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A NEW SUBCLASS OF MEROMORPHICALLY UNIFORMLY CONVEX FUNCTIONS WITH POSITIVE COEFFICIENTS

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Abstract. In this paper, we introduce and study a new subclass of meromorphically uniformly convex functions with positive coefficients defined by a differential operator and obtain coefficient estimates ,growth and distortion theorem, radius of convexity,integral transforms, convex linear combinations, convolution properties and δ - neighborhoods for the class $\sigma_\rho(\alpha, \beta)$.

1 Introduction

Let **A** denote the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1.1}$$

which are analytic in the open unit disk E ={$z \in C : |z| < 1$} and satisfy the following usual normalization condition $f(0) = f'(0) - 1 = 0$. We denote by S the subclass of **A** consisting of functions $f(z)$ which are all univalent in E. A function $f \in$ **A** is a starlike function by the order α, $0 \leq \alpha < 1$ if it satisfy

$$Re\left\{\frac{zf'(z)}{f(z)}\right\} > \alpha(z \in E). \tag{1.2}$$

We denote this class with$S^*(\alpha)$.
A function $f \in$ **A** is a convex function by the order α, $0 \leq \alpha < 1$ if it satisfy

$$Re\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > \alpha(z \in E). \tag{1.3}$$

We denote this class with $K(\alpha)$.
Let T denote the class of functions analytic in E that are of the form

$$f(z) = z - \sum_{n=2}^{\infty} a_n z^n, a_n \geq 0(z \in E) \tag{1.4}$$

and let $T^*(\alpha) = T \bigcap S^*(\alpha)$, $C(\alpha) = T \bigcap K(\alpha)$. The class $T^*(\alpha)$ and allied classes possess some interesting properties and have been extensively studied by Silverman[17] and others.

A function $f \in$ **A** is said to in the class of uniformly convex functions of order γ and type β, denoted by $UCV(\beta, \gamma)$, if

$$Re\left\{1 + \frac{zf''(z)}{f'(z)} - \gamma\right\} > \beta\left|\frac{zf''(z)}{f'(z)}\right|, \tag{1.5}$$

where $\beta \geq 0$, $\gamma \in [-1, 1)$ and $\beta + \gamma \geq 0$, and is said to be in the class corresponding class denoted by $SP(\beta, \gamma)$ if

$$Re\left\{\frac{zf'(z)}{f(z)} - \gamma\right\} > \beta\left|\frac{zf'(z)}{f(z)} - 1\right|, \tag{1.6}$$

where $\beta \geq 0$, $\gamma \in [-1, 1)$ and $\beta + \gamma \geq 0$.
Indeed it follows from (1.5) and (1.6) that

$$f \in UCV(\gamma, \beta) \Leftrightarrow zf' \in SP(\gamma, \beta). \tag{1.7}$$

For $\beta = 0$ we get respectively, the classes $K(\gamma)$ and $S^*(\gamma)$.The function of the class $UCV(1, 0) \equiv UCV$ are called uniformly convex functions and were introduced by Goodman with geometric interpretation in [5]. The class $SP(1, 0) \equiv SP$ is defined by Ronning [13].

The classes $UCV(1, \gamma) \equiv UCV(\gamma)$ and $SP(1, \gamma) \equiv SP(\gamma)$ are investgated by Ronning in [12]. For $\gamma = 0$, the classes $UCV(\beta, 0) \equiv \beta - UCV$ and $SP(\beta, 0) \equiv \beta - SP$ are defined respectively, by Kanas and Wisniowska in [8] and [9].

Further Ahuja et al [1], Bharathi et al [2], Murugusundaramurthy and Magesh [10] and others have studied and investigated interesting properties for the classes $UCV(\beta, \gamma)$ and $SP(\beta, \gamma)$.
Let $\sum$ denote the class the class of functions of the form

$$f(z) = \frac{1}{z} + \sum_{m=1}^{\infty} a_m z^m \tag{1.8}$$

which are regular in domain E = {z :0 $<$ |z|$<$ 1} with a simple pole at the origin with residue 1 there.
Let $\sum_s$, $\sum^*(\alpha)$ and $\sum_k(\alpha)$ $(0 \leq \alpha < 1)$ denote the subclasses of $\sum$ that are univalent, meromorphically starlike of order α and meromorphically convex of order α respectively. Analytically f(z) of the form (1.8) is in $\sum^*(\alpha)$ if and only if

$$Re\left\{-\frac{zf'(z)}{f(z)}\right\} > \alpha, z \in E \tag{1.9}$$

Similarly, $f \in \sum_k(\alpha)$ if and only if, f(z) is of the form (1.8) and satisfies

$$Re\left\{-\left(1 + \frac{zf''(z)}{f'(z)}\right)\right\} > \alpha, z \in E \tag{1.10}$$

It being understood that if α = 1 then $f(z) = \frac{1}{z}$ is the only function which is $\sum^*(1)$ and $\sum_k(1)$.
The classes $\sum^*(\alpha)$ and $\sum_k(\alpha)$ have been extensively studied by Pommerenke [11], Clunie [3], Royster [15] and others.

Since, to a certain extent the work in the meromorphic univalent case has paralleled that of regular univalent case, it is natural to search for a subclass of $\sum_s$ that has properties analogous to those of $T^*(\alpha)$. Juneja and Reddy [7] introduced the class $\sum_p$of functions of the form

$$f(z) = \frac{1}{z} + \sum_{m=1}^{\infty} a_m z^m, \quad a_m \geq 0, \tag{1.11}$$

$$\sum_p^*(\alpha) = \sum_p \bigcap \sum^*(\alpha).$$

For functions $f(z)$ in the class $\sum_p$, we define a linear operator D^n by the following form

$$D^0 f(z) = f(z)$$

$$D^1 f(z) = \frac{1}{z} + 3a_1 z + 4a_2 z^2 + ... = \frac{(z^2 f(z))'}{z}$$

$$D^2 f(z) = D(D' f(z))$$

and for n=1,2,3,. . . ..

$$D^n f(z) = D(D^{n-1} f(z)) = \frac{1}{z} + \sum_{m=1}^{\infty} (m+2)^n a_m z^m = \frac{(z^2 D^{n-1} f(z))'}{z} \tag{1.12}$$

Now, we define a new subclass $\sigma_p(\alpha, \beta)$ of $\sum_p$.

Definition 1.1. For $-1 \leq \alpha < 1$, and $\beta \geq 1$, we let $\sigma_p(\alpha, \beta)$ be the subclass of $\sum_p$ consisting of functions of the form (1.11) and satisfying the analytic criterion

$$Re\left\{\frac{D^{n+1} f(z)}{D^n f(z)} - \alpha\right\} > \beta \left|\frac{D^{n+1} f(z)}{D^n f(z)} - 1\right| \tag{1.13}$$

$D^n f(z)$ is given by (1.12).

The main object of the paper is to study some usual properties of the geometric function theory such as coefficient bounds ,growth and distortion properties ,radius of convexity ,convex linear combination and convolution properties, integral operators and δ- neighbourhoods for the class $\sigma_p(\alpha, \beta)$.

2 Coefficient Inequality

Theorem 2.1. A function $f(z)$ of the form (1.11) is in $\sigma_p(\alpha, \beta)$ if
$\sum_{m=1}^{\infty} (m+2)^n \left[(1+\beta)(m+1) + 1 - \alpha\right] |a_m| \leq (1-\alpha)$,$-1 \leq \alpha < 1$ and $\beta \geq 1$.

Proof: It suffices to show that

$$\beta \left|\frac{D^{n+1} f(z)}{D^n f(z)} - 1\right| - Re\left\{\frac{D^{n+1} f(z)}{D^n f(z)} - 1\right\} \leq 1 - \alpha.$$

We have

$$\beta \left|\frac{D^{n+1} f(z)}{D^n f(z)} - 1\right| - Re\left\{\frac{D^{n+1} f(z)}{D^n f(z)} - 1\right\}$$

$$\leq (1+\beta) \left|\frac{D^{n+1} f(z)}{D^n f(z)} - 1\right|$$

$$\leq \frac{(1+\beta) \sum_{m=1}^{\infty} (m+2)^n (m+1) |a_m| |z^m|}{\frac{1}{|z|} - \sum_{m=1}^{\infty} (m+2)^n |a_m| |z|^m}$$

Letting $z \to 1$ along the real axis, we obtain

$$\leq \frac{(1+\beta) \sum_{m=1}^{\infty} (m+2)^n (m+1) |a_m|}{1 - \sum_{m=1}^{\infty} (m+2)^n |a_m|}$$

This last expression is bounded by $(1-\alpha)$ if

$$\sum_{m=1}^{\infty} (m+2)^n \left[(1+\beta)(m+1) + 1 - \alpha\right] |a_m| \leq (1-\alpha).$$

Hence the theorem is completed.

Corollary 2.1. Let the function $f(z)$ defined by (1.11) be in the class $\sigma_p(\alpha, \beta)$, then

$$a_m \leq \frac{(1-\alpha)}{\sum_{m=1}^{\infty} (m+2)^n \left[(1+\beta)(m+1) + 1 - \alpha\right]}, (m \geq 1) \tag{2.1}$$

Equality holds for the functions of the form

$$f_m(z) = \frac{1}{z} + \frac{(1-\alpha)}{(m+2)^n \left[(1+\beta)(m+1) + 1 - \alpha\right]} z^m. \tag{2.2}$$

3 Distortion Theorems

Theorem 3.1. Let the function $f(z)$ defined by (1.11) be in the class $\sigma_p(\alpha,\beta)$. Then for $0 < |z| = r < 1$,

$$\frac{1}{r} - \frac{(1-\alpha)}{3^n(3+2\beta-\alpha)}r \le |f(z)| \le \frac{1}{r} + \frac{(1-\alpha)}{3^n(3+2\beta-\alpha)}r \tag{3.1}$$

with equality for the function

$$f(z) = \frac{1}{z} + \frac{(1-\alpha)}{3^n(3+2\beta-\alpha)}z, at z = r, ir. \tag{3.2}$$

Proof: Suppose $f(z)$ is in $\sigma_p(\alpha,\beta)$. In view of Theorem 2.1, we have

$$3^n(3+2\beta-\alpha)\sum_{m=1}^{\infty} a_m \le \sum_{m=1}^{\infty}(m+2)^n\left[(1+\beta)(m+1)+1-\alpha\right] \le (1-\alpha).$$

which evidently yields

$$\sum_{m=1}^{\infty} a_m \le \frac{1-\alpha}{3^n(3+2\beta-\alpha)}$$

Consequently, we obtain $|f(z)| = \left|\frac{1}{z} + \sum_{m=1}^{\infty} a_m z^m\right| \le \left|\frac{1}{z}\right| + \sum_{m=1}^{\infty} a_m |z|^m \le \frac{1}{r} + r\sum_{m=1}^{\infty} a_m$ $\le \frac{1}{r} + \frac{(1-\alpha)}{3^n(3+2\beta-\alpha)}r$

Also,

$$\begin{aligned}|f(z)| &= \left|\frac{1}{z} + \sum_{m=1}^{\infty} a_m z^m\right| \\ &\ge \left|\frac{1}{z}\right| - \sum_{m=1}^{\infty} a_m |z|^m \ge \frac{1}{r} - r\sum_{m=1}^{\infty} a_m \\ &\ge \frac{1}{r} - \frac{(1-\alpha)}{3^n(3+2\beta-\alpha)}r\end{aligned}$$

Hence the results (3.1) follow.

Theorem 3.2. Let the function $f(z)$ defined by (1.11) be in the class $\sigma_p(\alpha,\beta)$. Then for $0 < |z| = r < 1$,

$$\frac{1}{r^2} - \frac{(1-\alpha)}{3^n(3+2\beta-\alpha)} \le |f'(z)| \le \frac{1}{r^2} + \frac{(1-\alpha)}{3^n(3+2\beta-\alpha)}$$

The result is sharp, the extremal function being of the form (2.2).

Proof : From Theorem 2.1, we have

$$3^n(3+2\beta-\alpha)\sum_{m=1}^{\infty} m a_m \le \sum_{m=1}^{\infty}(m+2)^n\left[(1+\beta)(m+1)+1-\alpha\right] \le (1-\alpha)$$

which evidently yields

$$\sum_{m=1}^{\infty} m a_m \le \frac{1-\alpha}{3^n(3+2\beta-\alpha)}$$

Consequently, we obtain

$$\begin{aligned}|f'(z)| &\le \frac{1}{r^2} + \sum_{m=1}^{\infty} m a_m r^{m-1} \\ &\le \frac{1}{r^2} + \sum_{m=1}^{\infty} m a_m \\ &\le \frac{1}{r^2} + \frac{(1-\alpha)}{3^n(3+2\beta-\alpha)}\end{aligned}$$

Also,

$$|f'(z)| \geq \frac{1}{r^2} - \sum_{m=1}^{\infty} m a_m r^{m-1}$$

$$\geq \frac{1}{r^2} - \sum_{m=1}^{\infty} m a_m$$

$$\geq \frac{1}{r^2} - \frac{(1-\alpha)}{3^n(3+2\beta-\alpha)}$$

This completes the proof.

4 Class Preserving Integral operators

In this section we consider the class preserving integral operators of the form (1.11).
Theorem 4.1. Let the function $f(z)$ be defined by (1.11) be in the class $\sigma_p(\alpha,\beta)$. Then the integral operator

$$F(z) = cz^{-c-1}\int_0^z t^c f(t)dt = \frac{1}{z} + \sum_{m=}^{\infty} \frac{c}{c+m+1} a_m z^m, c > 0 \tag{4.1}$$

is in $\sigma_p(\delta,\beta)$, where

$$\delta(\alpha,\beta,c) = \frac{(m+1)(1+\beta)+\alpha c(1-\beta)+(1-\alpha)}{(c+m+1)(1+\beta)+(1-\alpha)} \tag{4.2}$$

The result is sharp for

$$f(z) = \frac{1}{z} + \frac{(1-\alpha)}{3^n(3+2\beta-\alpha)} z.$$

Proof. Suppose $f(z) = \frac{1}{z} + \sum_{m=1}^{\infty} a_m z^m$ is in $\sigma_p(\alpha,\beta)$.We have

$$F(z) = cz^{-c-1}\int_0^z t^c f(t)dt = \frac{1}{z} + \sum_{m=1}^{\infty} \frac{c}{c+m+1} a_m z^m, c > 0.$$

It is sufficient to show that

$$\tfrac{\sum_{m=1}^{\infty}(m+2)^n[(1+\beta)(m+1)+1-\delta]}{1-\delta} \tfrac{c}{c+m+1} a_n \leq 1 \tag{4.3}$$

Since $f(z)$ is in $\sigma_p(\alpha,\beta)$,we have

$$\tfrac{\sum_{m=1}^{\infty}(m+2)^n[(1+\beta)(m+1)+1-\alpha]|a_m|}{1-\alpha} \leq 1 \tag{4.4}$$

Thus (4.3) will be satisfied if

$$\tfrac{[(1+\beta)(m+1)+1-\delta]}{1-\delta} \tfrac{c}{c+m+1} \leq \tfrac{[(1+\beta)(m+1)+1-\alpha]}{1-\alpha}$$

Solving for δ, we obtain

$$\delta \leq \frac{(m+1)(1+\beta)+\alpha c(1-\beta)+(1-\alpha)}{(c+m+1)(1+\beta)+(1-\alpha)} = G(m)$$

A simple computation will show that $G(m)$ is increasing and $G(m) \geq G(1)$.Using this, the result follows.

5 Convex Linear Combinations and Convolution Properties

Theorem 5.1. If the function $f(z) = \frac{1}{z} + \sum_{m=1}^{\infty} a_m z^m$ is in $\sigma_p(\alpha, \beta)$then $f(z)$ is meromorphically convex of order $\delta(0 \leq \delta < 1)$ in $|z| < r = r(\alpha, \beta, \delta)$ where

$$r(\alpha, \beta, \delta) = \inf_{n \geq 1} \left\{ \frac{(1-\delta)(m+2)^n \left[(1+\beta)(1+m)+1-\alpha\right]}{(1-\alpha)m(m+2-\delta)} \right\}^{1/m+1}$$

.
The result is sharp.
Proof. Let $f(z)$ is in $\sigma_p(\alpha, \beta)$. Then ,by Theorem 2.1, we have

$$\sum_{m=1}^{\infty} (m+2)^n \left[(1+\beta)(m+1)+1-\alpha\right] |a_m| \leq (1-\alpha) \tag{5.1}$$

It is sufficient to show that

$$\left| 2 + \frac{zf''(z)}{f'(z)} \right| \leq 1 - \delta$$

for $|z| < r = r(\alpha, \beta, \delta)$,where $r(\alpha, \beta, \delta)$ is specified in the statement of the theorem. Then

$$\left| 2 + \frac{zf''(z)}{f'(z)} \right| = \left| \frac{\sum_{m=1}^{\infty} m(m+1)a_m z^{m-1}}{\frac{-1}{z^2} + \sum_{m=1}^{\infty} m a_m z^{m-1}} \right| \leq \sum_{m=1}^{\infty} \frac{m(m+1)a_m |z|^{m+1}}{1 - \sum_{m=1}^{\infty} m a_m |z|^{m+1}}$$

This will be bounded by $(1-\delta)$ if

$$\sum_{m=1}^{\infty} \frac{m(m+2-\delta)}{1-\delta} a_m |z|^{m+1} \leq 1. \tag{5.2}$$

By (5.1), it follow that (5.2) is true if

$$\frac{m(m+2-\delta)}{1-\delta} |z|^{m+1} \leq \frac{(m+2)^n[(1+\beta)(m+1)+1-\alpha]}{1-\alpha}, m \geq 1$$

or

$$|z| \leq \left\{ \frac{(1-\delta)(m+2)^n \left[(1+\beta)(1+m)+1-\alpha\right]}{(1-\alpha)m(m+2-\delta)} \right\}^{1/m+1} \tag{5.3}$$

Setting $|z| = r(\alpha, \beta, \delta)$ in (5.3) , the result follows. The result is sharp for the function

$$f_m(z) = \frac{1}{z} + \frac{(1-\alpha)}{(m+2)^n \left[(1+\beta)(m+1)+1-\alpha\right]} z^m, (m \geq 1).$$

Theorem 5.2. Let $f_0(z) = \frac{1}{z}$ and

$$f_m(z) = \frac{1}{z} + \frac{(1-\alpha)}{(m+2)^n \left[(1+\beta)(m+1)+1-\alpha\right]} z^m, (m \geq 1)$$

Then $f(z) = \frac{1}{z} + \sum_{m=1}^{\infty} a_m z^m$ is in the class $\sigma_p(\alpha, \beta)$ if and only if it can be expressed in the form

$$f(z) = \lambda_0 f_0(z) + \sum_{m=1}^{\infty} \lambda_m f_m(z)$$

where $\lambda_0 \geq 0$, $\lambda_m \geq 0$ $(m \geq 1)$ and $\lambda_0 + \sum_{m=1}^{\infty} \lambda_m = 1$.

Proof. Let $f(z) = \lambda_0 f_0(z) + \sum_{m=1}^{\infty} \lambda_m f_m(z)$ with $\lambda_0 \geq 0$, $\lambda_m \geq 0$ $(m \geq 1)$ and

$$\lambda_0 + \sum_{m=1}^{\infty} \lambda_m = 1$$

Then

$$f(z) = \lambda_0 f_0(z) + \sum_{m=1}^{\infty} \lambda_m f_m(z)$$

$$= \frac{1}{z} + \sum_{m=1}^{\infty} \lambda_m \frac{(1-\alpha)}{(m+2)^n \left[(1+\beta)(m+1)+1-\alpha\right]} z^m$$

Since

$$\textstyle\sum_{m=1}^{\infty} \frac{(m+2)^n[(1+\beta)(m+1)+1-\alpha]}{1-\alpha} \lambda_m \frac{1-\alpha}{(m+2)^n(1+\beta)(m+1)+1-\alpha}$$

$$= \sum_{m=1}^{\infty} \lambda_m = 1 - \lambda_0 \leq 1,$$

By Theorem 2.1 $f(z)$ is in the class $\sigma_p(\alpha, \beta)$. Conversely suppose that the function $f(z)$ is in the class $\sigma_p(\alpha, \beta)$, Since

$$a_m \leq \frac{(1-\alpha)}{(m+2)^n \left[(1+\beta)(m+1)+1-\alpha\right]}, (m \geq 1)$$

$$\lambda_m = \tfrac{(m+2)^n[(1+\beta)(m+1)+1-\alpha]}{1-\alpha} a_m,$$

and $\lambda_0 = 1 - \sum_{m=1}^{\infty} \lambda_m$, it follows that $f(z) = \lambda_0 f_0(z) + \sum_{m=1}^{\infty} \lambda_m f_m(z)$. This completes the proof of the theorem.

For the functions $f(z) = \frac{1}{z} + \sum_{m=1}^{\infty} a_m z^m$ and $g(z) = \frac{1}{z} + \sum_{m=1}^{\infty} b_m z^m$belong to $\sum_p$ we denote by $(f * g)(z)$ the convolution of $f(z)$ and $g(z)$ or

$$(f * g)(z) = \frac{1}{z} + \sum_{m=1}^{\infty} a_m b_m z^m$$

Theorem 5.3. If the functions $f(z) = \frac{1}{z} + \sum_{m=1}^{\infty} a_m z^m$ and $g(z) = \frac{1}{z} + \sum_{m=1}^{\infty} b_m z^m$ are in the class $\sigma_p(\alpha, \beta)$,then $(f * g)(z) = \frac{1}{z} + \sum_{m=1}^{\infty} a_m b_m z^m$ is in the class $\sigma_p(\alpha, \beta)$.
Proof. Suppose $f(z)$ and $g(z)$ are in $\sigma_p(\alpha, \beta)$. By Theorem 2.1, we have and $\sum_{m=1}^{\infty} \frac{(m+2)^n[(1+\beta)(m+1)+1-\alpha]}{1-\alpha} a$

$$\textstyle\sum_{m=1}^{\infty} \frac{(m+2)^n[(1+\beta)(m+1)+1-\alpha]}{1-\alpha} b_m \leq 1$$

Since $f(z)$ and $g(z)$ are regular are in E, so is $(f * g)(z)$. Furthermore,

$$\textstyle\sum_{m=1}^{\infty} \frac{(m+2)^n[(1+\beta)(m+1)+1-\alpha]}{1-\alpha} a_m b_m \quad \leq \sum_{m=1}^{\infty} \left\{ \frac{(m+2)^n[(1+\beta)(m+1)+1-\alpha]}{1-\alpha} \right\}^2 a_m b_m$$

$$\leq \left(\sum_{m=1}^{\infty} \frac{(m+2)^n \left[(1+\beta)(m+1)+1-\alpha\right]}{1-\alpha} a_m \right) \left(\sum_{m=1}^{\infty} \frac{(m+2)^n \left[(1+\beta)(m+1)+1-\alpha\right]}{1-\alpha} b_m \right)$$

$\leq 1.$

Hence by Theorem 2.1, $(f * g)(z)$ is in the class $\sigma_p(\alpha, \beta)$.

6 Neighborhoods for the class $\sigma_p(\alpha, \beta, \gamma)$ which we define as follows:

Definition 6.1. A function $f \in \sum_p$ is said to in the class $\sigma_p(\alpha, \beta, \gamma)$ if there exists a function $g \in \sigma_p(\alpha, \beta)$ such that

$$\left|\frac{f(z)}{g(z)} - 1\right| < 1 - \gamma, z \in E, (0 \leq \gamma < 1) \tag{6.1}$$

Following the earlier works on neighborhoods of analytic functions by Goodman [4] and Ruschweyh [16], we define the δ-neighborhood of a function $f \in \sum_p$by

$$N_\delta(f) := \left\{g \in \sum_p : g(z) = \frac{1}{z} + \sum_{m=1}^{\infty} b_m z^m : \sum_{m=1}^{\infty} m\,|a_m - b_m| \leq \delta\right\} \tag{6.2}$$

Theorem 6.1. If $g \in \sigma_p(\alpha, \beta)$ and

$$\gamma = 1 - \frac{\delta(3 + 2\beta - \alpha)}{2 + 2\beta} \tag{6.3}$$

Then

$$N_\delta(g) \subset \sigma_p(\alpha, \beta, \gamma).$$

Proof: Let $f \in N_\delta(g)$. Then we find from (6.2) that

$$\sum_{m=1}^{\infty} m\,|a_m - b_m| \leq \delta \tag{6.4}$$

which implies the coefficient inequality

$$\sum_{m=1}^{\infty} |a_m - b_m| \leq \delta, (m \in N) \tag{6.5}$$

Since $g \in \sigma_p(\alpha, \beta)$, we have

$$\sum_{m=1}^{\infty} b_m \leq \frac{1 - \alpha}{3 + 2\beta - \alpha} \tag{6.6}$$

So that

$$\left|\frac{f(z)}{g(z)} - 1\right| \leq \frac{\sum_{m=1}^{\infty} |a_m - b_m|}{1 - \sum_{m=1}^{\infty} b_m} \leq \frac{\delta(3 + 2\beta - \alpha)}{2 + 2\beta} = 1 - \gamma.$$

provided γ is given by (6.3). Hence ,by definition , $f \in \sigma_p(\alpha, \beta, \gamma)$ for γ given by (6.3), which completes the proof.

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