

# A Study of Unified Fractional Integral operators involving S-Generalized Gauss’s Hypergeometric Function as its Kernel

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**Abstract.** In the present Paper, we study a pair of a general class of fractional integral operators whose kernel involves the product of a Appell Polynomial, Fox H-function and S-Generalized Gauss’s Hypergeometric Function. First, we have given images of some useful functions under these fractional integral operators. Next, we obtain Mellin Transform, inversion formula and Mellin convolutions for these fractional operators. Finally, we study the Parseval-Goldstein Theorem related to operators of our study. The fractional integral operators studied by us are most general in nature and may be considered as generalizations of a number of unified fractional operators studied from time to time by several authors. For the sake of illustration, we give here exact references of the recent results obtained by Saxena and Kumbhat [15] , Saigö [13]. which follow as special case of our findings. The importance of the present study lies in the fact that it unifies and extends the recent results of a large number of authors.

## 1 Introduction

### Fox H-Function

A single Mellin-Barnes contour integral, occurring in the present work, is now popularly known as the *H*-function of Charles Fox (1897-1977). It will be defined and represented here in the following manner (see, for example, [9, p. 10]):

$$\begin{aligned}
 H_{P,Q}^{M,N}[z] &= H_{P,Q}^{M,N} \left[ z \left| \begin{matrix} (a_j, \alpha_j)_{1,P} \\ (b_j, \beta_j)_{1,Q} \end{matrix} \right. \right] = H_{P,Q}^{M,N} \left[ z \left| \begin{matrix} (a_1, \alpha_1), \dots, (a_P, \alpha_P) \\ (b_1, \beta_1), \dots, (b_Q, \beta_Q) \end{matrix} \right. \right] \\
 &= \frac{1}{2\pi i} \int_{\mathcal{L}} \Theta(\mathfrak{s}) z^{\mathfrak{s}} d\mathfrak{s}, \tag{1.1}
 \end{aligned}$$

where  $i = \sqrt{-1}$ ,  $z \in \mathbb{C} \setminus \{0\}$ ,  $\mathbb{C}$  being the set of complex numbers,

$$\Theta(\mathfrak{s}) = \frac{\prod_{j=1}^M \Gamma(b_j - \beta_j \mathfrak{s}) \prod_{j=1}^N \Gamma(1 - a_j + \alpha_j \mathfrak{s})}{\prod_{j=M+1}^Q \Gamma(1 - b_j + \beta_j \mathfrak{s}) \prod_{j=N+1}^P \Gamma(a_j - \alpha_j \mathfrak{s})}, \tag{1.2}$$

and

$$1 \leq M \leq Q \quad \text{and} \quad 0 \leq N \leq P \quad (M, Q \in \mathbb{N} = \{1, 2, 3, \dots\}; N, P \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}), \tag{1.3}$$

an empty product being interpreted to be 1. Here  $\mathcal{L}$  is a Mellin-Barnes type contour in the complex  $\mathfrak{s}$ -plane with appropriate indentations in order to separate the two sets of poles of the integrand  $\Theta(\mathfrak{s})$  (see, for details, [1] and [9]).

**Multivariable H-Function**

The Multivariable H-Function is defined and represented in the following manner [9, p. 251, Eqs. (C.1-C.3)]

$$\begin{aligned}
 & H_{C,D:C_1,D_1;\dots;C_r,D_r}^{0,B:A_1,B_1;\dots;A_r,B_r} \left[ \begin{array}{l} z_1 \\ \cdot \\ \cdot \\ z_r \end{array} \middle| \begin{array}{l} (a_j; \alpha_j^{(1)}, \dots, \alpha_j^{(r)})_{1,C} : (c_j^{(1)}, \gamma_j^{(1)})_{1,C_1}; \dots; (c_j^{(r)}, \gamma_j^{(r)})_{1,C_r} \\ (b_j; \beta_j^{(1)}, \dots, \beta_j^{(r)})_{1,D} : (d_j^{(1)}, \delta_j^{(1)})_{1,D_1}; \dots; (d_j^{(r)}, \delta_j^{(r)})_{1,D_r} \end{array} \right] \\
 &= \frac{1}{(2\pi\omega)^r} \int_{\mathfrak{L}_1} \int_{\mathfrak{L}_2} \dots \int_{\mathfrak{L}_r} \Phi(\xi_1, \xi_2, \dots, \xi_r) \prod_{i=1}^r \Theta_i(\xi_i) z^{\xi_i} d\xi_1 d\xi_2, \dots, d\xi_r, \tag{1.4}
 \end{aligned}$$

where  $\omega = \sqrt{-1}$ ,

$$\begin{aligned}
 \Phi(\xi_1, \xi_2, \dots, \xi_r) &= \frac{\prod_{j=1}^B \Gamma(1 - a_j + \sum_{i=1}^r \alpha_j^{(i)} \xi_i)}{\prod_{j=1}^D \Gamma(1 - b_j + \sum_{i=1}^r \beta_j^{(i)} \xi_i) \prod_{j=B+1}^C \Gamma(a_j - \sum_{i=1}^r \alpha_j^{(i)} \xi_i)} \\
 \Theta_i(\xi_i) &= \frac{\prod_{j=1}^{A_i} \Gamma(d_j^{(i)} - \delta_j^{(i)} \xi_i) \prod_{j=1}^{B_i} \Gamma(1 - c_j^{(i)} + \gamma_j^{(i)} \xi_i)}{\prod_{j=B_i+1}^{C_i} \Gamma(c_j^{(i)} - \gamma_j^{(i)} \xi_i) \prod_{j=A_i+1}^{D_i} \Gamma(1 - d_j^{(i)} + \delta_j^{(i)} \xi_i)} \quad (i = 1, 2, \dots, r), \tag{1.5}
 \end{aligned}$$

All the Greek letters occurring on the left and side of (1.4) are assumed to be positive real numbers for standardization purposes. The definition of the multivariable H-function will however be meaningful even if some of these quantities are zero. The details about the nature of contour  $\mathfrak{L}_1, \dots, \mathfrak{L}_r$ , conditions of convergence of the integral given by (1.4). Throughout the paper it is assumed that this function always satisfied its appropriate conditions of convergence [9, p. 251, Eqs. (C.4-C.6)]

**S-Generalized Gauss’s Hypergeometric Function**

The S-generalized Gauss hypergeometric function  $F_p^{(\alpha,\beta;\tau,\mu)}(a, b; c; z)$  introduced and defined by Srivastava et al. [11, p.350, Eq.(1.12)] is represented in the following manner:

$$F_p^{(\alpha,\beta;\tau,\mu)}(a, b; c; z) = \sum_{n=0}^{\infty} (a)_n \frac{B_p^{(\alpha,\beta;\tau,\mu)}(b+n, c-b)}{B(b, c-b)} \frac{z^n}{n!} \quad (|z| < 1) \tag{1.6}$$

provided that  $(\Re(p) \geq 0; \min \Re(\alpha, \beta, \tau, \mu) > 0; \Re(c) > \Re(b) > 0)$

where the S-generalized Beta function  $B_p^{(\alpha,\beta;\tau,\mu)}(x, y)$  was introduced and defined by Srivastava et al. [11, p.350, Eq.(1.13)]:

$$B_p^{(\alpha,\beta;\tau,\mu)}(x, y) = \int_0^1 t^{x-1} (1-t)^{y-1} {}_1F_1 \left( \alpha; \beta; \frac{-p}{t^\tau(1-t)^\mu} \right) dt \tag{1.7}$$

provided that  $(\Re(p) \geq 0; \min \Re\{x, y, \alpha, \beta\} > 0; \min \Re\{\tau, \mu\} > 0)$

and  $(\lambda)_n$  denotes the pochhammer symbol defined (for  $\lambda \in \mathbb{C}$ ) by (see [6, p. 2 and pp. 4-6]; see also [5, p. 2]):

$$(\lambda)_n := \frac{\Gamma(\lambda+n)}{\Gamma(\lambda)} = \begin{cases} 1, & (n = 0) \\ \lambda(\lambda+1)\dots(\lambda+n-1), & (n \in \mathbb{N} := \{1, 2, 3\}) \end{cases} \tag{1.8}$$

provided that the Gamma quotient exists (see, for details,[7, et seq.] and [8, p. 22 et seq.]).

**Generalized incomplete hypergeometric function**

The generalized incomplete hypergeometric function introduced and defined by Srivastava et al. [10, p.675, Eq.(4.1)] is represented in the following manner:

$$\begin{aligned}
 {}_p\gamma_q \left[ \begin{matrix} (E_p; \sigma); \\ F_q; \end{matrix} z \right] &= {}_p\gamma_q \left[ \begin{matrix} (e_1; \sigma), e_2, \dots, e_p; \\ f_1, \dots, f_q; \end{matrix} z \right] \\
 &:= \sum_{n=0}^{\infty} \frac{(E_p; \sigma)_n}{(F_q; 0)_n} \frac{z^n}{n!} = \sum_{n=0}^{\infty} \frac{(e_1; \sigma)_n, (e_2)_n, \dots, (e_p)_n}{(f_1)_n, (f_2)_n, \dots, (f_q)_n} \frac{z^n}{n!} \tag{1.9}
 \end{aligned}$$

where the incomplete Pochhammer symbols are defined as follows:

$$(a; \sigma)_n = \frac{\gamma(a + n, \sigma)}{\Gamma(a)} \quad (a, n \in \mathbb{C}; x \geq 0) \tag{1.10}$$

and the familiar incomplete gamma function  $\gamma(s, x)$  is

$$\gamma(s, x) = \int_0^x t^{s-1} e^{-t} dt \quad (\Re(s) > 0; x \geq 0) \tag{1.11}$$

provided that the defining of infinite series in each case is absolutely convergent.

**Appell Polynomial**

The Appell Polynomial introduced and defined by [14] is represented in the following manner:

$$A_n(z) = \sum_{k=0}^n a_{n-k} \frac{z^k}{k!} \quad n = 0, 1, 2, \dots \tag{1.12}$$

where  $a_{n-k}$  is the complex coefficient  $a_0 \neq 0$

**Fractional Integral Operators**

We study two unified fractional integral operators involving the Appell Polynomial, Fox H-function and S-Generalized Gauss’s Hypergeometric Function having general arguments

$$\begin{aligned}
 I_x^{\nu, \lambda} \{A_n, H, F_p; f(t)\} &= x^{-\nu-\lambda-1} \int_0^x t^\nu (x-t)^\lambda A_n \left[ z_1 \left(\frac{t}{x}\right)^{\nu_1} \left(1 - \frac{t}{x}\right)^{\lambda_1} \right] \\
 H_{P,Q}^{M,N} \left[ z_2 \left(\frac{t}{x}\right)^{\nu_2} \left(1 - \frac{t}{x}\right)^{\lambda_2} \middle| \begin{matrix} (a_j, \alpha_j)_{1,P} \\ (b_j, \beta_j)_{1,Q} \end{matrix} \right] &F_p^{(\alpha, \beta; \tau, \mu)} \left[ a, b; c; z_3 \left(\frac{t}{x}\right)^{\nu_3} \left(1 - \frac{t}{x}\right)^{\lambda_3} \right] f(t) dt \tag{1.13}
 \end{aligned}$$

where, The operators are defined for  $f(t) \in \Lambda$ ,  $\Lambda$  denotes the class of function  $f(t)$  for which

$$f(t) := \begin{cases} O\{|t|^\zeta\}; & \text{Max}\{|t|\} \rightarrow 0 \\ O\{|t|^{w_1} e^{-w_2|t|\}; & \text{Min}\{|t|\} \rightarrow \infty \end{cases} \tag{1.14}$$

provided that

$$\min_{1 \leq j \leq M} \Re \left( \nu + \nu_2 \frac{b_j}{\beta_j} + \zeta + 1, \lambda + \lambda_2 \frac{b_j}{\beta_j} + 1 \right) > 0 \quad \text{and} \quad \min\{\nu_1, \nu_3, \lambda_1, \lambda_3\} \geq 0 \tag{1.15}$$

$$\begin{aligned}
 J_x^{\nu,\lambda}\{A_n, H, F_p; f(t)\} &= x^\nu \int_x^\infty t^{-\nu-\lambda-1}(t-x)^\lambda A_n \left[ z_1 \left(\frac{x}{t}\right)^{\nu_1} \left(1-\frac{x}{t}\right)^{\lambda_1} \right] \\
 H_{P,Q}^{M,N} \left[ z_2 \left(\frac{x}{t}\right)^{\nu_2} \left(1-\frac{x}{t}\right)^{\lambda_2} \middle| \begin{matrix} (a_j, \alpha_j)_{1,P} \\ (b_j, \beta_j)_{1,Q} \end{matrix} \right] & F_p^{(\alpha,\beta;\tau,\mu)} \left[ a, b; c; z_3 \left(\frac{x}{t}\right)^{\nu_3} \left(1-\frac{x}{t}\right)^{\lambda_3} \right] f(t) dt
 \end{aligned}
 \tag{1.16}$$

provided that

$$\begin{aligned}
 \Re(w_2) > 0 \quad \text{or} \quad \Re(w_2) = 0 \quad \text{and} \quad \min_{1 \leq j \leq M} \Re\left(\nu - w_1 + \nu_2 \frac{b_j}{\beta_j}\right) > 0; \\
 \min_{1 \leq j \leq M} \Re\left(\lambda + \lambda_2 \frac{b_j}{\beta_j} + 1\right) > 0, \quad \min\{\nu_1, \nu_3, \lambda_1, \lambda_3\} \geq 0
 \end{aligned}
 \tag{1.17}$$

### 2 Special Cases

(i) In (1.13) and (1.16), if we reduce Appell polynomial to Laguerre polynomial [3, p.101, Eq.(5.1.6)] and Fox H-function reduced to Lorenzo Hartley G-function [12, p.64,Eq.(2.3)], we obtain the following integral

$$\begin{aligned}
 I_x^{\nu,\lambda}\{L_n, G_{q,\sigma,r}, F_p; f(t)\} &= x^{-\nu-\lambda-1} \int_0^x t^\nu (x-t)^\lambda L_n^{(\rho)} \left[ z_1 \left(\frac{t}{x}\right)^{\nu_1} \left(1-\frac{t}{x}\right)^{\lambda_1} \right] \\
 G_{q,\sigma,r} \left[ z_2, \left(\frac{t}{x}\right)^{\nu_2} \left(1-\frac{t}{x}\right)^{\lambda_2} \right] & F_p^{(\alpha,\beta;\tau,\mu)} \left[ a, b; c; z_3 \left(\frac{t}{x}\right)^{\nu_3} \left(1-\frac{t}{x}\right)^{\lambda_3} \right] f(t) dt
 \end{aligned}
 \tag{2.1}$$

and

$$\begin{aligned}
 J_x^{\nu,\lambda}\{L_n, G_{q,\sigma,r}, F_p; f(t)\} &= x^\nu \int_x^\infty t^{-\nu-\lambda-1}(t-x)^\lambda L_n^{(\rho)} \left[ z_1 \left(\frac{t}{x}\right)^{\nu_1} \left(1-\frac{t}{x}\right)^{\lambda_1} \right] \\
 G_{q,\sigma,r} \left[ z_2, \left(\frac{x}{t}\right)^{\nu_2} \left(1-\frac{x}{t}\right)^{\lambda_2} \right] & F_p^{(\alpha,\beta;\tau,\mu)} \left[ a, b; c; z_3 \left(\frac{x}{t}\right)^{\nu_3} \left(1-\frac{x}{t}\right)^{\lambda_3} \right] f(t) dt
 \end{aligned}
 \tag{2.2}$$

(ii) In (1.13) and (1.16), if we reduce Appell polynomial to Bessel polynomial [4, p.108, Eq.(34)] and Fox H-function reduced to Generalized Mittag Leffler function [2, p.25,Eq.(1.137)], we obtain the following integral

$$\begin{aligned}
 I_x^{\nu,\lambda}\{y_n, E_{\gamma,\delta}^\eta, F_p; f(t)\} &= x^{-\nu-\lambda-1} \int_0^x t^\nu (x-t)^\lambda y_n \left[ z_1 \left(\frac{t}{x}\right)^{\nu_1} \left(1-\frac{t}{x}\right)^{\lambda_1}, \rho, \sigma \right] \\
 E_{\gamma,\delta}^\eta \left[ z_2 \left(\frac{t}{x}\right)^{\nu_2} \left(1-\frac{t}{x}\right)^{\lambda_2} \right] & F_p^{(\alpha,\beta;\tau,\mu)} \left[ a, b; c; z_3 \left(\frac{t}{x}\right)^{\nu_3} \left(1-\frac{t}{x}\right)^{\lambda_3} \right] f(t) dt
 \end{aligned}
 \tag{2.3}$$

and

$$\begin{aligned}
 J_x^{\nu,\lambda}\{y_n, E_{\gamma,\delta}^\eta, F_p; f(t)\} &= x^\nu \int_x^\infty t^{-\nu-\lambda-1}(t-x)^\lambda y_n \left[ z_1 \left(\frac{t}{x}\right)^{\nu_1} \left(1-\frac{t}{x}\right)^{\lambda_1}, \rho, \sigma \right] \\
 E_{\gamma,\delta}^\eta \left[ z_2 \left(\frac{t}{x}\right)^{\nu_2} \left(1-\frac{t}{x}\right)^{\lambda_2} \right] & F_p^{(\alpha,\beta;\tau,\mu)} \left[ a, b; c; z_3 \left(\frac{x}{t}\right)^{\nu_3} \left(1-\frac{x}{t}\right)^{\lambda_3} \right] f(t) dt
 \end{aligned}
 \tag{2.4}$$

(iii) In (1.13) and (1.16), if we reduce Appell polynomial to Cesaro polynomial [8, p.449, Eq.(20)] and Fox H-function reduced to Bessel Maitland function [2, p.25,Eq.(1.139)], we obtain

the following integral

$$\begin{aligned}
 I_x^{\nu,\lambda}\{g_n^\rho, J_{\gamma,\delta}^\eta, F_p; f(t)\} &= x^{-\nu-\lambda-1} \int_0^x t^\nu (x-t)^\lambda g_n^{(\rho)} \left[ z_1 \left(\frac{t}{x}\right)^{\nu_1} \left(1-\frac{t}{x}\right)^{\lambda_1} \right] \\
 J_{\gamma,\delta}^\eta \left[ z_2 \left(\frac{t}{x}\right)^{\nu_2} \left(1-\frac{t}{x}\right)^{\lambda_2} \right] F_p^{(\alpha,\beta;\tau,\mu)} \left[ a, b, c; z_3 \left(\frac{t}{x}\right)^{\nu_3} \left(1-\frac{t}{x}\right)^{\lambda_3} \right] f(t) dt & \quad (2.5)
 \end{aligned}$$

and

$$\begin{aligned}
 J_x^{\nu,\lambda}\{g_n^\rho, J_{\gamma,\delta}^\eta, F_p; f(t)\} &= x^\nu \int_x^\infty t^{-\nu-\lambda-1} (t-x)^\lambda g_n^{(\rho)} \left[ z_1 \left(\frac{t}{x}\right)^{\nu_1} \left(1-\frac{t}{x}\right)^{\lambda_1} \right] \\
 J_{\gamma,\delta}^\eta \left[ z_2 \left(\frac{t}{x}\right)^{\nu_2} \left(1-\frac{t}{x}\right)^{\lambda_2} \right] F_p^{(\alpha,\beta;\tau,\mu)} \left[ a, b, c; z_3 \left(\frac{x}{t}\right)^{\nu_3} \left(1-\frac{x}{t}\right)^{\lambda_3} \right] f(t) dt & \quad (2.6)
 \end{aligned}$$

If we reduce Appell polynomial  $A_n(z)$  and Fox’s H-function to unity, S-generalized hypergeometric function into gauss hypergeometric function and  $\lambda_3 = 1, \nu_3 = 0$  in our fractional integral operators defined by (1.13) and (1.16), we easily arrive at the results which are same in essence as those obtained by Saxena and Kumbhat [15].

If we reduce Appell polynomial  $A_n(z)$  and Fox’s H-function to unity, S-generalized hypergeometric function into gauss hypergeometric function and  $\lambda_3 = 1, \nu_3 = \nu = 0$  in our fractional integral operators defined by (1.13) and (1.16), we easily arrive at the results which are same in essence as those obtained by Saigo [13].

### 3 Images

In this section we shall obtain the following images in our operators define by (1.13) and (1.16).  
 (i)

$$\begin{aligned}
 & I_x^{\nu,\lambda} \left( A_n, H, F_p; t^\rho H_{C,D;C_1,D_1;\dots;C_r,D_r}^{0,A;A_1,B_1;\dots;A_r,B_r} \left[ \begin{matrix} z^{(1)} t^{\nu^{(1)}} (x-t)^{\lambda^{(1)}} \\ \vdots \\ z^{(r)} t^{\nu^{(r)}} (x-t)^{\lambda^{(r)}} \end{matrix} \right] \right) \\
 &= \frac{\Gamma(\beta)x^\rho}{\Gamma(a)\Gamma(\alpha)B(b,c-b)} \sum_{k=0}^n \frac{a_{n-k}}{k!} z_1^k H_{C+3,D+2;C_1,D_1;\dots;C_r,D_r}^{0,B+3;A_1,B_1;\dots;A_r,B_r;M,N;1,1,1,2} \left[ \begin{matrix} z^{(1)} x^{\nu^{(1)}+\lambda^{(1)}} \\ \vdots \\ z^{(r)} x^{\nu^{(r)}+\lambda^{(r)}} \\ z_2 \\ z_3 \\ p^{-1} \end{matrix} \middle| \begin{matrix} A^* : C^* \\ B^* : D^* \end{matrix} \right] & \quad (3.1)
 \end{aligned}$$

where

$$\left. \begin{aligned}
 A^* &= (-\rho - \nu - \nu_1 k; \nu^{(1)}, \dots, \nu^{(r)}, \nu_2, \nu_3, 0), (-\lambda - \lambda_1 k; \lambda^{(1)}, \dots, \lambda^{(r)}, \lambda_2, \lambda_3, 0), \\
 &\quad (1 - b; \underbrace{0, 0, \dots, 0}_r, 0, 1, \tau), (a_j; \alpha_j^{(1)}, \dots, \alpha_j^{(r)}, 0, 0, 0)_{1,C} \\
 B^* &= (-1 - \rho - (\lambda + \nu)) - (\lambda_1 + \nu_1)k; (\lambda^{(1)} + \nu^{(1)}), \dots, (\lambda^{(r)} + \nu^{(r)}), (\lambda_2 + \nu_2), (\lambda_3 + \nu_3), 0), \\
 &\quad (1 - c; \underbrace{0, \dots, 0}_r, 0, 1, \tau + \mu), (b_j; \beta_j^{(1)}, \dots, \beta_j^{(r)}, 0, 0, 0)_{1,D} \\
 C^* &= (c_j^{(1)}, \gamma_j^{(1)})_{1,C_1}; \dots; (c_j^{(r)}, \gamma_j^{(r)})_{1,C_r}; (a_j, \alpha_j)_{1,P}; (1 - a, 1); (1, 1), (1 - c + b, \mu), (\beta, 1) \\
 D^* &= (d_j^{(1)}, \delta_j^{(1)})_{1,D_1}; \dots; (d_j^{(r)}, \delta_j^{(r)})_{1,D_r}; (b_j, \beta_j)_{1,Q}; (0, 1); (\alpha, 1)
 \end{aligned} \right\} \tag{3.2}$$

provided that conditions given by (1.15) are satisfied.

(ii)

$$\begin{aligned}
 & J_x^{\nu, \lambda} \left( A_n, H, F_p; t^\rho H_{C,D;C_1,D_1;\dots;C_r,D_r}^{0,A;A_1,B_1;\dots;A_r,B_r} \left[ \begin{matrix} z^{(1)} t^{-\nu^{(1)}} \left(1 - \frac{x}{t}\right)^{\lambda^{(1)}} \\ \cdot \\ \cdot \\ \cdot \\ z^{(r)} t^{-\nu^{(r)}} \left(1 - \frac{x}{t}\right)^{\lambda^{(r)}} \end{matrix} \right] \right) \\
 &= \frac{\Gamma(\beta) x^\rho}{\Gamma(a)\Gamma(\alpha)B(b, c - b)} \sum_{k=0}^n \frac{a_{n-k}}{k!} z_1^k H_{C+3,D+2;C_1,D_1;\dots;C_r,D_r;P,Q;1,1;3,1}^{0,B+3;A_1,B_1;\dots;A_r,B_r;M,N;1,1;1,2} \left[ \begin{matrix} z^{(1)} x^{-\nu^{(1)}} \\ \cdot \\ \cdot \\ z^{(r)} x^{-\nu^{(r)}} \\ z_2 \\ z_3 \\ p^{-1} \end{matrix} \middle| \begin{matrix} A^{**} : C^* \\ \\ \\ B^{**} : D^* \end{matrix} \right]
 \end{aligned} \tag{3.3}$$

where  $A^{**}$  and  $B^{**}$  can be obtained from  $A^*$  and  $B^*$  defined in (3.2) by replacing  $\rho$  by  $-1 - \rho$ , and provided that conditions given by (1.17) are satisfied.

(iii)

$$\begin{aligned}
 & I_x^{\nu, \lambda} \left( A_n, H, F_p; t^\rho {}_p\gamma_q \left[ \begin{matrix} (E_p; \sigma); \\ F_q; \end{matrix} z_4 \left(\frac{t}{x}\right)^{\nu_4} \left(1 - \frac{t}{x}\right)^{\lambda_4} \right] \right) \\
 &= \frac{\Gamma(\beta)}{\Gamma(a)\Gamma(\alpha)B(b, c - b)} \sum_{i=0}^\infty \sum_{k=0}^n \frac{a_{n-k}}{k!} \frac{(e_1; \sigma)_i, (e_2)_i, \dots, (e_p)_i}{(f_1)_i, (f_2)_i, \dots, (f_q)_i i!} z_1^k z_4^i x^\rho \\
 & H_{3,2;P,Q;1,1;3,1}^{0,3;M,N;1,1;1,2} \left[ \begin{matrix} z_2 \\ z_3 \\ p^{-1} \end{matrix} \middle| \begin{matrix} E^*; & (a_j, \alpha_j)_{1,P}; & (1 - a, 1); & (1, 1), (1 - c + b, \mu), (\beta, 1) \\ F^*; & (b_j, \beta_j)_{1,Q}; & (0, 1); & (\alpha, 1) \end{matrix} \right]
 \end{aligned} \tag{3.4}$$

$$E^* = (-\rho - \nu - \nu_1 k - \nu_4 i, \nu_2, \nu_3, 0), (-\lambda - \lambda_1 k - \lambda_4 i, \lambda_2, \lambda_3, 0), (1 - b; 0, 1, \tau)$$

$$F^* = (-1 - \rho - (\lambda + \nu)) - (\lambda_1 + \nu_1)k - (\lambda_4 + \nu_4)i, (\lambda_2 + \nu_2), (\lambda_3 + \nu_3), 0), (1 - c; 0, 1, \tau + \mu) \tag{3.5}$$

provided that the conditions given by (1.15) are satisfied.  
(iv)

$$\begin{aligned}
 & J_x^{\nu, \lambda} \left( A_n, H, F_p; t^\rho {}_p\gamma_q \left[ \begin{matrix} (E_p; \sigma); \\ F_q; \end{matrix} \begin{matrix} z_4 t^{-\nu_4} \left(1 - \frac{x}{t}\right)^{\lambda_4} \end{matrix} \right] \right) \\
 &= \frac{\Gamma(\beta)}{\Gamma(a)\Gamma(\alpha)B(b, c-b)} \sum_{i=0}^{\infty} \sum_{k=0}^n \frac{a_{n-k} (e_1; \sigma)_i (e_2)_i, \dots, (e_p)_i}{k! (f_1)_i (f_2)_i, \dots, (f_q)_i i!} z_1^k z_4^i x^{\rho - \nu_4 i} \\
 & H_{3,2:P,Q;1,1;3,1}^{0,3:M,N;1,1;1,2} \left[ \begin{matrix} z_2 \\ z_3 \\ p^{-1} \end{matrix} \left| \begin{matrix} E^{**}; & (a_j, \alpha_j)_{1,P}; & (1-a, 1); & (1, 1), (1-c+b, \mu), (\beta, 1) \\ F^{**}; & (b_j, \beta_j)_{1,Q}; & (0, 1); & (\alpha, 1) \end{matrix} \right. \right] \tag{3.6}
 \end{aligned}$$

where  $E^{**}$  and  $F^{**}$  can be obtained from  $E^*$  and  $F^*$  defined in (3.5) by replacing  $\rho$  by  $-1 - \rho$ , and provided that conditions given by (1.17) are satisfied.

**Proof:** To prove (3.1), first of all express the I-operator involved in its left hand side in the integral form with the help of (1.13). Then we express Appell polynomial in terms of series with the help of (1.12). Now we interchange the order of series and t-integral, express both the Fox H-function, multivariable H-function and S-Generalized Gauss’s Hypergeometric Function in terms of Mellin-Barnes type contour integral with the help of (1.1), (1.4) and (1.6) respectively. Then we interchange the order of  $\xi_i (i = 1, 2, \dots, r + 3)$ -integral and t-integral, (which is permissible under the condition stated). Finally, on evaluating the t-integral and reinterpreting the result thus obtained in terms of multivariable H-function, we easily arrive at the required result after a little simplification.

**The proof of (3.3), (3.4) and (3.6), can be obtained by proceeding on similar lines.**

Due to general nature of the H-function in several variables occur in (3.1) and (3.3), we can obtain a large number of special cases by specializing the parameters in them.

### 4 Mellin Transform, Inversion Formulas and Mellin Convolution

In this section we shall obtain the Mellin Transform, Inversion Formulas and Mellin Convolution in our operators of study.

**Theorem 4.1.** *If  $\mathfrak{M}\{[f(t)]; \mathfrak{s}\}$ ,  $\mathfrak{M}\{I_x^{\nu, \lambda}[A_n, H, F_p; f(t)]; \mathfrak{s}\}$  exist,  $\Re(1 + \lambda) > 0$ ,  $\Re(1 + \nu - s) > 0$  and the conditions of the existence of the operator  $I_x^{\nu, \lambda}[A_n, H, F_p; f(t)]$  are satisfied then*

$$\mathfrak{M}\{I_x^{\nu, \lambda}[A_n, H, F_p; f(t)]; \mathfrak{s}\} = \mathfrak{M}\{f(t); \mathfrak{s}\}G(\mathfrak{s}) \tag{4.1}$$

**Theorem 4.2.** *If  $\mathfrak{M}\{[f(t)]; \mathfrak{s}\}$ ,  $\mathfrak{M}\{J_x^{\nu, \lambda}[A_n, H, F_p; f(t)]; \mathfrak{s}\}$  exist,  $\Re(1 + \lambda) > 0$ ,  $\Re(\nu + s) > 0$  and the conditions of the existence of the operator  $J_x^{\nu, \lambda}[A_n, H, F_p; f(t)]$  are satisfied then*

$$\mathfrak{M}\{J_x^{\nu, \lambda}[A_n, H, F_p; f(t)]; \mathfrak{s}\} = \mathfrak{M}\{f(t); \mathfrak{s}\}G((1 - \mathfrak{s})) \tag{4.2}$$

where

$$G(\mathfrak{s}) = \frac{\Gamma(\beta)}{\Gamma(a)\Gamma(\alpha)B(b, c-b)} \sum_{k=0}^n a_{n-k} \frac{z_1^k}{k!}$$

$$H_{3,2:P,Q;1,1;3,1}^{0,3:M,N;1,1;1,2} \left[ \begin{array}{c} z_2 \\ z_3 \\ p^{-1} \end{array} \middle| \begin{array}{l} G^*; (a_j, \alpha_j)_{1,P}; (1-a, 1); (1, 1), (1-c+b, \mu), (\beta, 1) \\ H^*; (b_j, \beta_j)_{1,Q}; (0, 1); (\alpha, 1) \end{array} \right] \quad (4.3)$$

$$G^* = (-\lambda - \lambda_1 k; \lambda_2, \lambda_3, 0), (\mathfrak{s} - \nu - \nu_1 k; \nu_2, \nu_3, 0), (1-b; 0, 1, \tau)$$

$$H^* = (\mathfrak{s} - (\nu + \lambda) - (\nu_1 + \lambda_1)k - 1; (\nu_2 + \lambda_2), (\nu_3 + \lambda_3), 0), (1-c; 0, 1, \tau + \mu)$$

provided that conditions given by (1.15) and (1.17) are satisfied and  $\mathfrak{M}\{f(t); \mathfrak{s}\}$  stands for the well known Mellin transform of function f(t) defined by the following equation

$$\mathfrak{M}\{f(t); \mathfrak{s}\} = \int_0^\infty t^{\mathfrak{s}-1} f(t) dt \quad (4.4)$$

**Proof:** To prove Theorem 4.1 first we write the Mellin transform of the I-operator defined by (1.13) with the help of (4.4)

$$\begin{aligned} \mathfrak{M}\{I_x^{\nu,\lambda}[A_n, H, F_p; f(t)]; \mathfrak{s}\} &= \int_0^\infty x^{\mathfrak{s}-1} I_x^{\nu,\lambda}[f(t)] dx \\ &= \int_0^\infty x^{\mathfrak{s}-1} x^{-\nu-\lambda-1} \int_0^x t^\nu (x-t)^\lambda A_n \left[ z_1 \left(\frac{t}{x}\right)^{\nu_1} \left(1-\frac{t}{x}\right)^{\lambda_1} \right] \\ &\quad H_{P,Q}^{M,N} \left[ z_2 \left(\frac{t}{x}\right)^{\nu_2} \left(1-\frac{t}{x}\right)^{\lambda_2} \middle| \begin{array}{l} (a_j, \alpha_j)_{1,P} \\ (b_j, \beta_j)_{1,Q} \end{array} \right] F_p^{(\alpha,\beta;\tau,\mu)} \left[ a, b; c; z_3 \left(\frac{t}{x}\right)^{\nu_3} \left(1-\frac{t}{x}\right)^{\lambda_3} \right] f(t) dt \end{aligned}$$

Next, we change the order of x- and t-integrals. Now, we replace the Fox H-function and S-generalized hypergeometric function occurring in it in terms of Mellin Barnes Contour integral with the help of equation (1.1) and (1.6) respectively and Appell polynomial in terms of series with the help of equation (1.12) and interchange the order of summation and integration in the result thus obtained. Next we evaluate the t-integral and interpret the result in terms of multi-variable H-function and finally with the help of (4.4), we easily arrive at the desired result (4.1) after a little simplification.

**The proof of theorem 2 can be developed on similar lines.**

**Inversion Formulas**

**Formula 1**

$$f(t) = \frac{1}{2\pi i} \int_{\mathfrak{L}} \frac{t^{-\mathfrak{s}}}{G(\mathfrak{s})} \mathfrak{M}\{I_x^{\nu,\lambda}[A_n, H, F_p; f(t)]; \mathfrak{s}\} d\mathfrak{s} \quad (4.5)$$

**Formula 2**

$$f(t) = \frac{1}{2\pi i} \int_{\mathfrak{L}} \frac{t^{-\mathfrak{s}}}{G(-1-\mathfrak{s})} \mathfrak{M}\{J_x^{\nu,\lambda}[A_n, H, F_p; f(t)]; \mathfrak{s}\} d\mathfrak{s} \quad (4.6)$$

where  $G(s)$  is given by (4.3)

**Mellin Convolution**

The Mellin convolution of two functions  $f(t)$  and  $g(t)$  will be defined by

$$(f * g)(x) = (g * f)(x) = \int_0^\infty t^{-1} g\left(\frac{x}{t}\right) f(t) dt \tag{4.7}$$

provided that the integral involved in (4.7) exists.

The fractional integral operators defined by (1.13) and (1.16) can readily be expressed as Mellin convolutions. we have the following interesting results involving the Mellin convolutions :

**Result 1**

$$I_x^{\nu,\lambda}[A_n, H, F_p; f(t)] = (g * f)(x) \tag{4.8}$$

where

$$g(x) = x^{-\nu-\lambda-1}(x-1)^\lambda A_n [z_1 x^{-\nu_1-\lambda_1}(x-1)^{\lambda_1}] H_{P,Q}^{M,N} \left[ \begin{matrix} z_2 x^{-\nu_2-\lambda_2}(x-1)^{\lambda_2} \\ (a_j, \alpha_j)_{1,P} \\ (b_j, \beta_j)_{1,Q} \end{matrix} \right] F_p^{(\alpha,\beta;\tau,\mu)} [a, b; c; z_3 x^{-\nu_3-\lambda_3}(x-1)^{\lambda_3}] U(x-1) \tag{4.9}$$

**Result 2**

$$J_x^{\nu,\lambda}[A_n, H, F_p; f(t)] = (h * f)(x) \tag{4.10}$$

where

$$h(x) = x^\nu(1-x)^\lambda A_n [z_1 x^{\nu_1}(1-x)^{\lambda_1}] H_{P,Q}^{M,N} \left[ \begin{matrix} z_2 x^{\nu_2}(1-x)^{\lambda_2} \\ (a_j, \alpha_j)_{1,P} \\ (b_j, \beta_j)_{1,Q} \end{matrix} \right] F_p^{(\alpha,\beta;\tau,\mu)} [a, b; c; z_3 x^{\nu_3}(1-x)^{\lambda_3}] U(1-x) \tag{4.11}$$

$U(x)$  being the Heaviside's unit function.

**Proof:** To prove Result 1 we first write the  $I_x^{\nu,\lambda}$ -operator defined by (1.13) in the following form using the definition of Heaviside's unit function:

$$I_x^{\nu,\lambda}[f(t)] = \int_0^\infty t^{-1} \left(\frac{x}{t}\right)^{-\nu-\lambda-1} \left(\frac{x}{t}-1\right)^\lambda A_n \left[ z_1 \left(\frac{x}{t}\right)^{-\nu_1-\lambda_1} \left(\frac{x}{t}-1\right)^{\lambda_1} \right] H_{P,Q}^{M,N} \left[ \begin{matrix} z_2 \left(\frac{x}{t}\right)^{-\nu_2-\lambda_2} \left(\frac{x}{t}-1\right)^{\lambda_2} \\ (a_j, \alpha_j)_{1,P} \\ (b_j, \beta_j)_{1,Q} \end{matrix} \right] F_p^{(\alpha,\beta;\tau,\mu)} \left[ a, b; c; z_3 \left(\frac{x}{t}\right)^{-\nu_3-\lambda_3} \left(\frac{x}{t}-1\right)^{\lambda_3} \right] U\left(\frac{x}{t}-1\right) f(t) dt \tag{4.12}$$

Now making use of the equation (4.9) and the definition of the Mellin convolution given by (4.7) in the above equation, we easily arrive at the required Result 1 after a little simplification.

**The proof of Result 2 can be developed on similar lines.**

## 5 Analogue of Parseval Goldstein Theorem

If

$$\phi_1(x) = I_x^{\nu,\lambda}[A_n, H, F_p; f(t)] \tag{5.1}$$

and

$$\phi_2(x) = J_x^{\nu,\lambda}[A_n, H, F_p; f(t)] \tag{5.2}$$

then

$$\int_0^\infty \phi_1(x) f_2(x) dx = \int_0^\infty \phi_2(x) f_1(x) dx \tag{5.3}$$

provided that the integral involved in (5.1), (5.2) and (5.3) exists.

**Proof:** To prove the above theorem, we substitute the value of  $\phi_1(x)$  from (5.1) in the left hand side of (5.3) and expressing the I-operator in its integral form by using (1.13). Now interchange the order of x and t-integrals (which is permissible under given conditions) and interpret the expression thus obtained in term of J-operator with the help of (1.16), we arrive at the desired result by (5.2) after a little simplification.

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