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Tow estimates for the Generalized Fourier-Bessel Transform in the Space $L^2_{\alpha,n}$

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Abstract. Two estimates are proved for the generalized Fourier-Bessel transform in the space $L^2_{\alpha,n}$ on certain classes of functions characterized by the generalized continuity modulus.

1 Introduction

In [5], Abilov et al. proved two estimates for the Fourier transform in the space of square integrable functions on certain classes of functions characterized by the generalized continuity modulus, using a translation operator.
In this paper, we consider a second-order singular differential operator $\mathcal{B}$ on the half line which generalizes the Bessel operator $\mathcal{B}_\alpha$, we prove two estimates in certain classes of functions characterized by a generalized continuity modulus and connected with the generalized Fourier-Bessel transform associated to $\mathcal{B}$ in $L^2_{\alpha,n}$ analogs of the statements proved in [5]. For this purpose, we use a generalized translation operator.
In section 2, we give some definitions and preliminaries concerning the generalized Fourier-Bessel transform. Two estimates are proved in section 3.

2 Preliminaries on the generalized Fourier-Bessel transform

Integral transforms and their inverses the Bessel transform are widely used to solve various problems in calculus, mechanics, mathematical physics, and computational mathematics (see [3, 4]).
We briefly overview the theory of generalized Fourier-Bessel transform and related harmonic analysis (see [1, 6])
Consider the second-order singular differential operator on the half line defined by

$$\mathcal{B}f(x) = \frac{d^2 f(x)}{dx^2} + \frac{(2\alpha+1)}{x}\frac{df(x)}{dx} - \frac{4n(\alpha+n)}{x^2}f(x),$$

where $\alpha > \frac{-1}{2}$ and $n = 0, 1, 2, ...$ For $n = 0$, we obtain the classical Bessel operator

$$\mathcal{B}_\alpha f(x) = \frac{d^2 f(x)}{dx^2} + \frac{(2\alpha+1)}{x}\frac{df(x)}{dx}.$$

Let M be the map defined by

$$Mf(x) = x^{2n}f(x), \quad n = 0, 1, ..$$

Let $L^p_{\alpha,n}$, $1 \le p < \infty$, be the class of measurable functions f on $[0, \infty[$ for which

$$\|f\|_{p,\alpha,n} = \|M^{-1}f\|_{p,\alpha+2n} < \infty,$$

where

$$\|f\|_{p,\alpha} = \left(\int_0^\infty |f(x)|^p x^{2\alpha+1} dx\right)^{1/p}.$$

If $p=2$, then we have $L^2_{\alpha,n}=L^2([0,\infty[,x^{2\alpha+1})$.
For $\alpha>\frac{-1}{2}$, we introduce the normalized spherical Bessel function j_α defined by

$$j_\alpha(x)=\frac{2^\alpha\Gamma(\alpha+1)J_\alpha(x)}{x^\alpha}, \tag{2.1}$$

where $J_\alpha(x)$is the Bessel function of the first Kind and $\Gamma(x)$ is the gamma-function (see [7]). The function $y=j_\alpha(x)$ satisfies the differential equation

$$\mathcal{B}_\alpha y+y=0$$

with the condition initial $y(0)=0$ and $y'(0)=0$. The function $j_\alpha(x)$ is infinitely differentiable, even and moreover entire analytic.
In the terms of $j_\alpha(x)$, we have (see [2])

$$\begin{aligned} 1-j_\alpha(x) &= O(1), \quad x\geq 1. & (2.2)\\ 1-j_\alpha(x) &= O(x^2), \quad 0\leq x\leq 1. & (2.3)\\ \sqrt{hx}J_\alpha(hx) &= O(1), \quad hx\geq 0. & (2.4)\end{aligned}$$

For $\lambda\in\mathbb{C}$, and $x\in\mathbb{R}$, put

$$\varphi_\lambda(x)=x^{2n}j_{\alpha+2n}(\lambda x). \tag{2.5}$$

From [1, 6] recall the following properties.

Proposition 2.1.
(1) φ_λ satisfies the differential equation

$$\mathcal{B}\varphi_\lambda=-\lambda^2\varphi_\lambda.$$

(2) For all $\lambda\in\mathbb{C}$, and $x\in\mathbb{R}$

$$|\varphi_\lambda(x)|\leq x^{2n}e^{|Im\lambda||x|}.$$

The generalized Fourier-Bessel transform that we call it the integral transform defined by

$$\mathcal{F}_\mathcal{B}f(\lambda)=\int_0^\infty f(x)\varphi_\lambda(x)x^{2\alpha+1}dx, \lambda\geq 0, f\in L^1_{\alpha,n},$$

(see [1]). Let $f\in L^1_{\alpha,n}$ such that $\mathcal{F}_\mathcal{B}(f)\in L^1_{\alpha+2n}=L^1([0,\infty[,x^{2\alpha+4n+1}dx)$. Then the inverse generalized Fourier-Bessel transform is given by the formula

$$f(x)=\int_0^\infty \mathcal{F}_\mathcal{B}f(\lambda)\varphi_\lambda(x)d\mu_{\alpha+2n}(\lambda),$$

where

$$d\mu_{\alpha+2n}(\lambda)=a_{\alpha+2n}\lambda^{2\alpha+4n+1}d\lambda, \quad a_\alpha=\frac{1}{4^\alpha(\Gamma(\alpha+1))^2},$$

(see [1]).

Proposition 2.2. *[1]*
(1) For every $f\in L^1_{\alpha,n}\cap L^2_{\alpha,n}$ we have the Plancherel formula

$$\int_0^{+\infty}|f(x)|^2x^{2\alpha+1}dx=\int_0^{+\infty}|\mathcal{F}_\mathcal{B}f(\lambda)|^2d\mu_{\alpha+2n}(\lambda).$$

(2) The generalized Fourier-Bessel transform $\mathcal{F}_\mathcal{B}$ extends uniquely to an isometric isomorphism from $L^2_{\alpha,n}$ onto $L^2([0,+\infty[,\mu_{\alpha+2n})$.

Define the generalized translation operator T^h, $h \geq 0$ by the relation

$$T^h f(x) = (xh)^{2n}\tau^h_{\alpha+2n}(M^{-1}f)(x), x \geq 0,$$

where $\tau^h_{\alpha+2n}$ is the Bessel translation operators of order $\alpha + 2n$ defined by

$$\tau^h_\alpha f(x) = c_\alpha \int_0^\pi f(\sqrt{x^2+h^2-2xh\cos t})\sin^{2\alpha} t dt,$$

where

$$c_\alpha = \left(\int_0^\pi \sin^{2\alpha} t dt\right)^{-1} = \frac{\Gamma(\alpha+1)}{\Gamma(\pi)\Gamma(\alpha+\frac{1}{2})}.$$

For $f \in L^2_{\alpha,n}$, we have

$$\mathcal{F}_\mathcal{B}(T^h f)(\lambda) = \varphi_\lambda(h)\mathcal{F}_\mathcal{B}(f)(\lambda), \tag{2.6}$$

$$\mathcal{F}_\mathcal{B}(\mathcal{B}f)(\lambda) = -\lambda^2\mathcal{F}_\mathcal{B}(f)(\lambda), \tag{2.7}$$

(see [1, 6] for details) .
Let $f \in L^2_{\alpha,n}$. The quantity

$$w(f,\delta)_{2,\alpha,n} = \sup_{0<h\leq\delta} \|T^h f(x) - h^{2n}f(x)\|_{2,\alpha,n}, \delta > 0,$$

is called the modulus of continuity of the function f.
Let $W^r_{2,\phi}(\mathcal{B})$, $r = 0, 1, ...,$ denote the class of functions $f \in L^2_{\alpha,n}$ that have generalized derivatives satisfying the estimate

$$\omega(\mathcal{B}^r f, \delta)_{2,\alpha,n} = O(\phi(\delta)), \quad \delta \to 0,$$

where $\phi(x)$ is any nonnegative function given on $[0,\infty)$, and $\mathcal{B}^0 f = f$, $\mathcal{B}^r f = \mathcal{B}(\mathcal{B}^{r-1}f)$, $r = 1, 2, ...$
i.e.,

$$W^r_{2,\phi}(\mathcal{B}) = \{f \in L^2_{\alpha,n}, \mathcal{B}^r f \in L^2_{\alpha,n} \text{and} \quad \omega(\mathcal{B}^r f,\delta)_{2,\alpha,n} = O(\phi(\delta)), \delta \to 0\}.$$

3 Main Results

The goal of this work is to prove several new estimates for the integral

$$J^2_N(f) = \int_N^\infty |\mathcal{F}_\mathcal{B} f(\lambda)|^2 d\mu_{\alpha+2n}(\lambda),$$

in certain classes of functions in $L^2_{\alpha,n}$.

Lemma 3.1. *For $f \in W^r_{2,\phi}(\mathcal{B})$, we have*

$$\|T^h\mathcal{B}^r f(x) - h^{2n}\mathcal{B}^r f(x)\|^2_{2,\alpha,n} = h^{4n}\int_0^\infty \lambda^{4r}|j_{\alpha+2n}(\lambda h) - 1|^2|\mathcal{F}_\mathcal{B} f(\lambda)|^2 d\mu_{\alpha+2n}(\lambda).$$

where $r = 0, 1, 2, ...$

Proof. From formula (2.7), we obtain

$$\mathcal{F}_\mathcal{B}(\mathcal{B}^r f)(\lambda) = (-1)^r\lambda^{2r}\mathcal{F}_\mathcal{B} f(\lambda); r = 0, 1, ... \tag{3.1}$$

By using the formula (2.6), we conclude that

$$\mathcal{F}_\mathcal{B}(T^h f - h^{2n}f)(\lambda) = h^{2n}(j_{\alpha+2n}(\lambda h) - 1)\mathcal{F}_\mathcal{B} f(\lambda). \tag{3.2}$$

Now by formulas (3.1), (3.2) and Plancherel equality, we have the result. □

Theorem 3.2. *Given r and $f \in W^r_{2,\phi}(\mathcal{B})$. Then there exist constant $c > 0$ such that, for all $N > 0$,*

$$J_N(f) = O(N^{-2r+2n}\phi(c/N)).$$

Proof. Firstly, we have

$$J_N^2(f) \leq \int_N^\infty |j| d\mu + \int_N^\infty |1-j| d\mu, \tag{3.3}$$

with $j = j_p(\lambda h)$, $p = \alpha + 2n$ and $d\mu = |\mathcal{F}_\mathcal{B} f(\lambda)|^2 d\mu_{\alpha+2n}(\lambda)$. The parameter $h > 0$ will be chosen in an instant.
In view of formulas (2.1) and (2.4), there exist a constant $c_1 > 0$ such that

$$|j| \leq c_1(\lambda h)^{-p-\frac{1}{2}}.$$

Then

$$\int_N^\infty |j| d\mu \leq c_1(hN)^{-p-\frac{1}{2}} J_N^2(f).$$

Choose a constant c_2 such that the number $c_3 = 1 - c_1 c_2^{-p-\frac{1}{2}}$ is positif.
Setting $h = c_2/N$ in the inequality (3.3), we have

$$c_3 J_N^2(f) \leq \int_N^\infty |1-j| d\mu. \tag{3.4}$$

By Hölder inequality the second term in (3.4) satisfies

$$\begin{aligned}
\int_N^\infty |1-j| d\mu &= \int_N^\infty |1-j|.1.d\mu \\
&\leq \left(\int_N^\infty |1-j|^2 d\mu\right)^{1/2} \left(\int_N^\infty d\mu\right)^{1/2} \\
&\leq \left(\int_N^\infty \lambda^{-4r}|1-j|^2 \lambda^{4r} d\mu\right)^{1/2} J_N(f) \\
&\leq N^{-2r}\left(\int_N^\infty |1-j|^2 \lambda^{4r} d\mu\right)^{1/2} J_N(f).
\end{aligned}$$

From Lemma 3.1, we conclude that

$$\int_N^\infty |1-j|^2 \lambda^{4r} d\mu \leq h^{-4n} \|T^h \mathcal{B}^r f(x) - h^{2n}\mathcal{B}^r f(x)\|^2_{2,\alpha,n}.$$

Therefore

$$\int_N^\infty |1-j| d\mu \leq N^{-2r} h^{-2n} \|T^h \mathcal{B}^r f(x) - h^{2n}\mathcal{B}^r f(x)\|_{2,\alpha,n} J_N(f).$$

For $f \in W^r_{2,\phi}(\Lambda)$ there exist a constant $c_4 > 0$ such that

$$\|T^h \mathcal{B}^r f(x) - h^{2n}\mathcal{B}^r f(x)\|_{2,\alpha,n} \leq c_4 \phi(h).$$

For $h = c_2/N$, we obtain

$$c_3 J_N^2(f) \leq c_2^{-2n} N^{2n-2r} c_4 \phi(c_2/N) J_N(f).$$

Hence

$$c_2^{2n} c_3 J_N(f) \leq c_4 N^{-2r+2n}\phi(c_2/N).$$

for all $N > 0$. The theorem is proved with $c = c_2$. $\square$

Theorem 3.3. *Let $\phi(t) = t^\nu$, then*

$$J_N(f) = O(N^{-2r-\nu+2n}) \Leftrightarrow f \in W^r_{2,\phi}(\mathcal{B}),$$

where, $r = 0, 1, ...; 0 < \nu < 2.$

Proof. We prove sufficiency by using Theorem 3.2 let $f \in W^r_{2,\phi}(\mathcal{B})$ then

$$J_N(f) = O(N^{-2r-\nu+2n}).$$

To prove necessity let

$$J_N(f) = O(N^{-2r-\nu+2n})$$

i.e.

$$\int_N^\infty |\mathcal{F}_\mathcal{B} f(\lambda)|^2 d\mu_{\alpha+2n}(\lambda) = O(N^{-4r-2\nu+4n})$$

It is easy to show, that there exists a function $f \in L^2_{\alpha,n}$ such that $\mathcal{B}^r f \in L^2_{\alpha,n}$ and

$$\mathcal{B}^r f(x) = (-1)^r \int_0^\infty \lambda^{2r} \mathcal{F}_\mathcal{B} f(\lambda)\varphi_\lambda(x) d\mu_{\alpha+2n}(\lambda). \tag{3.5}$$

From formula (3.5) and Plancherel equality, we have

$$\|T^h \mathcal{B}^r f(x) - h^{2n}\mathcal{B}^r f(x)\|^2_{2,\alpha,n} = h^{4n} \int_0^\infty \lambda^{4r} |j_{\alpha+2n}(\lambda h) - 1|^2 |\mathcal{F}_\mathcal{B} f(\lambda)|^2 d\mu_{\alpha+2n}(\lambda).$$

This integral is divided into two

$$\int_0^\infty = \int_0^N + \int_N^\infty = I_1 + I_2,$$

where $N = [h^{-1}]$, We estimate them separately.
From (2.2), we have the estimate

$$\begin{aligned}
I_2 &\leq c_5 \int_{|\lambda| \geq N} \lambda^{4r} |\mathcal{F}_\mathcal{B} f(\lambda)|^2 d\mu_{\alpha+2n}(\lambda) \\
&= c_5 \sum_{l=0}^{\infty} \int_{N+l \leq |\lambda| \leq N+l+1} \lambda^{4r} |\mathcal{F}_\mathcal{B} f(\lambda)|^2 d\mu_{\alpha+2n}(\lambda) \\
&\leq c_5 \sum_{l=0}^{\infty} a_l (u_l - u_{l+1}),
\end{aligned}$$

with $a_l = (N + l + 1)^{4r}$ and $u_l = \displaystyle\int_{|\lambda| \geq N+l} |\mathcal{F}_\mathcal{B} f(\lambda)|^2 d\mu_{\alpha+2n}(\lambda).$
For all integers $m \geq 1$, the Abel transformation shows

$$\begin{aligned}
\sum_{l=0}^{m} a_l (u_l - u_{l+1}) &= a_0 u_0 + \sum_{l=1}^{m} (a_l - a_{l-1}) u_l - a_m u_{m+1} \\
&\leq a_0 u_0 + \sum_{l=1}^{n} (a_l - a_{l-1})\, u_l,
\end{aligned}$$

because $a_m u_{m+1} \geq 0$. Moreover by the finite increments theorem, we have

$$a_l - a_{l-1} \leq 4r(N + l + 1)^{4r-1}$$

Furthermore by the hypothesis of f there exists $c_6 > 0$ such that, for all $N > 0$

$$J^2_N(f) \leq c_6 N^{-4r-2\nu+4n},$$

For $N \geq 1$, we have

$$\begin{aligned}\sum_{l=1}^{m}(a_l - a_{l-1})\,u_l &\leq c_6\left(1+\frac{1}{N}\right)^{4r} N^{-2\nu+4n} + 4rc_6\sum_{l=1}^{m}\left(1+\frac{1}{N+l}\right)^{4r-1}(N+l)^{-1-2\nu+4n}\\ &\leq 2^{2r}c_6N^{-2\nu+4n} + 4r2^{4r-1}c_6\sum_{l=1}^{m}(N+l)^{-1-2\nu+4n}.\end{aligned}$$

Finally, by the integral comparison test we have

$$\sum_{l=1}^{m}(N+l)^{-1-2\nu+4n} \leq \int_N^\infty x^{-1-2\nu+4n}dx = \frac{1}{2\nu-4n}N^{-2\nu+4n}.$$

Letting $m \to \infty$ we see that, for $r \geq 0$ and $\nu > 0$, there exists a constant c_7 such that, for all $N \geq 1$ and for $h > 0$,

$$I_2 \leq c_7 N^{-2\nu+4n}.$$

Now, we estimate I_1. From formula (2.3), we have

$$\begin{aligned}I_1 &\leq c_8h^4\int_{|\lambda|\leq N}\lambda^{4r+4}|\mathcal{F}_{\mathcal{B}}f(\lambda)|^2d\mu_{\alpha+2n}(\lambda)\\ &= c_8h^4\sum_{l=0}^{N-1}\int_{l\leq|\lambda|\leq l+1}\lambda^{4r+4}|\mathcal{F}_{\mathcal{B}}f(\lambda)|^2d\mu_{\alpha+2n}(\lambda)\\ &\leq c_8h^4\sum_{l=0}^{N-1}(l+1)^{4r+4}(v_l - v_{l+1}),\end{aligned}$$

with $v_l = \int_{|\lambda|\geq l}|\mathcal{F}_{\mathcal{B}}f(\lambda)|^2d\mu_{\alpha+2n}(\lambda)$.

Using an Abel transformation and proceeding as with I_2 we obtain

$$\begin{aligned}I_1 &\leq c_8h^4\left(v_0 + \sum_{l=1}^{N-1}((l+1)^{4r+4} - l^{4r+4})v_l\right)\\ &\leq c_8h^4\left(v_0 + (4r+4)c_6\sum_{l=1}^{N-1}(l+1)^{4r+3}l^{-4r-2\nu+4n}\right),\end{aligned}$$

since $v_l \leq c_6 l^{-4r-2\nu+4n}$ by hypothesis. From the inequality $l+1 \leq 2l$ we conclude

$$I_1 \leq c_8h^4\left(v_0 + c_9\sum_{l=1}^{N-1}l^{3-2\nu+4n}\right).$$

As a consequence of a series comparison for $\mu \geq 1$ and $\mu < 1$ we have the inequality,

$$\mu\sum_{l=1}^{N-1}l^{\mu-1} < N^\mu, \text{for} \quad \mu > 0 \quad \text{and} \quad N \geq 2.$$

If $\mu = 4 - 2\nu + 4n > 0$ for $\nu < 2$ then we obtain

$$I_1 \leq c_8h^4\left(v_0 + c_{10}N^{4-2\nu+4n}\right) \leq c_8h^4\left(v_0 + c_{10}h^{-4+2\nu-4n}\right),$$

since $N \leq 1/h$. If h is sufficiently small then $v_0 \leq c_{10}h^{-4+2\nu-4n}$. Then we have

$$I_1 \leq c_{11}h^{2\nu-4n}$$

Combining the estimates for I_1 and I_2 gives

$$\|T^h\mathcal{B}^rf(x) - h^{2n}\mathcal{B}^rf(x)\|_{2,\alpha,n} = O(h^\nu),$$

The necessity is proved. □

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