_R.0) = F(ka_R.0) = \sum_{m\in\wedge} \Phi_{\lambda,m}(a_R.0) f_m(k.e).$$

Then, replacing $F$ by the above series expansion we get

$$||F||_*^2 = \sup_{t>0} \frac{1}{t^2} \int_0^t \Big( \int_0^{r_1} \sum_{m\in\wedge} \Big|\Phi_{\lambda,m}(a_R.0)\Big|^2 ||f_m||_2^2 \sinh(r_1 - r_2) \sinh(r_1 + r_2) dr_2 \Big) dr_1.$$

Next, we use the Lemma 1.2 to obtain

$$\sum_{m\in\wedge} \frac{||f_m||_2^2}{t^2} \int_0^t \Big( \int_0^{r_1} \Big|\Phi_{\lambda,m}(a_R.0)\Big|^2 \sinh(r_1 - r_2) \sinh(r_1 + r_2) dr_2 \Big) dr_1 \leq \gamma_2^2(\lambda) \sum_{m\in\wedge} ||f_m||_2^2 < \infty.$$

Henceforth

$$||P_\lambda f||_* \leq \gamma_2(\lambda) ||f||_2,$$

this gives the right hand side of estimate (2.3) in Theorem 2.1.
Now, to prove the sufficiency condition. Let $F = P_\lambda f$, $f \in A'(S)$ such that $||F||_* < \infty$. Let $f = \sum_{m\in\wedge} f_m$ be its K-type decomposition, then using Proposition 3.1, we get

$$F(ka_R.0) = \sum_{m\in\wedge} \Phi_{\lambda,m}(a_R.0) f_m(k.e).$$

Since $||F||_* < \infty$, we have

$$\sum_{m\in\wedge} ||f_m||_2^2 \frac{1}{t^2} \int_0^t \Big( \int_0^{r_1} |\Phi_{\lambda,m}(a_R.0)|^2 \sinh(r_1 - r_2) \sinh(r_1 + r_2) dr_2 \Big) dr_1 < \infty$$

Let $\wedge_\circ$ be a finite subset of $\wedge$, then we have

$$\sum_{m\in\wedge_\circ} ||f_m||_2^2 \frac{1}{t^2} \int_0^t \Big( \int_0^{r_1} |\Phi_{\lambda,m}(a_R.0)|^2 \sinh(r_1 - r_2) \sinh(r_1 + r_2) dr_2 \Big) dr_1 \leq ||F||_*^2 < \infty,$$

for every $t > 0$.
Next, using the asymptotic behaviour of $\Phi_{\lambda,m}$ given by Lemma 1.1. we obtain

$$\gamma_1^2 |C(\lambda)|^2 \sum_{m\in\wedge_\circ} ||f_m||_2^2 \leq ||F||_*^2 < \infty.$$

Since $\wedge_\circ$ is arbitrary, we get

$$\gamma_1^2 |C(\lambda)|^2 \sum_{m\in\wedge} ||f_m||_2^2 \leq ||F||_*^2 < \infty.$$

Thus $\gamma_1^2 |C(\lambda)|^2 ||f||_2^2 \leq ||F||_*^2 < \infty$ and $f \in L^2(S)$ this finishes the proof of Theorem 2.1.

### 4.2 Proof of Theorem 2.2

In this section we try to prove the $L^2$-inversion formula.
Let $F = P_\lambda f$, $f \in A'(S)$ such that $||F||_* < \infty$. By the Theorem 2.1, we know that $f$ in $L^2(S)$. Expanding $f$ into its K-type series, $f = \sum_{m\in\wedge} f_m$ and using Proposition 3.1, we get the series expansion of $F$,

$$F(ka_R.0) = \sum_{m\in\wedge} \Phi_{\lambda,m}(a_R.0) f_m(k.e), \quad f_m \in V_m, \tag{4.1}$$

with $\sum_{m\in\wedge} |\Phi_{\lambda,m}(a_R.0)|^2 ||f_m||_2^2 < \infty$, for all $R = (r_1, r_1)$, $r_1 > r_2 > 0$. Next, set for each $t > 0$, the following $\mathbb{C}$-valued function on $S$

$$g_t(h.e) = \frac{1}{t^2}\int_0^t \Big(\int_0^{r_1}\int_K F(ka_R.0)\overline{P_\lambda(ha_R.0, k.e)}|\sinh(r_1 - r_2)|\sinh(r_1 + r_2) dk dr_2\Big) dr_1.$$

Then, replacing $F$ by its above series expansion in (4.1), the function $g_t$ can be rewritten as:

$$g_t(h.e) = \frac{1}{t^2}\int_0^t \Big(\int_0^{r_1}\int_K \sum_{m\in\wedge} \Phi_{\lambda,m}(a_R.0) f_m(k.e)\overline{P_\lambda(ha_R.0, k.e)}|\sinh(r_1 - r_2)|\sinh(r_1 + r_2) dk dr_2\Big) dr_1.$$

Since, for every fixed $r_1 > r_2 > 0$, the series $\sum_{m\in\wedge} \Phi_{\lambda,m}(a_R.0) f_m(k.e)$ is uniformly convergent on $S$, we get

$$g_t(h.e) = \frac{1}{t^2}\sum_{m\in\wedge}\int_0^t \Big(\int_0^{r_1}\int_K \Phi_{\lambda,m}(a_R.0) f_m(k.e)\overline{P_\lambda(ha_R.0, k.e)}|\sinh(r_1 - r_2)|\sinh(r_1 + r_2) dk dr_2\Big) dr_1$$

and by proposition 3.1, we have

$$g_t(h.e) = \frac{1}{t^2}\sum_{m\in\wedge}\Big[\int_0^t \Big(\int_0^{r_1} \Big|\Phi_{\lambda,m}(a_R.0)\Big|^2 |\sinh(r_1 - r_2)|\sinh(r_1 + r_2) dr_2\Big) dr_1\Big] f_m(h.e).$$

Hence the $L^2(S)$-norm of the function $g_t$ is given by:

$$||g_t||_2^2 = \\ (\frac{1}{t^2})^2 \sum_{m\in\wedge}\Big[\int_0^t \Big(\int_0^{r_1} \Big|\Phi_{\lambda,m}(a_R.0)\Big|^2 |\sinh(r_1 - r_2)|\sinh(r_1 + r_2) dr_2\Big) dr_1\Big]^2 ||f_m||_2^2.$$

Now using the fact that

$$\begin{aligned} & \frac{1}{t^2}\int_0^t \Big(\int_0^{r_1} \Big|\Phi_{\lambda,m}(a_R.0)\Big|^2 |\sinh(r_1 - r_2)|\sinh(r_1 + r_2) dr_2\Big) dr_1 \\ = & \frac{1}{t^2}\int_0^t \Big(\int_0^{r_1} \Big|P_\lambda \phi_m(a_R.0)\Big|^2 |\sinh(r_1 - r_2)|\sinh(r_1 + r_2) dr_2\Big) dr_1 \le \gamma_2^2 \end{aligned}$$

we obtain that

$$\begin{aligned} & ||\gamma_1^2 |C(\lambda)|^{-2} g_t - f||_2^2 \\ = & \sum_{m\in\wedge}\Big[\frac{\gamma_1^2 |C(\lambda)|^{-2}}{t^2}\int_0^t \Big(\int_0^{r_1} \Big|\Phi_{\lambda,m}(a_R.0)\Big|^2 |\sinh(r_1 - r_2)|\sinh(r_1 + r_2) dr_2\Big) dr_1 - 1\Big]^2 ||f_m||_2^2 \end{aligned}$$

and, using the uniform pointwises boundedness of $\Phi_{\lambda,m}$ given by Lemma 1.1, we see that

$$\lim_{t\longrightarrow\infty} ||\gamma_1^2 |C(\lambda)|^{-2} g_t - f|||_2^2 = 0$$

which given the desired result.

## 5 Appendix The asymptotic behaviour of $\Phi_{\lambda,m}$.

We will now establish the asymptotic behavior of the generalized spherical function $\Phi_{\lambda,m}$. Recall that $\Phi_{\lambda,m}$ is given by

$$\begin{aligned}
\Phi_{\lambda,m}(a_R.0) &= 4\pi^2\Big[\cosh(\frac{r_1-r_2}{2})\cosh(\frac{r_1+r_2}{2})\Big]^{-(i\lambda+1)}\\
&\times \tanh^{|m_1|}(\frac{r_1-r_2}{2})\tanh^{|m_2|}(\frac{r_1+r_2}{2})\frac{(\frac{i\lambda+1}{2})_{|m_1|}}{(1)_{|m_1|}}\frac{(\frac{i\lambda+1}{2})_{|m_2|}}{(1)_{|m_2|}}\\
&\times F(\frac{i\lambda+1}{2},\frac{i\lambda+1}{2}+|m_1|,|m_1|+1;\tanh^2(\frac{r_1-r_2}{2}))\\
&\times F(\frac{i\lambda+1}{2},\frac{i\lambda+1}{2}+|m_2|,|m_2|+1;\tanh^2(\frac{r_1+r_2}{2})).
\end{aligned}$$

**Lemma 1.1** There exists a constant $\gamma_1>0$ such that we have:

$$\lim_{t\longrightarrow\infty}\frac{1}{t^2}\int_0^t\Big(\int_0^{r_1}\Big|\Phi_{\lambda,m}(a_R.0)\Big|^2|\sinh(r_1-r_2)|\sinh(r_1+r_2)dr_2\Big)dr_1=\gamma_1^2\Big|\frac{\Gamma^2(i\lambda)}{\Gamma^4(\frac{i\lambda+1}{2})}\Big|^2$$

for every $\lambda\in\mathbb{R}\backslash\{0\}$ and for every $m=(m_1,m_2)\in\wedge$.

Proof. Using the following identity on hypergeometric function (see[6]):

$$\begin{aligned}
F(a,b,c;x) &= \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}F(a,b,a+b-c+1,1-x)\\
&+ \frac{\Gamma(c)\Gamma(a+b-c)}{\Gamma(a)\Gamma(b)}(1-x)^{c-a-b}\\
&\times F(c-a,c-b,c-a-b+1,1-x)
\end{aligned}$$

Then, the hypergeometric function $\varphi_{\lambda,k}(x)$ can be written as follows

$$\begin{aligned}
\varphi_{\lambda,k}(\tanh^2 x) &= \cosh^{-(i\lambda+1)}(x)\tanh^k(x)\frac{(\frac{i\lambda+1}{2})_k}{(1)_k}F(\frac{i\lambda+1}{2},\frac{i\lambda+1}{2}+k,k+1;\tanh^2(x))\\
&= \cosh^{-(i\lambda+1)}(x)\tanh^k(x)\frac{(\frac{i\lambda+1}{2})_k\Gamma(-i\lambda)}{\Gamma(k+\frac{1-i\lambda}{2})\Gamma(\frac{1-i\lambda}{2})}F(\frac{i\lambda+1}{2},\frac{i\lambda+1}{2}+k,i\lambda+1;1-\tanh^2(x)\\
&+ \cosh^{i\lambda-1}(x)\tanh^k(x)\frac{\Gamma(i\lambda)}{\Gamma^2(\frac{i\lambda+1}{2})}F(\frac{1-i\lambda}{2},\frac{1-i\lambda}{2}+k,1-i\lambda;1-\tanh^2(x)).
\end{aligned}$$

Therefore for $\lambda\in\mathbb{R}\backslash\{0\}$, we have

$$|\varphi_{\lambda,k}(\tanh^2 x)|^2\cosh(x)\sinh(x)\simeq_{x\longrightarrow\infty}2\Big|\frac{\Gamma(i\lambda)}{\Gamma^2(\frac{i\lambda+1}{2})}\Big|^2+\overline{A(\lambda,k)}\cosh^{-2i\lambda}(x)+A(\lambda,k)\cosh^{2i\lambda}(x)$$

with $A(\lambda,k)=\dfrac{\Gamma^2(i\lambda)(\frac{1-i\lambda}{2})_k}{(\frac{1+i\lambda}{2})_k\Gamma^4(\frac{1+i\lambda}{2})}$.

To complete the proof , we are going to establish that

$$\lim_{t\longrightarrow\infty}\frac{I_1}{t^2}=\lim_{t\longrightarrow\infty}\frac{I_2}{t^2}=\lim_{t\longrightarrow\infty}\frac{I_3^{\pm}}{t^2}=0$$

where

$$\begin{aligned}
I_1 &= \int_0^t\Big[\int_0^{r_1}\cosh^{2i\lambda}(\frac{r_1-r_2}{2})\cosh^{2i\lambda}(\frac{r_1+r_2}{2})dr_2\Big]dr_1\\
I_2 &= \int_0^t\Big[\int_0^{r_1}\cosh^{2i\lambda}(\frac{r_1-r_2}{2})\cosh^{-2i\lambda}(\frac{r_1+r_2}{2})dr_2\Big]dr_1.\\
I_3^{\pm} &= \int_0^t\Big[\int_0^{r_1}\cosh^{2i\lambda}(\frac{r_1\pm r_2}{2})dr_2\Big]dr_1
\end{aligned}$$

Indeed,

**For the integral $I_1$**

By using the fact that

$$\int_0^{r_1} \cosh^{2i\lambda}(\frac{r_1-r_2}{2})\cosh^{2i\lambda}(\frac{r_1+r_2}{2})dr_2 = \frac{1}{2}\int_0^{r_1}\Big(\cosh(r_1)+\cosh(r_2)\Big)^{2i\lambda}dr_2$$

and the fact that for every $s>0$

$$\begin{aligned}\int_0^s \cosh(x)(\cosh(x)+\cosh(y))^{2i\lambda-1}dx &= \frac{(\cosh(s)+\cosh(y))^{2i\lambda}-(\cosh(y)+1)^{2i\lambda}}{2i\lambda}\\ &+ \int_0^s e^{-x}(\cosh(x)+\cosh(y))^{2i\lambda-1}dx,\end{aligned}$$

which imply that

$$\begin{aligned}\Big|\int_0^s e^{-x}(\cosh(x)+\cosh(y))^{2i\lambda-1}dx\Big| &\leq \int_0^s e^{-x}(\cosh(x)+\cosh(y))^{-1}dx\\ &\leq \int_0^s e^{-x}dx = 1-e^{-s}<1, \quad \lambda\in\mathbb{R}\backslash\{0\},\end{aligned}$$

we have

$$\lim_{t\longrightarrow\infty}\frac{I_1}{t^2}=0.$$

**For the integral $I_2$**

The integral $I_2$ is equal

$$\begin{aligned}I_2 &= \int_0^t\Big[\int_0^{r_1}\cosh^{2i\lambda}(\frac{r_1-r_2}{2})\cosh^{-2i\lambda}(\frac{r_1+r_2}{2})dr_2\Big]dr_1\\ &= \frac{1}{2}\int_0^t\Big[\int_0^{r_1}\frac{\cosh(r_1)+\cosh(r_2)}{\cosh^2(\frac{r_1+r_2}{2})}\Big[\frac{\cosh(\frac{r_1-r_2}{2})}{\cosh(\frac{r_1+r_2}{2})}\Big]^{2i\lambda-1}dr_2\Big]dr_1\\ &= \frac{1}{2}\int_0^t\Big[\int_0^{r_1}\frac{\sinh(r_1)+e^{-r_1}+\cosh(r_2)}{\cosh^2(\frac{r_1+r_2}{2})}\Big[\frac{\cosh(\frac{r_1-r_2}{2})}{\cosh(\frac{r_1+r_2}{2})}\Big]^{2i\lambda-1}dr_2\Big]dr_1\\ &= \int_0^t\frac{1-\Big(\frac{1}{\cosh(r_1)}\Big)^{2i\lambda}}{2i\lambda}dr_1\\ &+ \frac{1}{2}\int_0^t\Big[\int_0^{r_1}\frac{e^{-r_1}+\cosh(r_2)}{\cosh^2(\frac{r_1+r_2}{2})}\Big[\frac{\cosh(\frac{r_1-r_2}{2})}{\cosh(\frac{r_1+r_2}{2})}\Big]^{2i\lambda-1}dr_2\Big]dr_1.\end{aligned}$$

Then, by using the fact that for every $\lambda\in\mathbb{R}\backslash\{0\}$

$$\begin{aligned}&\Big|\int_0^t\Big[\int_0^{r_1}\frac{e^{-r_1}+\cosh(r_2)}{\cosh^2(\frac{r_1+r_2}{2})}\Big[\frac{\cosh(\frac{r_1-r_2}{2})}{\cosh(\frac{r_1+r_2}{2})}\Big]^{2i\lambda-1}dr_2\Big]dr_1\Big|\\ \leq\ &\int_0^t\Big[\int_0^{r_1}\frac{e^{-r_1}+e^{-r_2}+\sinh(r_2)}{\cosh(r_1)+\cosh(r_2)}dr_2\Big]dr_1 \leq \int_0^t\Big[\int_0^{r_1}(e^{-r_1}+e^{-r_2})dr_2\Big]dr_1+\int_0^t\log\Big(\frac{2\cosh(r_1)}{\cosh(r_1)+1}\Big)d\end{aligned}$$

we have

$$\lim_{t\longrightarrow\infty}\frac{I_2}{t^2}=0.$$

**For the integral $I_3^{\pm}$**

By using the fact that

$$\begin{aligned}\int_0^{r_1}\cosh^{2i\lambda}(\frac{r_1-r_2}{2})dr_2 &= \int_0^{r_1}\cosh(\frac{r_1-r_2}{2})\cosh^{2i\lambda-1}(\frac{r_1-r_2}{2})dr_2\\ &= \int_0^{r_1}\Big[\sinh(\frac{r_1-r_2}{2})+e^{(\frac{r_2-r_1}{2})}\Big]\cosh^{2i\lambda-1}(\frac{r_1-r_2}{2})dr_2\\ &= \frac{\cosh^{2i\lambda}(\frac{r_1}{2})-1}{i\lambda}+2\int_0^{r_1}\Big[\frac{e^{(r_2-r_1)}}{1+e^{(r_2-r_1)}}\Big]\cosh^{2i\lambda}(\frac{r_1-r_2}{2})dr_2\end{aligned}$$

and the fact that

$$\begin{aligned}&\left|\int_0^{r_1}\Big[\frac{e^{(r_2-r_1)}}{1+e^{(r_2-r_1)}}\Big]\cosh^{2i\lambda}(\frac{r_1-r_2}{2})dr_2\right|\\ \leq\ &\int_0^{r_1}\frac{e^{(r_2-r_1)}}{1+e^{(r_2-r_1)}}dr_2=\log\Big(\frac{2}{1+e^{-r_1}}\Big)=r_1+\log\Big(\frac{2}{1+e^{r_1}}\Big),\end{aligned}$$

we have, for every $\lambda\in\mathbb{R}\backslash\{0\}$ that

$$\lim_{t\longrightarrow\infty}\frac{I_3^-}{t^2}=0.$$

and analogously

$$\lim_{t\longrightarrow\infty}\frac{I_3^+}{t^2}=0.$$

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