SOME NEW MODULAR EQUATIONS OF RATIO’S OF RAMANUJAN QUANTITY

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Abstract. Recently, In [5],[6] Nikos Bagis defined Ramanujan Quantities $R(a, b, p; q)$ as

$$R(a, b, p; q) = q^{-(a-b)/2+(a^2-b^2)/(2p)} \prod_{n=0}^{\infty} \frac{(1 - q^n p^n)(1 - q^p - q^n p^n)}{(1 - q^n b^n)(1 - q^b - q^n b^n)},$$ (0.1)

where $a$, $b$ and $p$ are positive rationals such that $a + b < p$. In this paper, we establish some modular equations of ratios for Ramanujan quantity $R(q^n) := R(1, 2, 4; q^n)$ for $n = 2, 3, 4, 5, 7, 9$ and some of their evaluations.

1 Introduction

In Chapter 16 of his second notebook [1], Ramanujan develops the theory of theta-function and is defined by

$$f(a, b) := \sum_{n=-\infty}^{\infty} a^{n(n+1)/2} b^{-(n-1)/2} |ab| < 1,$$ (1.1)

$$= (-a; ab)_{\infty}(-b; ab)_{\infty}(ab; ab)_{\infty}$$

where $(a; q)_{\infty} = 1$ and $(a; q)_{\infty} = (1 - a)(1 - aq)(1 - aq^2)\cdots$.

Following Ramanujan, we define

$$\varphi(q) := f(q, q) = \sum_{n=-\infty}^{\infty} q^{n^2} = \frac{(q^2; q)_{\infty}}{(q; q)_{\infty}},$$ (1.2)

$$\psi(q) := f(q, q^2) = \sum_{n=0}^{\infty} q^{n(n+1)/2} = \frac{(q^3, q^3; q)_{\infty}}{(q, q^2)_{\infty}},$$ (1.3)

$f(-q) := f(-q, -q^2) = \sum_{n=-\infty}^{\infty} (-1)^n q^{n(n+1)/2} = (q; q)_{\infty}$ (1.4)

and

$$\chi(q) := (-q; q^2)_{\infty}.$$ (1.5)

Now we define a modular equation in brief. The ordinary hypergeometric series

$$_2 F_1(a, b; c; x)$$

is defined by

$$-_2 F_1(a, b; c; x) := \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n n!} x^n,$$
where \((a)_0 = 1, (a)_n = a(a + 1)(a + 2) \cdots (a + n - 1)\) for any positive integer \(n\), and \(|x| < 1\).

Let

\[ z := z(x) := 2F1 \left( \frac{1}{2}, 1; \frac{1}{2}; x \right) \quad (1.6) \]

and

\[ q := q(x) := \exp \left( -\pi \frac{2F1 \left( \frac{1}{2}, 1; \frac{1}{2}; 1 - x \right)}{2F1 \left( \frac{1}{2}, \frac{1}{2}; \frac{1}{2}; 1 \right)} \right), \quad (1.7) \]

where \(0 < x < 1\).

Let \(r\) denote a fixed natural number and assume that the following relation holds:

\[
\frac{r}{2} \frac{2F1 \left( \frac{1}{2}, 1; 1 - \alpha \right)}{2F1 \left( \frac{1}{2}, \frac{1}{2}; \frac{1}{2}; 1 \right)} = \frac{2F1 \left( \frac{1}{2}, 1; 1 - \beta \right)}{2F1 \left( \frac{1}{2}, \frac{1}{2}; \frac{1}{2}; 1 \right)}. \quad (1.8)
\]

Then a modular equation of degree \(r\) in the classical theory is a relation between \(\alpha\) and \(\beta\) induced by \((1.8)\). We often say that \(\beta\) is of degree \(r\) over \(\alpha\) and \(m := \frac{z(\alpha)}{z(\beta)}\) is called the multiplier. We also use the notations \(z_1 := z(\alpha)\) and \(z_r := z(\beta)\) to indicate that \(\beta\) has degree \(r\) over \(\alpha\).

Using Ramanujan continued fraction \([1]\), Nikos Bagis \([5,6]\) deduced the following result:

\[
\frac{q^{B-A}}{1 - a_1 b_1} + \frac{(a_1 - b_1 q_1)(b_1 - a_1 q_1)}{(1 - a_1 b_1)(q_1^2 + 1)} + \frac{(a_1 - b_1 q_1^2)(b_1 - a_1 q_1^2)}{(1 - a_1 b_1)(q_1^2 + 1)} + \ldots
\]

\[
= \prod_{n=0}^{\infty} \left( 1 - q^n q^{np} \right) \left( 1 - q^n q^{np} \right) \quad (1.9)
\]

where \(a_1 = q^A, b_1 = q^B, q_1 = q^{A+B}, a = 2A + 3p/4, 2B + p/4, \) and \(p = 4(A + B), |q| < 1\).

In this paper, we establish several new modular relations between \(\frac{R(-q)}{R(q)}\) and \(\frac{R(-q^n)}{R(q^n)}\) for \(n = 2, 3, 4, 5, 7, 9\) and values of \(\frac{R(-q)}{R(q)}\) for \(q = e^{-\pi}, e^{-2\pi}\).

2 Preliminary results

Definition 2.1. \([6]\)

\([a, p; q] = (q^a; q)^\infty(q^a; q^p)^\infty \quad (2.1)\]

where \(q = e^{-\pi\sqrt{7}}\) and \(a, p, r > 0\).

Definition 2.2. \([6]\)

\[R(a, b; p; q) := q^{-(a-b)/2+(a^2-b^2)/(2p)} \frac{[a, p; q]}{[b, p; q]}. \quad (2.2)\]

Lemma 2.3. \([1, Ch. 17, Entry 10-11, p.122-123]\)

\[\psi(q) = \sqrt{\frac{\Gamma}{2}} z(\alpha q^{-1})^{1/8} \quad (2.3)\]

\[\psi(-q) = \sqrt{\frac{\Gamma}{2}} z(\alpha(1 - \alpha)q^{-1})^{1/8} \quad (2.4)\]

where \(q = e^{-\nu}\).

Lemma 2.4. \([4, Entry 17.3.1, p.385]\) If \(\beta\) is of degree 2 over \(\alpha\), then

\[(1 - \sqrt{1 - \alpha})(1 - \sqrt{\beta}) = 2\sqrt{\beta(1 - \alpha)}. \quad (2.5)\]

Lemma 2.5. \([1, Entry 5(ii), p.230]\) If \(\beta\) has degree 3 over \(\alpha\), then

\[(\alpha \beta)^{1/4} + \{(1 - \alpha)(1 - \beta)\}^{1/4} = 1. \quad (2.6)\]
Lemma 2.6. [4, Entry 17.3.2, p.385] If $\beta$ has degree 4 over $\alpha$, then
\[
(1 - \sqrt{1 - \alpha})(1 - \sqrt{\beta}) = 2\sqrt{\beta(1 - \alpha)}.
\] (2.7)

Lemma 2.7. [1, Entry 13(i), p.280] If $\beta$ has degree 5 over $\alpha$, then
\[
(\alpha \beta)^{1/2} + \{(1 - \alpha)(1 - \beta)\}^{1/2} + 2\{16\alpha \beta(1 - \alpha)(1 - \beta)\}^{1/6} = 1.
\] (2.8)

Lemma 2.8. [1, Entry 19(i), p.314] If $\beta$ has degree 7 over $\alpha$, then
\[
(\alpha \beta)^{1/8} + \{(1 - \alpha)(1 - \beta)\}^{1/8} = 1.
\] (2.9)

Lemma 2.9. [1, Entry 3(x)(xi), p.352] If $\beta$ has degree 9 over $\alpha$, then
\[
\left(\frac{\beta}{\alpha}\right)^{1/8} + \left(\frac{1 - \beta}{1 - \alpha}\right)^{1/8} - \left(\frac{\beta(1 - \beta)}{\alpha(1 - \alpha)}\right)^{1/8} = \sqrt{m}.
\] (2.10)
\[
\left(\frac{\alpha}{\beta}\right)^{1/8} + \left(\frac{1 - \alpha}{1 - \beta}\right)^{1/8} - \left(\frac{\alpha(1 - \alpha)}{\beta(1 - \beta)}\right)^{1/8} = \frac{3}{\sqrt{m}}.
\] (2.11)

Lemma 2.10. [1, Entry 24(i), p.39]
\[
\frac{\psi(-q)}{\varphi(q)} = \sqrt{\frac{\varphi(-q)}{\varphi(q)}}
\] (2.12)

3 Modular relations of Ratio’s of Ramanujan Quantities of $R(q)$

In this section, we obtain certain modular relations between $\frac{a}{b} := \frac{R(-q)}{R(q)}$ and $\frac{c}{d} := \frac{R(-q^c)}{R(q^d)}$.

Theorem 3.1. If $\frac{a}{b} := \frac{R(-q)}{R(q)}$ and $\frac{c}{d} := \frac{R(-q^c)}{R(q^d)}$ then
\[
\left(\frac{ac}{bd}\right)^2 + \left(\frac{bc}{ad}\right)^2 = 2\left(\frac{d}{c}\right)^2.
\] (3.1)

Proof. Putting $a = 1, b = 2, p = 4$ in (0.1), we obtain
\[
R(q) := R(1, 2, 4; q) = \frac{(q^2; q^4)_{\infty}(q^2; q^4)_{\infty}}{(q^4; q^4)_{\infty}(q^2; q^4)_{\infty}}.
\] (3.2)

Using the equations (1.1), (1.2) and (1.3), the above equation can be written as
\[
R(q) = \frac{f(-q^2, -q^3)}{f(-q, -q^3)} = \frac{\varphi(-q^2)}{\psi(-q)}.
\] (3.3)

Replacing $q$ by $-q$ in the above equation, we obtain
\[
R(-q) = \frac{f(-q^2, -q^3)}{f(q, q^3)} = \frac{\varphi(-q^2)}{\psi(q)}.
\] (3.4)

Dividing (3.4) by (3.3), we obtain
\[
\frac{R(-q)}{R(q)} = \frac{\psi(-q)}{\psi(q)}.
\] (3.5)

Employing the equations (2.3) and (2.4), we obtain
\[
\frac{\psi(-q)}{\psi(q)} = (1 - \alpha)^{1/8}.
\] (3.6)
Dividing (3.4) by (3.3) and then using (3.6), we obtain

\[(1 - a)^{1/8} = \frac{R(-q)}{R(q)}.\]  

(3.7)

From the equation (3.7) and lemma (2.4), we obtain

\[(b^6 c^6 + 2b^4 d^2 a^2 + c^6 d^4)(b^6 c^6 - 2b^4 d^2 a^2 + c^6 d^4) = 0.\]  

(3.8)

By examining the behavior of the above factors near \(q = 0\), we can find a neighborhood about the origin, where the second factor is zero; whereas first factor are not zero in this neighborhood. By the Identity Theorem second factor vanishes identically. This completes the proof. □

**Theorem 3.2.** If \(\frac{a}{b} := \frac{R(-q)}{R(q)}\) and \(\frac{c}{d} := \frac{R(-q^d)}{R(q^d)}\), then

\[
\left(\frac{ad}{bc}\right)^2 + 2\frac{bd}{ac} = \left(\frac{bc}{ad}\right)^2 + 2\frac{ac}{bd}.
\]

(3.9)

**Proof.** From the equation (3.7) and lemma (2.5), we obtain

\[
(d^4 a^4 + 2d^2 c^2 b^3 a - c^2 b^4)(d^4 a^4 - 2d^2 c^2 b^3 a + c^2 b^4) = 0.
\]

(3.10)

By examining the behavior of the above factors near \(q = 0\), we can find a neighborhood about the origin, where the second factor is zero; whereas first factor are not zero in this neighborhood. By the Identity Theorem second factor vanishes identically. This completes the proof. □

**Theorem 3.3.** If \(\frac{a}{b} := \frac{R(-q)}{R(q)}\) and \(\frac{c}{d} := \frac{R(-q^d)}{R(q^d)}\), then

\[
\left(\frac{ac}{bd}\right)^4 + \left(\frac{bc}{ad}\right)^4 + \frac{c^4}{d^4} \left(4 \left[\frac{a^2}{b^2} + \frac{b^2}{a^2}\right] + 6\right) = 8\frac{d^4}{c^4} \left(\frac{a^2}{b^2} + \frac{b^2}{a^2}\right).
\]

(3.11)

**Proof.** From the equation (3.7) and lemma (2.6), we obtain(3.11). □

**Theorem 3.4.** If \(\frac{a}{b} := \frac{R(-q)}{R(q)}\) and \(\frac{c}{d} := \frac{R(-q^d)}{R(q^d)}\), then

\[
\left(\frac{ad}{bc}\right)^3 + 4 \left(\frac{bd}{ac}\right)^2 + 5\frac{ad}{bc} = \left(\frac{bc}{ad}\right)^3 + 4 \left(\frac{ac}{bd}\right)^2 + 5\frac{bc}{ad}.
\]

(3.12)

**Proof.** From the equation (3.7) and lemma (2.9), we obtain

\[
(-5b^4 c^4 a^2 - 4d^5 b^5 c a + d^6 a^6 + 4bd^5 c^5 a^5 + 5d^4 b^2 c^2 a^4 - b^6 c^6)
\]

\[
(\text{Simplify terms as given in the text})
\]

\[
(15b^4 d^6 c^8 - 10b^{10} d^4 c^{10} a^2 + 15b^4 d^6 c^8 a^4 + 20b^4 d^6 c^6 a^6 + b^4 d^6 a^6 + d^{12} a^{12} + 16b^6 d^6 a^{10} + 16b^4 d^6 c^6 a^2 - 10d^{10} b^2 c^4 a^4)(b^4 d^4 c^{24})
\]

\[
+ 58b^{20} d^4 c^{20} a^4 - 320b^{20} d^2 c^{12} a^4 + 256a^4 c^4 b^4 a^4 + 1423 b^{16} d^8 c^6 a^4 - 1048 b^{16} d^8 a^{12} c^{12} a^{12} + 1423 b^{16} d^8 c^6 a^4 + 1423 b^{16} a^{12} c^{12} a^{12} + 58b^8 d^4 c^2 a^{20} - 320b^8 d^4 c^2 a^{20}
\]

By examining the behavior of the above factors near \(q = 0\), we can find a neighborhood about the origin, where the second factor is zero; whereas other factors are not zero in this neighborhood. By the Identity Theorem second factor vanishes identically. This completes the proof. □
Theorem 3.5. If $\frac{a}{b} := \frac{R(-q)}{R(q)}$ and $\frac{c}{d} := \frac{R(-q^d)}{R(q^d)}$ then
\[
\left( \frac{ad}{bc} \right)^4 + \left( \frac{bc}{ad} \right)^4 + 28 \left( \frac{ac}{bd} \right)^2 + \left( \frac{bd}{ac} \right)^2 + 70
\]
\[
= 8 \left[ \left( \frac{ac}{bd} \right)^3 + \left( \frac{bd}{ac} \right)^3 \right] + 56 \left[ \left( \frac{ac}{bd} \right) + \left( \frac{bd}{ac} \right) \right].
\]
(3.14)

Proof. From the equation (3.7) and lemma (2.8), we obtain (3.14). □

Theorem 3.6. If $\frac{a}{b} := \frac{R(-q)}{R(q)}$ and $\frac{c}{d} := \frac{R(-q^n)}{R(q^n)}$ then
\[
\left( \frac{ad}{bc} \right)^8 + \left( \frac{bc}{ad} \right)^6 + 8 \left[ \left( \frac{ad}{bc} \right)^5 + \left( \frac{bc}{ad} \right)^5 \right] + 10 \left[ \left( \frac{ad}{bc} \right)^4 + \left( \frac{bc}{ad} \right)^4 \right]
\]
\[
+ 16 \left[ \left( \frac{ac}{bd} \right)^4 + \left( \frac{bd}{ac} \right)^4 \right] + 15 \left[ \left( \frac{ad}{bc} \right)^2 + \left( \frac{bc}{ad} \right)^2 \right] + 48 \left[ \left( \frac{ad}{bc} \right) + \left( \frac{bc}{ad} \right) \right]
\]
\[
= 24 \left[ \left( \frac{ad}{bc} \right)^3 + \left( \frac{bc}{ad} \right)^3 \right] + 16 \left( \frac{bd}{ac} \right)^3 \left[ \frac{b^2}{a^2} + \frac{d^2}{c^2} \right] + 16 \left( \frac{ac}{bd} \right)^3 \left[ \frac{a^2}{b^2} + \frac{c^2}{d^2} \right] + 84.
\]
(3.15)

Proof. From the equation (3.7) and lemma (2.9), we obtain (3.15). □

Remark 3.7. Similarly, we obtain the Modular relations between $\frac{a}{b} := \frac{R(-q)}{R(q)}$ and $\frac{c}{d} := \frac{R(-q^n)}{R(q^n)}$, for $n = 8, 11, 13, 15, 17, 19, 23, 25$.

4 Explicit values of Ratio’s of Ramanujan Quantities of $R(q)$

In this section, we obtain explicit values of Ratio’s of Ramanujan Quantities of $R(q)$.

Theorem 4.1. We have
\[
\frac{R(-e^{-n\pi})}{R(e^{-n\pi})} = \frac{\psi(-e^{-n\pi})}{\psi(e^{-n\pi})}
\]
(4.1)

Proof. Put $q = e^{-n\pi}$ in equation (3.5), we get (4.1) □

In his first notebook, Ramanujan’s second notebook [7] recorded many elementary values of $\varphi(q)$. In particularly, he recorded $\varphi(e^{-n\pi})$ and $\varphi(-e^{-n\pi})$ for $n = 1, 2, 4$ and etc. Noting from [[3, Entry 1, p.325], we have
\[
\varphi(e^{-\pi}) = \frac{\pi^{1/4}}{\Gamma(3/4)} \quad \text{and} \quad \varphi(-e^{-\pi}) = \frac{\pi^{1/4}}{\Gamma(3/4)} 2^{-1/4}.
\]
(4.2)

Employing the equations (4.1), (4.2) and (2.12), we obtain
\[
\frac{R(-e^{-\pi})}{R(e^{-\pi})} = 2^{-1/8}.
\]
(4.3)

Using the above value in Theorem (3.1), we get
\[
\frac{R(-e^{-2\pi})}{R(e^{-2\pi})} = 2^{5/16} [\sqrt{2} - 1]^{1/4}.
\]
(4.4)
References


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