

Tow estimates for the Generalized Fourier-Bessel Transform in the Space $L_{\alpha,n}^2$

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Abstract. Two estimates are proved for the generalized Fourier-Bessel transform in the space $L_{\alpha,n}^2$ on certain classes of functions characterized by the generalized continuity modulus.

1 Introduction

In [5], Abilov et al. proved two estimates for the Fourier transform in the space of square integrable functions on certain classes of functions characterized by the generalized continuity modulus, using a translation operator.

In this paper, we consider a second-order singular differential operator \mathcal{B} on the half line which generalizes the Bessel operator \mathcal{B}_α , we prove two estimates in certain classes of functions characterized by a generalized continuity modulus and connected with the generalized Fourier-Bessel transform associated to \mathcal{B} in $L_{\alpha,n}^2$ analogs of the statements proved in [5]. For this purpose, we use a generalized translation operator.

In section 2, we give some definitions and preliminaries concerning the generalized Fourier-Bessel transform. Two estimates are proved in section 3.

2 Preliminaries on the generalized Fourier-Bessel transform

Integral transforms and their inverses the Bessel transform are widely used to solve various problems in calculus, mechanics, mathematical physics, and computational mathematics (see [3, 4]).

We briefly overview the theory of generalized Fourier-Bessel transform and related harmonic analysis (see [1, 6])

Consider the second-order singular differential operator on the half line defined by

$$\mathcal{B}f(x) = \frac{d^2 f(x)}{dx^2} + \frac{(2\alpha + 1)}{x} \frac{df(x)}{dx} - \frac{4n(\alpha + n)}{x^2} f(x),$$

where $\alpha > \frac{-1}{2}$ and $n = 0, 1, 2, \dots$. For $n = 0$, we obtain the classical Bessel operator

$$\mathcal{B}_\alpha f(x) = \frac{d^2 f(x)}{dx^2} + \frac{(2\alpha + 1)}{x} \frac{df(x)}{dx}.$$

Let M be the map defined by

$$Mf(x) = x^{2n} f(x), \quad n = 0, 1, \dots$$

Let $L_{\alpha,n}^p$, $1 \leq p < \infty$, be the class of measurable functions f on $[0, \infty[$ for which

$$\|f\|_{p,\alpha,n} = \|M^{-1}f\|_{p,\alpha+2n} < \infty,$$

where

$$\|f\|_{p,\alpha} = \left(\int_0^\infty |f(x)|^p x^{2\alpha+1} dx \right)^{1/p}.$$

If $p = 2$, then we have $L^2_{\alpha,n} = L^2([0, \infty[, x^{2\alpha+1})$.

For $\alpha > \frac{-1}{2}$, we introduce the normalized spherical Bessel function j_α defined by

$$j_\alpha(x) = \frac{2^\alpha \Gamma(\alpha + 1) J_\alpha(x)}{x^\alpha}, \tag{2.1}$$

where $J_\alpha(x)$ is the Bessel function of the first Kind and $\Gamma(x)$ is the gamma-function (see [7]).

The function $y = j_\alpha(x)$ satisfies the differential equation

$$\mathcal{B}_\alpha y + y = 0$$

with the condition initial $y(0) = 0$ and $y'(0) = 0$. The function $j_\alpha(x)$ is infinitely differentiable, even and moreover entire analytic.

In the terms of $j_\alpha(x)$, we have (see [2])

$$1 - j_\alpha(x) = O(1), \quad x \geq 1. \tag{2.2}$$

$$1 - j_\alpha(x) = O(x^2), \quad 0 \leq x \leq 1. \tag{2.3}$$

$$\sqrt{hx} J_\alpha(hx) = O(1), \quad hx \geq 0. \tag{2.4}$$

For $\lambda \in \mathbb{C}$, and $x \in \mathbb{R}$, put

$$\varphi_\lambda(x) = x^{2n} j_{\alpha+2n}(\lambda x). \tag{2.5}$$

From [1, 6] recall the following properties.

Proposition 2.1.

(1) φ_λ satisfies the differential equation

$$\mathcal{B}\varphi_\lambda = -\lambda^2 \varphi_\lambda.$$

(2) For all $\lambda \in \mathbb{C}$, and $x \in \mathbb{R}$

$$|\varphi_\lambda(x)| \leq x^{2n} e^{|\text{Im}\lambda||x|}.$$

The generalized Fourier-Bessel transform that we call it the integral transform defined by

$$\mathcal{F}_B f(\lambda) = \int_0^\infty f(x) \varphi_\lambda(x) x^{2\alpha+1} dx, \lambda \geq 0, f \in L^1_{\alpha,n},$$

(see [1]). Let $f \in L^1_{\alpha,n}$ such that $\mathcal{F}_B(f) \in L^1_{\alpha+2n} = L^1([0, \infty[, x^{2\alpha+4n+1} dx)$. Then the inverse generalized Fourier-Bessel transform is given by the formula

$$f(x) = \int_0^\infty \mathcal{F}_B f(\lambda) \varphi_\lambda(x) d\mu_{\alpha+2n}(\lambda),$$

where

$$d\mu_{\alpha+2n}(\lambda) = a_{\alpha+2n} \lambda^{2\alpha+4n+1} d\lambda, \quad a_\alpha = \frac{1}{4^\alpha (\Gamma(\alpha + 1))^2},$$

(see [1]).

Proposition 2.2. [1]

(1) For every $f \in L^1_{\alpha,n} \cap L^2_{\alpha,n}$ we have the Plancherel formula

$$\int_0^{+\infty} |f(x)|^2 x^{2\alpha+1} dx = \int_0^{+\infty} |\mathcal{F}_B f(\lambda)|^2 d\mu_{\alpha+2n}(\lambda).$$

(2) The generalized Fourier-Bessel transform \mathcal{F}_B extends uniquely to an isometric isomorphism from $L^2_{\alpha,n}$ onto $L^2([0, +\infty[, \mu_{\alpha+2n})$.

Define the generalized translation operator $T^h, h \geq 0$ by the relation

$$T^h f(x) = (xh)^{2n} \tau_{\alpha+2n}^h(M^{-1}f)(x), x \geq 0,$$

where $\tau_{\alpha+2n}^h$ is the Bessel translation operators of order $\alpha + 2n$ defined by

$$\tau_{\alpha}^h f(x) = c_{\alpha} \int_0^{\pi} f(\sqrt{x^2 + h^2 - 2xh \cos t}) \sin^{2\alpha} t dt,$$

where

$$c_{\alpha} = \left(\int_0^{\pi} \sin^{2\alpha} t dt \right)^{-1} = \frac{\Gamma(\alpha + 1)}{\Gamma(\pi)\Gamma(\alpha + \frac{1}{2})}.$$

For $f \in L_{\alpha,n}^2$, we have

$$\mathcal{F}_{\mathcal{B}}(T^h f)(\lambda) = \varphi_{\lambda}(h)\mathcal{F}_{\mathcal{B}}(f)(\lambda), \tag{2.6}$$

$$\mathcal{F}_{\mathcal{B}}(\mathcal{B}f)(\lambda) = -\lambda^2 \mathcal{F}_{\mathcal{B}}(f)(\lambda), \tag{2.7}$$

(see [1, 6] for details).

Let $f \in L_{\alpha,n}^2$. The quantity

$$w(f, \delta)_{2,\alpha,n} = \sup_{0 < h \leq \delta} \|T^h f(x) - h^{2n} f(x)\|_{2,\alpha,n}, \delta > 0,$$

is called the modulus of continuity of the function f .

Let $W_{2,\phi}^r(\mathcal{B})$, $r = 0, 1, \dots$, denote the class of functions $f \in L_{\alpha,n}^2$ that have generalized derivatives satisfying the estimate

$$\omega(\mathcal{B}^r f, \delta)_{2,\alpha,n} = O(\phi(\delta)), \quad \delta \rightarrow 0,$$

where $\phi(x)$ is any nonnegative function given on $[0, \infty)$, and $\mathcal{B}^0 f = f, \mathcal{B}^r f = \mathcal{B}(\mathcal{B}^{r-1} f), r = 1, 2, \dots$

i.e.,

$$W_{2,\phi}^r(\mathcal{B}) = \{f \in L_{\alpha,n}^2, \mathcal{B}^r f \in L_{\alpha,n}^2 \text{ and } \omega(\mathcal{B}^r f, \delta)_{2,\alpha,n} = O(\phi(\delta)), \delta \rightarrow 0\}.$$

3 Main Results

The goal of this work is to prove several new estimates for the integral

$$J_N^2(f) = \int_N^{\infty} |\mathcal{F}_{\mathcal{B}} f(\lambda)|^2 d\mu_{\alpha+2n}(\lambda),$$

in certain classes of functions in $L_{\alpha,n}^2$.

Lemma 3.1. For $f \in W_{2,\phi}^r(\mathcal{B})$, we have

$$\|T^h \mathcal{B}^r f(x) - h^{2n} \mathcal{B}^r f(x)\|_{2,\alpha,n}^2 = h^{4n} \int_0^{\infty} \lambda^{4r} |j_{\alpha+2n}(\lambda h) - 1|^2 |\mathcal{F}_{\mathcal{B}} f(\lambda)|^2 d\mu_{\alpha+2n}(\lambda).$$

where $r = 0, 1, 2, \dots$

Proof. From formula (2.7), we obtain

$$\mathcal{F}_{\mathcal{B}}(\mathcal{B}^r f)(\lambda) = (-1)^r \lambda^{2r} \mathcal{F}_{\mathcal{B}} f(\lambda); r = 0, 1, \dots \tag{3.1}$$

By using the formula (2.6), we conclude that

$$\mathcal{F}_{\mathcal{B}}(T^h f - h^{2n} f)(\lambda) = h^{2n} (j_{\alpha+2n}(\lambda h) - 1) \mathcal{F}_{\mathcal{B}} f(\lambda). \tag{3.2}$$

Now by formulas (3.1), (3.2) and Plancherel equality, we have the result. \square

Theorem 3.2. *Given r and $f \in W_{2,\phi}^r(\mathcal{B})$. Then there exist constant $c > 0$ such that, for all $N > 0$,*

$$J_N(f) = O(N^{-2r+2n}\phi(c/N)).$$

Proof. Firstly, we have

$$J_N^2(f) \leq \int_N^\infty |j|d\mu + \int_N^\infty |1 - j|d\mu, \tag{3.3}$$

with $j = j_p(\lambda h)$, $p = \alpha + 2n$ and $d\mu = |\mathcal{F}_{\mathcal{B}}f(\lambda)|^2 d\mu_{\alpha+2n}(\lambda)$. The parameter $h > 0$ will be chosen in an instant.

In view of formulas (2.1) and (2.4), there exist a constant $c_1 > 0$ such that

$$|j| \leq c_1(\lambda h)^{-p-\frac{1}{2}}.$$

Then

$$\int_N^\infty |j|d\mu \leq c_1(hN)^{-p-\frac{1}{2}}J_N^2(f).$$

Choose a constant c_2 such that the number $c_3 = 1 - c_1c_2^{-p-\frac{1}{2}}$ is positif. Setting $h = c_2/N$ in the inequality (3.3), we have

$$c_3J_N^2(f) \leq \int_N^\infty |1 - j|d\mu. \tag{3.4}$$

By Hölder inequality the second term in (3.4) satisfies

$$\begin{aligned} \int_N^\infty |1 - j|d\mu &= \int_N^\infty |1 - j|.1.d\mu \\ &\leq \left(\int_N^\infty |1 - j|^2d\mu\right)^{1/2} \left(\int_N^\infty d\mu\right)^{1/2} \\ &\leq \left(\int_N^\infty \lambda^{-4r}|1 - j|^2\lambda^{4r}d\mu\right)^{1/2} J_N(f) \\ &\leq N^{-2r} \left(\int_N^\infty |1 - j|^2\lambda^{4r}d\mu\right)^{1/2} J_N(f). \end{aligned}$$

From Lemma 3.1, we conclude that

$$\int_N^\infty |1 - j|^2\lambda^{4r}d\mu \leq h^{-4n}\|T^h\mathcal{B}^r f(x) - h^{2n}\mathcal{B}^r f(x)\|_{2,\alpha,n}^2.$$

Therefore

$$\int_N^\infty |1 - j|d\mu \leq N^{-2r}h^{-2n}\|T^h\mathcal{B}^r f(x) - h^{2n}\mathcal{B}^r f(x)\|_{2,\alpha,n}J_N(f).$$

For $f \in W_{2,\phi}^r(\Lambda)$ there exist a constant $c_4 > 0$ such that

$$\|T^h\mathcal{B}^r f(x) - h^{2n}\mathcal{B}^r f(x)\|_{2,\alpha,n} \leq c_4\phi(h).$$

For $h = c_2/N$, we obtain

$$c_3J_N^2(f) \leq c_2^{-2n}N^{2n-2r}c_4\phi(c_2/N)J_N(f).$$

Hence

$$c_3^{2n}c_3J_N(f) \leq c_4N^{-2r+2n}\phi(c_2/N).$$

for all $N > 0$. The theorem is proved with $c = c_2$. \square

Theorem 3.3. Let $\phi(t) = t^\nu$, then

$$J_N(f) = O(N^{-2r-\nu+2n}) \Leftrightarrow f \in W_{2,\phi}^r(\mathcal{B}),$$

where, $r = 0, 1, \dots; 0 < \nu < 2$.

Proof. We prove sufficiency by using Theorem 3.2 let $f \in W_{2,\phi}^r(\mathcal{B})$ then

$$J_N(f) = O(N^{-2r-\nu+2n}).$$

To prove necessity let

$$J_N(f) = O(N^{-2r-\nu+2n})$$

i.e.

$$\int_N^\infty |\mathcal{F}_{\mathcal{B}}f(\lambda)|^2 d\mu_{\alpha+2n}(\lambda) = O(N^{-4r-2\nu+4n})$$

It is easy to show, that there exists a function $f \in L_{\alpha,n}^2$ such that $\mathcal{B}^r f \in L_{\alpha,n}^2$ and

$$\mathcal{B}^r f(x) = (-1)^r \int_0^\infty \lambda^{2r} \mathcal{F}_{\mathcal{B}}f(\lambda) \varphi_\lambda(x) d\mu_{\alpha+2n}(\lambda). \tag{3.5}$$

From formula (3.5) and Plancherel equality, we have

$$\|T^h \mathcal{B}^r f(x) - h^{2n} \mathcal{B}^r f(x)\|_{2,\alpha,n}^2 = h^{4n} \int_0^\infty \lambda^{4r} |j_{\alpha+2n}(\lambda h) - 1|^2 |\mathcal{F}_{\mathcal{B}}f(\lambda)|^2 d\mu_{\alpha+2n}(\lambda).$$

This integral is divided into two

$$\int_0^\infty = \int_0^N + \int_N^\infty = I_1 + I_2,$$

where $N = [h^{-1}]$, We estimate them separately.

From (2.2), we have the estimate

$$\begin{aligned} I_2 &\leq c_5 \int_{|\lambda| \geq N} \lambda^{4r} |\mathcal{F}_{\mathcal{B}}f(\lambda)|^2 d\mu_{\alpha+2n}(\lambda) \\ &= c_5 \sum_{l=0}^\infty \int_{N+l \leq |\lambda| \leq N+l+1} \lambda^{4r} |\mathcal{F}_{\mathcal{B}}f(\lambda)|^2 d\mu_{\alpha+2n}(\lambda) \\ &\leq c_5 \sum_{l=0}^\infty a_l (u_l - u_{l+1}), \end{aligned}$$

with $a_l = (N + l + 1)^{4r}$ and $u_l = \int_{|\lambda| \geq N+l} |\mathcal{F}_{\mathcal{B}}f(\lambda)|^2 d\mu_{\alpha+2n}(\lambda)$.

For all integers $m \geq 1$, the Abel transformation shows

$$\begin{aligned} \sum_{l=0}^m a_l (u_l - u_{l+1}) &= a_0 u_0 + \sum_{l=1}^m (a_l - a_{l-1}) u_l - a_m u_{m+1} \\ &\leq a_0 u_0 + \sum_{l=1}^m (a_l - a_{l-1}) u_l, \end{aligned}$$

because $a_m u_{m+1} \geq 0$. Moreover by the finite increments theorem, we have

$$a_l - a_{l-1} \leq 4r(N + l + 1)^{4r-1}$$

Furthermore by the hypothesis of f there exists $c_6 > 0$ such that, for all $N > 0$

$$J_N^2(f) \leq c_6 N^{-4r-2\nu+4n},$$

For $N \geq 1$, we have

$$\begin{aligned} \sum_{l=1}^m (a_l - a_{l-1}) u_l &\leq c_6 \left(1 + \frac{1}{N}\right)^{4r} N^{-2\nu+4n} + 4rc_6 \sum_{l=1}^m \left(1 + \frac{1}{N+l}\right)^{4r-1} (N+l)^{-1-2\nu+4n} \\ &\leq 2^{2r} c_6 N^{-2\nu+4n} + 4r2^{4r-1} c_6 \sum_{l=1}^m (N+l)^{-1-2\nu+4n}. \end{aligned}$$

Finally, by the integral comparison test we have

$$\sum_{l=1}^m (N+l)^{-1-2\nu+4n} \leq \int_N^\infty x^{-1-2\nu+4n} dx = \frac{1}{2\nu-4n} N^{-2\nu+4n}.$$

Letting $m \rightarrow \infty$ we see that, for $r \geq 0$ and $\nu > 0$, there exists a constant c_7 such that, for all $N \geq 1$ and for $h > 0$,

$$I_2 \leq c_7 N^{-2\nu+4n}.$$

Now, we estimate I_1 . From formula (2.3), we have

$$\begin{aligned} I_1 &\leq c_8 h^4 \int_{|\lambda| \leq N} \lambda^{4r+4} |\mathcal{F}_{\mathcal{B}} f(\lambda)|^2 d\mu_{\alpha+2n}(\lambda) \\ &= c_8 h^4 \sum_{l=0}^{N-1} \int_{l \leq |\lambda| \leq l+1} \lambda^{4r+4} |\mathcal{F}_{\mathcal{B}} f(\lambda)|^2 d\mu_{\alpha+2n}(\lambda) \\ &\leq c_8 h^4 \sum_{l=0}^{N-1} (l+1)^{4r+4} (v_l - v_{l+1}), \end{aligned}$$

with $v_l = \int_{|\lambda| \geq l} |\mathcal{F}_{\mathcal{B}} f(\lambda)|^2 d\mu_{\alpha+2n}(\lambda)$.

Using an Abel transformation and proceeding as with I_2 we obtain

$$\begin{aligned} I_1 &\leq c_8 h^4 \left(v_0 + \sum_{l=1}^{N-1} ((l+1)^{4r+4} - l^{4r+4}) v_l \right) \\ &\leq c_8 h^4 \left(v_0 + (4r+4)c_6 \sum_{l=1}^{N-1} (l+1)^{4r+3} l^{-4r-2\nu+4n} \right), \end{aligned}$$

since $v_l \leq c_6 l^{-4r-2\nu+4n}$ by hypothesis. From the inequality $l+1 \leq 2l$ we conclude

$$I_1 \leq c_8 h^4 \left(v_0 + c_9 \sum_{l=1}^{N-1} l^{3-2\nu+4n} \right).$$

As a consequence of a series comparison for $\mu \geq 1$ and $\mu < 1$ we have the inequality,

$$\mu \sum_{l=1}^{N-1} l^{\mu-1} < N^\mu, \text{ for } \mu > 0 \text{ and } N \geq 2.$$

If $\mu = 4 - 2\nu + 4n > 0$ for $\nu < 2$ then we obtain

$$I_1 \leq c_8 h^4 (v_0 + c_{10} N^{4-2\nu+4n}) \leq c_8 h^4 (v_0 + c_{10} h^{-4+2\nu-4n}),$$

since $N \leq 1/h$. If h is sufficiently small then $v_0 \leq c_{10} h^{-4+2\nu-4n}$. Then we have

$$I_1 \leq c_{11} h^{2\nu-4n}$$

Combining the estimates for I_1 and I_2 gives

$$\|T^h \mathcal{B}^r f(x) - h^{2n} \mathcal{B}^r f(x)\|_{2,\alpha,n} = O(h^\nu),$$

The necessity is proved. \square

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