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PARAMETER AUGMENTATION FOR BASIC HYPERGEOMETRIC SERIES BY CAUCHY OPERATOR

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Abstract. The Cauchy augmentation operator has been used to establish certain new summation and transformation formulae for basic hypergeometric series. Some interesting applications of the results have also been discussed.

1 Introduction

Chen et. al. [3] have introduced an operator defined as

$$T(a, b; D_q) = \sum_{n=0}^{\infty} \frac{(a; q)_n}{(q; q)_n} (bD_q)^n \tag{1.1}$$

and termed the operator as Cauchy augmentation operator. They have used (1.1) to obtain an extensions of the Askey-Wilson and the Askey-Roy integrals and Sear's two term summation formula. It is easy to see that this operator possess the following basic property

$$T(a, b; D_q) \frac{1}{(ct; q)_\infty} = \frac{(abt; q)_\infty}{(bt, ct; q)_\infty}, \tag{1.2}$$

where

$$D_q f(a) = \frac{f(a) - f(aq)}{a}, \tag{1.3}$$

is the q-differential operator or q-derivative [9].

The operator (1.1) is a generalization of the following operator earlier introduced by Chen and Liu [5]

$$T(bD_q) = \sum_{n=0}^{\infty} \frac{(bD_q)^n}{(q; q)_n}. \tag{1.4}$$

Using (1.4), Chen and Liu [5, 6] have derived a number of known q-identities from their special cases. Zhang and Wang [10] have used the operator (1.4) for obtaining two new operator identities and established some q-series identities. Recently, Ali and Agnihotri [1, 2] have also established some new identities of basic hypergeometric series using the operator (1.4).

In the present work, we have used Cauchy augmentation operator (1.1) to produce some interesting summation and transformation formulae for basic hypergeometric series. In what follows, we have used the following notations and definitions [7].

Let $|q| < 1$ and the q-shifted factorial be defined by

$$(a)_\infty = (a; q)_\infty = \prod_{n=0}^{\infty} (1 - aq^n). \tag{1.5}$$

Clearly,

$$(a)_n = \frac{(a)_\infty}{(aq^n)_\infty}. \tag{1.6}$$

The generalized basic hypergeometric series is defined by

$$_{r+1}\phi_r\left(\begin{array}{cccccc} a_1, & a_2, & .. & a_{r+1} & & \\ & & & & ;q, & z \\ b_1, & b_2, & .. & b_r & & \end{array}\right) = \sum_{n=0}^{\infty} \frac{(a_1)_n(a_2)_n..(a_{r+1})_n}{(q)_n(b_1)_n..(b_r)_n} z^n,$$

where $|z| < 1, |q| < 1$.

The q-binomial coefficient is defined by

$$\left(\begin{array}{c} n \\ k \end{array}\right) = \frac{(q)_n}{(q)_k(q)_{n-k}}.$$

We shall also use the following well known Heine's transformations [7]

$$\sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{(q)_n(c)_n} z^n = \frac{(b)_\infty(az)_\infty}{(c)_\infty(z)_\infty} \sum_{n=0}^{\infty} \frac{(c/b)_n(z)_n}{(q)_n(az)_n} b^n \tag{1.7}$$

$$= \frac{(c/b)_\infty(bz)_\infty}{(c)_\infty(z)_\infty} \sum_{n=0}^{\infty} \frac{(abz/c)_n(b)_n}{(q)_n(bz)_n} (c/b)^n \tag{1.8}$$

$$= \frac{(abz/c)_\infty}{(z)_\infty} \sum_{n=0}^{\infty} \frac{(c/a)_n(c/b)_n}{(q)_n(c)_n} (abz/c)^n, \tag{1.9}$$

Jackson's transformation [7]

$$\sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{(q)_n(c)_n} z^n = \frac{(abz/c)_\infty}{(bz/c)_\infty} \sum_{n=0}^{\infty} \frac{(a)_n(c/b)_n(0)_n}{(q)_n(c)_n(cq/bz)_n} q^n$$
$$+ \frac{(a, bz, c/b)_\infty}{(c, z, c/bz)_\infty} \sum_{n=0}^{\infty} \frac{(z)_n(abz/c)_n(0)_n}{(q)_n(bz)_n(bzq/c)_n} q^n, \tag{1.10}$$

the transformation of $_3\phi_2$ series

$$_3\phi_2\left(\begin{array}{ccccc} a, & b, & c & & \\ & & & ;q, & \frac{de}{abc} \\ & d, & e & & \end{array}\right)$$
$$= \frac{(e/a, de/bc)_\infty}{(e, de/abc)_\infty}\, {}_3\phi_2\left(\begin{array}{ccccc} a, & d/b, & d/c & & \\ & & & ;q, & \frac{e}{a} \\ & d, & de/bc & & \end{array}\right) \tag{1.11}$$

and the q-Gauss sum

$$_2\phi_1\left(\begin{array}{cccc} a, & b & & \\ & & ;q, & \frac{c}{ab} \\ & c & & \end{array}\right) = \frac{(c/a, c/b)_\infty}{(c, c/ab)_\infty}. \tag{1.12}$$

2 Main Results

Theorem 2.1. *We have*

$$
{}_3\phi_2\left(\begin{matrix} a, & b, & e \\ & & & ;q, & aq \\ & c, & de \end{matrix}\right)
= \frac{(b,e,a^2q,deq)_\infty}{(c,aq,eq,de)_\infty}\,{}_3\phi_2\left(\begin{matrix} c/b, & aq, & eq \\ & & & ;q, & b \\ & a^2q, & deq \end{matrix}\right). \tag{2.1}
$$

Proof. Taking $z = aq$ in (1.7), we obtain

$$
\sum_{n=0}^{\infty}\frac{(a)_\infty(b)_n}{(q)_n(c)_n(aq^n)_\infty}(aq)^n = \frac{(b,a^2q)_\infty}{(aq,c)_\infty}\sum_{n=0}^{\infty}\frac{(c/b)_n(aq)_\infty}{(q)_n(a^2q)_n(aq^{n+1})_\infty}b^n.
$$

Applying the operator $T(d,e;D_q)$ on both sides with respect to a and using (1.2), after some simplification we obtain (2.1). □

Theorem 2.2. *We have*

$$
{}_3\phi_2\left(\begin{matrix} a, & b, & e \\ & & & ;q, & z \\ & aq, & de \end{matrix}\right)
= \frac{(bz/q,e,deq/b)_\infty}{(z,eq/b,de)_\infty}\,{}_3\phi_2\left(\begin{matrix} q, & aq/b, & eq/b \\ & & & ;q, & bz/q \\ & aq, & deq/b \end{matrix}\right). \tag{2.2}
$$

$$
\begin{aligned}
&= \frac{(e,bz/q,deq^2/bz)_\infty}{(de,bz/aq,eq^2/bz)_\infty}\,{}_3\phi_2\left(\begin{matrix} a, & aq/b, & 0 \\ & & & ;q, & q \\ & aq, & aq^2/bz \end{matrix}\right) \\
&\quad + \frac{(1-a)(1-e)(bz,aq/b)_\infty}{(1-de)(z,aq/bz)_\infty}\,{}_3\phi_2\left(\begin{matrix} z, & bz/q, & 0 \\ & & & ;q, & q \\ & bz, & bz/a \end{matrix}\right).
\end{aligned} \tag{2.3}
$$

Proof. Putting $c = aq$ in (1.9), we obtain

$$
\sum_{n=0}^{\infty}\frac{(b)_n}{(q)_n(aq)_n(aq^n)_\infty}z^n = \frac{(bz/q,aq/b)_\infty}{(a)_\infty(z)_\infty}\sum_{n=0}^{\infty}\frac{(q)_n}{(q)_n(aq)_n(aq^{n+1}/b)_\infty}(bz/q)^n.
$$

Applying the operator $T(d,e;D_q)$ on both sides with respect to a and using (1.2), we obtain (2.2).

Again, taking $c = aq$ in (1.10), we get

$$
\begin{aligned}
\sum_{n=0}^{\infty}\frac{(a)_\infty(b)_n z^n}{(q)_n(aq)_n(aq^n)_\infty} &= \frac{(bz/q)_\infty}{(bz/aq)_\infty}\sum_{n=0}^{\infty}\frac{(a)_n(0)_n(aq/b)_n(aq^{n+2}/bz)_\infty q^n}{(q)_n(aq)_n(aq^2/bz)_\infty} \\
&\quad + \frac{(a,bz,aq/b)_\infty}{(z,aq/bz)_\infty}\sum_{n=0}^{\infty}\frac{(z)_n(0)_n(bz/q)_n q^n}{(q)_n(bz)_n(bz/a)_n(aq)_\infty}.
\end{aligned}
$$

Applying the operator $T(d,e;D_q)$ on both sides with respect to a and using (1.2), we obtain (2.3). □

Theorem 2.3. *We have*

$$
{}_3\phi_2\left(\begin{matrix} a, & b, & e \\ & c, & de \end{matrix}; q, \quad z\right)
= \frac{(c/b, bz, adez/c, e)_\infty}{(c, de, aez/c, z)_\infty}\, {}_3\phi_2\left(\begin{matrix} b, & abz/c, & aez/c \\ & bz, & adez/c \end{matrix}; q, \quad c/b\right). \tag{2.4}
$$

Proof. Equation (1.8) can be written as

$$
\sum_{n=0}^{\infty} \frac{(a)_n (b)_\infty z^n}{(q)_n (c)_n} \{\frac{1}{(bq^n)_\infty}\} = \frac{(c/b, bz)_\infty}{(c, z)_\infty} \sum_{n=0}^{\infty} \frac{(b)_n (abz/c)_\infty (c/b)^n}{(q)_n (bz)_n} \{\frac{1}{(abzq^n/c)_\infty}\}.
$$

Applying the operator $T(d, e; D_q)$ on both sides with respect to b and using (1.2), after simple manipulation we obtain (2.4). □

Theorem 2.4. *We have*

$$
{}_4\phi_3\left(\begin{matrix} a, & b, & c, & g \\ & aq, & e, & fg \end{matrix}; q, \quad eq/bc\right)
= \frac{(e/a, aeq/bc, g, fgq/c)_\infty}{(e, eq/bc, fg, gq/c)_\infty}\, {}_4\phi_3\left(\begin{matrix} a, & aq/b, & aq/c, & gq/c \\ & aq, & aeq/bc, & fgq/c \end{matrix}; q, \quad e/a\right). \tag{2.5}
$$

Proof. Taking $d = aq$ in (1.11) and using (1.6), we have

$$
(a)_\infty \sum_{n=0}^{\infty} \frac{(b)_n (c)_n}{(q)_n (aq)_n (e)_n (aq^n)_\infty} (\frac{qe}{bc})^n
= \frac{(e/a, aqe/bc)_\infty}{(e, eq/bc)_\infty} \sum_{n=0}^{\infty} \frac{(a)_n (aq/b)_n (aq/c)_\infty}{(q)_n (aq)_n (aqe/bc)_n (aq^{n+1}/c)_\infty} (\frac{e}{a})^n.
$$

Applying the operator $T(f, g; D_q)$ and using (1.2), we obtain (2.5). □

Theorem 2.5. *We have*

$$
{}_3\phi_2\left(\begin{matrix} a, & b, & e \\ & aq, & de \end{matrix}; q, \quad q/b\right) = \frac{(q, aq/b, deq, e)_\infty}{(eq, aq, q/b, de)_\infty}. \tag{2.6}
$$

Proof. Taking $c = aq$ in (1.12) and then using the operator $T(d, e; D_q)$ on both sides along with the use of (1.2), we obtain (2.6).

□

3 Some Special Cases

I. If $|e/ab| < 1$, then for $c = aq$ in (2.5), we get

$$
{}_3\phi_2\left(\begin{matrix} a, & b, & g \\ & e, & fg \end{matrix}; q, \quad e/ab\right) = \frac{(e/a, e/b, g, fg/a)_\infty}{(e, e/ab, fg, g/a)_\infty}. \tag{3.1}
$$

If we take $f \to 1$ in (3.1), we get the following identity which is the well-known q-analogue of the Gauss summation formula

$$\sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{(q)_n(e)_n}(e/ab)^n = \frac{(e/a, e/b)_\infty}{(e, e/ab)_\infty}. \tag{3.2}$$

II. For $|z| < |q| < 1$, with $a = 0$ and then taking $b \to 1$ in (2.2), we obtain

$$\sum_{n=0}^{\infty} \frac{(eq)_n}{(deq)_n}(z/q)^n = \frac{q(1-de)}{(1-e)(q-z)}. \tag{3.3}$$

III. If we take $b = c$ in (2.1), we get the following summation formula

$${}_2\phi_1\left(\begin{matrix} a, & e \\ & & ;q, & aq \\ & de \end{matrix}\right) = \frac{(1-e)(a^2q)_\infty}{(1-de)(aq)_\infty}. \tag{3.4}$$

which on taking $e = 0$ and then $aq = z$ gives the q-binomial theorem.

IV. Taking $b \to \infty$ in (2.6), we get

$$\sum_{n=0}^{\infty} \frac{(a)_n(e)_n(-)^n q^{n(n+1)/2}}{(q)_n(aq)_n(de)_n} = \frac{(1-e)(q)_\infty}{(1-de)(aq)_\infty}$$

which on putting $a = e = 0$ gives

$$\sum_{n=0}^{\infty} \frac{(-)^n q^{n(n+1)/2}}{(q)_n} = (q)_\infty. \tag{3.5}$$

V. Taking $a = de$ in (2.6), we obtain

$${}_2\phi_1\left(\begin{matrix} b, & e \\ & & ;q, & q/b \\ & deq \end{matrix}\right) = \frac{(1-e)(q, deq/b)_\infty}{(q/b, de)_\infty}. \tag{3.6}$$

VI. Taking $e = aq$ in (2.6), we obtain

$${}_2\phi_1\left(\begin{matrix} a, & b \\ & & ;q, & q/b \\ & daq \end{matrix}\right) = \frac{(1-aq)(q, aq/b)_\infty}{(1-daq)(aq, q/b)_\infty}. \tag{3.7}$$

which on taking $b \to \infty$ gives

$$\sum_{n=0}^{\infty} \frac{(a)_n(-1)^n q^{n(n+1)/2}}{(q)_n(daq)_n} = \frac{(1-aq)(q)_\infty}{(1-daq)(aq)_\infty}. \tag{3.8}$$

If we take $d \to 1$, we obtain

$$\sum_{n=0}^{\infty} \frac{(a)_n(-1)^n q^{n(n+1)/2}}{(q)_n(aq)_n} = \frac{(q)_\infty}{(aq)_\infty}. \tag{3.9}$$

which is the well known sum of ${}_1\phi_1$ series [7, p. 354, Eq.II.5].

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