Palestine Journal of Mathematics

Vol. 6(1)(2017) , 211–218

On parity combination cordial graphs

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Communicated by Ayman Badawi

MSC 2010 Classifications: Primary 05C78.

Keywords and phrases: path, cycle, star, helms, dragon.

Abstract. Let G be a (p, q) graph. Let f be an injective map from $V(G)$ to $\{1, 2, \dots, p\}$. For each edge xy, assign the label $\binom{x}{y}$ or $\binom{y}{x}$ according as $x > y$ or $y > x$. f is called a parity combination cordial labeling (PCC-labeling) if f is a one to one map and $|e_f(0) - e_f(1)| \leq 1$ where $e_f(0)$ and $e_f(1)$ denote the number of edges labeled with an even number and odd number respectively. A graph with a parity combination cordial labeling is called a parity combination cordial graph (PCC-graph). In this paper, we investigate the parity combination cordial labeling behavior of helms, P_n^2, dragon, $C_n \hat{\circ} K_{1,m}$ and some more graphs.

1 Introduction

All graphs in this paper are finite, undirected and simple. The symbols $V(G)$ and $E(G)$ will denote the vertex set and edge set of a graph G. A general reference for graph theoretic ideas can be seen in [3]. A labeling of a graph is a map that carries graph elements to numbers (usually to positive or non-negative integers). Most graph labeling methods trace their origin to one introduced by Rosa [4] in year 1967. Labeled graphs serves as a useful mathematical model for a broad range of application such as coding theory, X-ray crystallography analysis, communication network addressing systems, astronomy, radar, circuit design and database management [2]. The union of two graphs G_1 and G_2 is the graph $G_1 \cup G_2$ with $V(G_1 \cup G_2) = V(G_1) \cup V(G_2)$ and $E(G_1 \cup G_2) = E(G_1) \cup E(G_2)$. In 1980, Cahit [1] introduced the cordial labeling of graphs. In [5], ponraj et al. introduced a notion, called combination parity cordial labeling. In this paper we present combination parity cordial labelings for helms, P_n^2, dragon, $C_n \hat{\circ} K_{1,m}$ and some more graphs.

2 Some basic results and definitions

Definition 2.1. Let G be a (p, q) graph. Let f be an injective map from $V(G)$ to $\{1, 2, \dots, p\}$. For each edge xy, assign the label $\binom{x}{y}$ or $\binom{y}{x}$ according as $x > y$ or $y > x$. f is called a parity combination cordial labeling (PCC-labeling) if f is a one to one map and $|e_f(0) - e_f(1)| \leq 1$ where $e_f(0)$ and $e_f(1)$ denote the number of edges labeled with an even number and odd number respectively. A graph with a parity combination cordial labeling is called a parity combination cordial graph (PCC-graph).

Result 2.2. $\binom{n}{n-1} = \binom{n}{1}$ is even if n is even and odd if n is odd.

Result 2.3. $\binom{n}{2}$ is even if $n \equiv 0, 1 \pmod 4$ and odd if $n \equiv 2, 3 \pmod 4$.

Proof. **Case 1.** $n \equiv 0 \pmod 4$.
Let $n = 4t$. Then $\binom{n}{2} = \frac{4t(4t-1)}{2} = 2t(4t-1)$. Hence $\binom{n}{2}$ is even.
Case 2. $n \equiv 1 \pmod 4$.
Let $n = 4t+1$. Then $\binom{n}{2} = \frac{(4t+1)4t}{2} = 2t(4t+1)$. Hence $\binom{n}{2}$ is even.
Case 3. $n \equiv 2 \pmod 4$.
Let $n = 4t+2$. Then $\binom{n}{2} = \frac{(4t+2)(4t+1)}{2} = (2t+1)(4t+1)$. Hence $\binom{n}{2}$ is odd.
Case 4. $n \equiv 3 \pmod 4$.
Let $n = 4t+3$. Then $\binom{n}{2} = \frac{(4t+3)(4t+2)}{2} = (2t+1)(4t+3)$. Hence $\binom{n}{2}$ is odd. □

Result 2.4. If $n \equiv 0 \pmod 4$ then $\binom{n}{3}$ is even.

Proof. **Case 1.** $n \equiv 0 \pmod{12}$.
Let $n = 12t$. Then $\binom{n}{3} = \frac{12t(12t-1)(12t-2)}{6} = 2t(12t-1)(12t-2)$. Hence $\binom{n}{3}$ is even.
Case 2. $n \equiv 4 \pmod{12}$.
Let $n = 12t + 4$. Then $\binom{n}{3} = \frac{(12t+4)(12t+3)(12t+2)}{6} = 2(3t+1)(4t+1)(12t+2)$. Hence $\binom{n}{3}$ is even.
Case 3. $n \equiv 8 \pmod{12}$.
Let $n = 12t + 8$. Then $\binom{n}{3} = \frac{(12t+8)(12t+7)(12t+6)}{6} = 2(3t+2)(12t+7)(4t+2)$. Hence $\binom{n}{3}$ is even. □

Result 2.5. $\binom{n}{r} = \binom{n}{n-r}$.

Definition 2.6. The graph P_n^2 is obtained from the path P_n by adding edges that joins all vertices u and v with $d(u, v) = 2$.

Definition 2.7. The helm H_n is the graph obtained from a wheel by attaching a pendant edge at each vertex of the n-cycle.

Definition 2.8. The bistar $B_{m,n}$ is the graph obtained by making adjacent the two central vertices of $K_{1,m}$ and $K_{1,n}$.

Definition 2.9. The dragon $C_m@P_n$ is the graph obtained by unifying an end vertex of a path P_n and a vertex of a cycle C_n.

Definition 2.10. The graph $C_n\widehat{\circ}K_{1,m}$ is obtained from C_n and $K_{1,m}$ by unifying a vertex of C_n and the central vertex of $K_{1,m}$.

Definition 2.11. The graph $C_n\widetilde{\circ}K_{1,m}$ is obtained from C_n and $K_{1,m}$ by unifying a vertex of C_n and a pendent vertex of $K_{1,m}$.

Definition 2.12. Two even cycles of the same order, say C_n, sharing a common vertex with m pendent edges attached at the common vertex is called a butterfly graph $By_{m,n}$.

3 Main Results

First we look into the graph $G \cup P_n$ where G is a parity combination cordial graph.

Theorem 3.1. Let G be a (p, q) parity combination cordial graph. Then $G \cup P_n$ is also parity combination cordial if $n \neq 2, 4$.

Proof. Let $P_n : u_1u_2 \ldots u_n$ be the path and $v_1, v_2, \ldots, v_p$ be the vertices of G. Since G is a parity combination cordial graph, there exists a parity combination cordial labeling, say f.
Therefore $e_f(0) = e_f(1) = \frac{q}{2}$ if q is even, and if q is odd then $e_f(0) = \frac{q+1}{2}$ and $e_f(1) = \frac{q-1}{2}$ (or) $e_f(0) = \frac{q-1}{2}$ and $e_f(1) = \frac{q+1}{2}$
Now, define an injective function $g : V(G \cup P_n) \to \{1, 2, \ldots, p+n\}$ by $g(v_i) = f(v_i)$, $1 \leq i \leq p$ and $g(u_j) = p + j$, $1 \leq j \leq n$.
Case 1. p is even.

Then $p + 1$ is odd. Now $\binom{p+i+1}{p+i} = p + i + 1$ where $1 \leq i \leq n - 1$. But p is even. Therefore the path contributes $\frac{n-1}{2}$ zeros and $\frac{n-1}{2}$ 1's if n is odd and $\frac{n}{2}$ zeros, $\frac{n}{2} - 1$ 1's if n is even.

$$\text{(i.e) Number of edges with label zero in } P_n = \begin{cases} \frac{n-1}{2} & \text{if} \quad \text{n is odd} \\ \frac{n}{2} & \text{if} \quad \text{n is even} \end{cases}$$

$$\text{Number of edges with label 1 in } P_n = \begin{cases} \frac{n-1}{2} & \text{if} \quad \text{n is odd} \\ \frac{n}{2} - 1 & \text{if} \quad \text{n is even} \end{cases}$$

Subcase 1a. q is even. If n is odd, then

$$|e_g(0) - e_g(1)| = \left|\left(\frac{q}{2} + \frac{n-1}{2}\right) - \left(\frac{q}{2} + \frac{n-1}{2}\right)\right| = 0.$$

For the case when n is even, we have $|e_g(0) - e_g(1)| = \left|\left(\frac{n}{2} + \frac{q}{2}\right) - \left(\frac{n}{2} - 1 + \frac{q}{2}\right)\right| = 1$.
Subcase 1b. q is odd. Here we have the following possible cases in G.

(i) $e_f(0) = \frac{q+1}{2}$ and $e_f(1) = \frac{q-1}{2}$.

(ii) $e_f(0) = \frac{q-1}{2}$ and $e_f(1) = \frac{q+1}{2}$.

Consider the first case. Suppose n is odd, then

$$|e_g(0) - e_g(1)| = \left|\left(\frac{n-1}{2} + \frac{q+1}{2}\right) - \left(\frac{n-1}{2} + \frac{q-1}{2}\right)\right| = 1$$

If n is even and $p \equiv 0 \pmod 4$ then relabel the vertices u_2, u_3 by $p+3$, $p+2$ respectively. Now $\binom{p+3}{p+1} = \binom{p+3}{2}$ and since $p \equiv 0 \pmod 4$, $\binom{p+3}{2}$ is odd and hence $\binom{p+3}{p+1}$ is odd. Also $\binom{p+3}{p+2} = \binom{p+3}{1} = p+3$, which is odd, $\binom{p+4}{p+2} = \binom{p+4}{2}$, and since $p \equiv 0 \pmod 4$, $\binom{p+4}{2}$ is even. Hence $e_g(0) = \frac{q+1}{2} + \frac{n}{2} - 1$, $e_g(1) = \frac{q-1}{2} + \frac{n}{2}$. This forces $|e_g(0) - e_g(1)| = 0$.

If n is even and $p \equiv 2 \pmod 4$ then assign the labels as in before and then interchange the labels of the vertices u_2 and u_3. That is, label the vertices of P_n as in previous case, $p \equiv 0 \pmod 4$. Now $\binom{p+3}{p+1} = \binom{p+3}{2}$. Since $p \equiv 2 \pmod 4$, $p+3 \equiv 1 \pmod 4$ and by the result 2.3, $\binom{p+3}{p+1} = \binom{p+3}{2}$ is even. Also $\binom{p+3}{p+2} = \binom{p+3}{1} = p+3$, which is odd. Finally, $\binom{p+4}{p+2} = \binom{p+4}{2}$ and since $p \equiv 2 \pmod 4$, $p+4 \equiv 2 \pmod 4$ and therefore $\binom{p+4}{2}$ is odd. Hence $e_g(0) = \frac{q+1}{2} + \frac{n}{2} - 1$, $e_g(1) = \frac{q-1}{2} + \frac{n}{2}$. This implies $|e_g(0) - e_g(1)| = 0$.

Now we look into the second case. If n is odd, then

$$|e_g(0) - e_g(1)| = \left|\left(\frac{n-1}{2} + \frac{q-1}{2}\right) - \left(\frac{n-1}{2} + \frac{q+1}{2}\right)\right| = 1.$$

For even values of n, we have $|e_g(0) - e_g(1)| = \left|\left(\frac{n}{2} + \frac{q-1}{2}\right) - \left(\frac{n}{2} + \frac{q+1}{2} - 1\right)\right| = 0$.

Case 2. p is odd.

In this case $p+1$ is even. Hence $\binom{p+i+1}{p+i} = p+i+1$ where $1 \le i \le n-1$. But p is odd. Hence the path P_n contributes $\frac{n-1}{2}$ zero's and $\frac{n-1}{2}$ 1's if n is odd and $\frac{n}{2} - 1$ zero's, $\frac{n}{2}$ 1's if n is even.

$$\text{(i.e) Number of edges with the label zero in } P_n = \begin{cases} \frac{n-1}{2} & \text{if} \quad \text{n is odd} \\ \frac{n}{2} - 1 & \text{if} \quad \text{n is even} \end{cases}$$

$$\text{Number of edges with the label 1 in } P_n = \begin{cases} \frac{n-1}{2} & \text{if} \quad \text{n is odd} \\ \frac{n}{2} & \text{if} \quad \text{n is even} \end{cases}$$

Subcase 2a. q is odd. Here we have the following possible cases in G.

(i) $e_f(0) = \frac{q+1}{2}$ and $e_f(1) = \frac{q-1}{2}$.

(ii) $e_f(0) = \frac{q-1}{2}$ and $e_f(1) = \frac{q+1}{2}$.

Consider the first case. Suppose n is odd, then

$$|e_g(0) - e_g(1)| = \left|\left(\frac{n-1}{2} + \frac{q+1}{2}\right) - \left(\frac{n-1}{2} + \frac{q-1}{2}\right)\right| = 1.$$

For the case when n is even,

$$|e_g(0) - e_g(1)| = \left|\left(\frac{n}{2} - 1 + \frac{q+1}{2}\right) - \left(\frac{n}{2} + \frac{q-1}{2}\right)\right| = 0.$$

Now consider the second case. If n is odd, then
$|e_g(0) - e_g(1)| = \left|\left(\frac{n-1}{2} + \frac{q-1}{2}\right) - \left(\frac{n-1}{2} + \frac{q+1}{2}\right)\right| = 1.$

Suppose n is even and $p \equiv 1 \pmod 4$ then relabel u_1, u_2, u_3 by $p+3$, $p+1$, $p+2$ respectively. Since $p \equiv 1 \pmod 4$, $p+3 \equiv 0 \pmod 4$. Hence $\binom{p+3}{p+1} = \binom{p+3}{2}$ is even. Also $\binom{p+2}{p+1} = p+2$, which is odd and $\binom{p+4}{p+2} = \binom{p+4}{2}$. But $p+4 \equiv 1 \pmod 4$. Therefore $p+4 \equiv 2 \pmod 4$ is even. Hence $e_g(0) = \frac{q-1}{2} + \frac{n}{2}$ and $e_g(1) = \frac{q+1}{2} + \frac{n}{2} - 1$.

If n is even and $p \equiv 3 \pmod 4$, $n > 4$, then relabel the vertices u_1, u_2, u_3, u_4, u_5 by $p+1$, $p+3$, $p+5$, $p+2$, $p+4$ respectively. Since $p+3 \equiv 2 \pmod 4$, by the result 2.3, $\binom{p+3}{p+1} = \binom{p+3}{2}$

is odd. Since $p+5 \equiv 0 \pmod 4$, by the result 2.3, $\binom{p+5}{p+3} = \binom{p+5}{2}$ is even. Since $p+5 \equiv 0 \pmod 4$, by the result 2.4, $\binom{p+5}{p+2} = \binom{p+5}{3}$ is even. Since $p+4 \equiv 3 \pmod 4$, by the result 2.3, $\binom{p+4}{p+2} = \binom{p+4}{2}$ is odd. Since $p+6 \equiv 1 \pmod 4$, by the result 2.3, $\binom{p+6}{p+4} = \binom{p+6}{2}$ is even. This implies $e_g(0) = \frac{q-1}{2} + \frac{n}{2}$ and $e_g(1) = \frac{q+1}{2} + \frac{n}{2} - 1$. Hence $|e_g(0) - e_g(1)| = 0$.
Subcase 2b. q is even. If n is even, then

$$|e_g(0) - e_g(1)| = \left|\left(\frac{n}{2} + \frac{q}{2} - 1\right) - \left(\frac{n}{2} + \frac{q}{2}\right)\right| = 1.$$

Suppose n is odd, then $|e_g(0) - e_g(1)| = \left|\left(\frac{n-1}{2} + \frac{q}{2}\right) - \left(\frac{n-1}{2} + \frac{q}{2}\right)\right| = 0$.

By the cases 1 and 2, $G \cup P_n$ is parity combination cordial, if $n \neq 2, 4$. □

We now investigate the square of a path.

Theorem 3.2. The graph P_n^2 is parity combination cordial.

Proof. Let $V(P_n^2) = \{u_i : 1 \leq i \leq n\}$ and $E(P_n^2) = \{u_i u_{i+1} : 1 \leq i \leq n-1\} \cup \{u_i u_{i+2} : 1 \leq i \leq n-2\}$. Clearly the number of vertices and edges of P_n^2 are n and $2n-3$ respectively. Define a function $f : V(P_n^2) \to \{1, 2, \ldots, n\}$ by $f(u_i) = i$, $1 \leq i \leq n$. Using the results 2.2 and 2.3, it is evident that $e_f(0) = n-1$ and $e_f(1) = n-2$.

Hence P_n^2 is a parity combination cordial graph. □

Next investigation is about helms and dragon.

Theorem 3.3. The helm H_n is parity combination cordial.

Proof. Let $V(H_n) = \{u\} \cup \{u_i, v_i : 1 \leq i \leq n\}$ and $E(H_n) = \{u_i u_{i+1} : 1 \leq i \leq n-1\} \cup \{uu_i, u_i v_i : 1 \leq i \leq n\}$. The number of vertices and edges of H_n are $2n+1$ and $3n$ respectively.
Case 1. n is odd.
Define a map $f : V(H_n) \to \{1, 2, \ldots, 2n+1\}$ by $f(u) = 1$,

$$\begin{aligned} f(u_i) &= 2i, & 1 \leq i \leq n \\ f(v_i) &= 2i+1, & 1 \leq i \leq n. \end{aligned}$$

From the results 2.2 and 2.3, we notice that $e_f(0) = \frac{3n-1}{2}$ and $e_f(1) = \frac{3n+1}{2}$.
Case 2. n is even.
Assign the labels to the vertices of H_n as in case 1. Then interchange the labels of the vertices u_3 and v_3. In this case $e_f(0) = e_f(1) = \frac{3n}{2}$.

Hence H_n is a parity combination cordial graph. □

Example 3.4. A parity combination cordial labeling of H_8 is given in FIGURE 1.

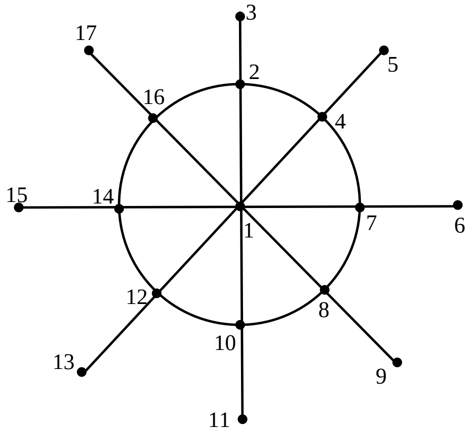

Figure 1.

Theorem 3.5. The dragon $C_m@P_n$ is a parity combination cordial graph.

Proof. Let $v_1, v_2, \dots, v_m$ be the vertices of C_m and $u_1, u_2, \dots, u_n$ be the vertices of P_n. Without loss of generality unify the vertices u_1 and v_1.
Case 1. m and n are odd.
Define an injective map $f : V(C_m@P_n) \to \{1, 2, \dots, m+n-1\}$ as follows:

$$\begin{aligned} f(v_i) &= i-1, & 2 \le i \le m \\ f(u_i) &= m-1+i, & 1 \le i \le n. \end{aligned}$$

Case 2. m is odd and n is even.
Assign the labels to the vertices of the dragon, as in case 1.
Case 3. m is even and n is odd.
Assign the labels to the vertices as in case 1. Then interchange the labels of the vertices u_3 and u_4.
Case 4. m and n are even.
Assign the labels to the vertices of the dragon, as in case 1.
Table 1 shows that f is a parity combination cordial labeling of $C_m@P_n$.

Nature of m and n	$e_f(0)$	$e_f(1)$
m and n are odd	$\frac{m+n}{2} - 1$	$\frac{m+n}{2}$
m is odd and n is even	$\frac{m+n-1}{2}$	$\frac{m+n-1}{2}$
m is even and n is odd	$\frac{m+n-1}{2}$	$\frac{m+n-1}{2}$
m and n are even	$\frac{m+n}{2}$	$\frac{m+n}{2} - 1$

Table 1.

□

Now we investigate the parity combination cordial labeling behavior of bistar and butterfly graphs.

Theorem 3.6. The bistar $B_{m,n}$ is parity combination cordial.

Proof. Let $V(B_{m,n}) = \{u, v, u_i, v_j : 1 \le i \le m, 1 \le j \le n\}$ and $E(B_{m,n}) = \{uv, uu_i, vv_j : 1 \le i \le m, 1 \le j \le n\}$.
Case 1. $m \equiv 0, 4 \pmod{12}$ and $m+n \equiv 3 \pmod 4$.
Define a map $f : V(B_{m,n}) \to \{1, 2, \dots, m+n+2\}$ by $f(u) = 1$, $f(v) = 2$, $f(v_1) = n+3$, $f(u_1) = 3$,

$$\begin{aligned} f(u_i) &= n+2+i, & 2 \le i \le m \\ f(v_j) &= j+2, & 2 \le j \le n. \end{aligned}$$

In this case $e_f(0) = e_f(1) = \frac{m+n+1}{2}$.
Case 2. $m \equiv 8 \pmod{12}$ and $m+n \equiv 1 \pmod 4$.
Similar to case 1.
Case 3. $m \equiv 1, 3 \pmod 6$ and $m+n \equiv 1 \pmod 4$.
Define a map $f : V(B_{m,n}) \to \{1, 2, \dots, m+n+2\}$ by $f(u) = 1$, $f(v) = 2$,

$$\begin{aligned} f(u_i) &= n+2+i, & 1 \le i \le m \\ f(v_j) &= j+2, & 1 \le j \le n. \end{aligned}$$

In this case $e_f(0) = e_f(1) = \frac{m+n+1}{2}$.
Case 4. $m \equiv 2 \pmod{12}$ and $m+n \equiv 1 \pmod 4$.
Similar to case 3.
Case 5. $m \equiv 5 \pmod 6$ and $m+n \equiv 3 \pmod 4$.
Similar to case 3.

Case 6. $m \equiv 6, 10 \pmod{12}$ and $m + n \equiv 3 \pmod 4$.
Similar to case 3.
Case 7. m and n are not in the previous cases.
Define a map $f : V(B_{m,n}) \to \{1, 2, \dots, m + n + 2\}$ by $f(u) = 1$, $f(v) = 2$,

$$\begin{aligned} f(u_i) &= i + 2, & 1 \le i \le m \\ f(v_j) &= m + 2 + j, & 1 \le j \le n. \end{aligned}$$

It is easy to verify that $|e_f(0) - e_f(1)| \le 1$. □

Theorem 3.7. The butterfly graph $By_{m,n}$ is parity combination cordial.

Proof. Let $u_1u_2 \dots u_nu_1$ and $v_1v_2 \dots v_nv_1$ be the two copies of the cycle C_n. Without loss of generality, unify the vertices u_1 and v_1. Let $w_1, w_2, \dots, w_m$ be the pendent vertices.
Case 1. $n \equiv 1 \pmod 2$ and $m \equiv 1 \pmod 2$.
Define a one to one map $f : V(By_{m,n}) \to \{1, 2, \dots, 2n + m - 1\}$ by

$$\begin{aligned} f(u_i) &= i, & 1 \le i \le n \\ f(v_i) &= n + i - 1, & 2 \le i \le n \\ f(w_i) &= 2n - 1 + i, & 1 \le i \le m. \end{aligned}$$

Case 2. $n \equiv 1 \pmod 2$ and $m \equiv 0 \pmod 2$.
Subcase 2a. $n = 3$.
Assign the label 1 to u_1, then put the labels 2, 3 to the vertices u_2, u_3 respectively. For the other side vertices v_2, v_3, we put the labels 6 and 5 respectively. Now, the remaining vertices are labeled with the labels from $\{4, 7, 8, \dots, 2n + m - 1\}$ in any order.
Subcase 2b. $n > 3$.
Assign the labels to the vertices as in case 1. Then relabel the vertices u_2, u_3 and u_4 with the labels 3, 4 and 2 respectively.
Case 3. $n \equiv 0 \pmod 2$ and $m \equiv 1 \pmod 2$.
Assign the labels to the vertices as in case 1. Then interchange the labels of the vertices u_2 and u_3.
Case 4. $n \equiv 0 \pmod 2$ and $m \equiv 0 \pmod 2$.
Similar to case 1.
Table 2 establish that f is a parity combination cordial labeling of $By_{m,n}$.

Values of m and n	$e_f(0)$	$e_f(1)$
$n \equiv 1 \pmod 2$ and $m \equiv 1 \pmod 2$	$\frac{2n+m-1}{2}$	$\frac{2n+m+1}{2}$
$n = 3$ and $m \equiv 0 \pmod 2$	$\frac{m+6}{2}$	$\frac{m+6}{2}$
$n \equiv 1 \pmod 2$, $m \equiv 0 \pmod 2$, and $n > 3$	$\frac{2n+m}{2}$	$\frac{2n+m}{2}$
$n \equiv 0 \pmod 2$, $m \equiv 1 \pmod 2$	$\frac{2n+m+1}{2}$	$\frac{2n+m-1}{2}$
$n \equiv 0 \pmod 2$, $m \equiv 0 \pmod 2$	$\frac{2n+m}{2}$	$\frac{2n+m}{2}$

Table 2.

□

Final investigation is about the graphs which are obtained from a cycle and a star.

Theorem 3.8. The graph $C_n \widehat{\circ} K_{1,m}$ is a parity combination cordial graph.

Proof. Let $u_1u_2 \dots u_nu_1$ be the cycle C_n and let u be the central vertex of the star $K_{1,m}$ and v_i $(1 \le i \le m)$ be the pendent vertices. Now unify the vertices u and u_1.
Case 1. n is even and m is odd.

Define an injective map $f : V(C_n \widehat{\circ} K_{1,m}) \to \{1, 2, \ldots, m+n\}$ as follows:

$$\begin{aligned} f(v_i) &= i, & 1 \le i \le n \\ f(u_i) &= n+i, & 1 \le i \le m. \end{aligned}$$

Case 2. m and n are even.
Assign the labels to the vertices as in case 1. Then interchange the labels of the vertices u_2 and u_3.
Case 3. m and n are odd.
Assign the labels to the vertices as in case 1.
Case 4. m is odd and n is even.
Assign the labels to the vertices as in case 1.
Table 3 shows that f is a parity combination cordial labeling of $C_n \widehat{\circ} K_{1,m}$.

Nature of m and n	$e_f(0)$	$e_f(1)$
n is even and m is odd	$\frac{m+n+1}{2}$	$\frac{m+n-1}{2}$
m and n are even	$\frac{m+n}{2}$	$\frac{m+n}{2}$
m and n are odd	$\frac{m+n}{2}$	$\frac{m+n}{2}$
m is odd and n is even	$\frac{m+n-1}{2}$	$\frac{m+n+1}{2}$

Table 3.

□

Theorem 3.9. The graph $C_n \widetilde{\circ} K_{1,m}$ is parity combination cordial.

Proof. Let $u_1 u_2 \ldots u_n u_1$ be the cycle C_n and let v be the central vertex of the star $K_{1,m}$ and v_i $(1 \le i \le m)$ be the pendent vertices. Now unify the vertices v_1 and u_1.
Case 1. $n \equiv 0 \pmod 4$ and $m \equiv 1 \pmod 2$.
Define an injective map $f : V(C_n \widetilde{\circ} K_{1,m}) \to \{1, 2, \ldots, m+n\}$ by $f(v) = 1$,

$$\begin{aligned} f(u_i) &= i+1, & 1 \le i \le n \\ f(v_i) &= n+i, & 2 \le i \le m. \end{aligned}$$

Case 2. $n \equiv 0 \pmod 4$ and $m \equiv 0 \pmod 2$.
Assign the labels to the vertices as in case 1. Then interchange the labels of the vertices u_1 and u_2.
Case 3. $n \equiv 1 \pmod 4$ and $m \equiv 1 \pmod 2$.
Similar to case 1.
Case 4. $n \equiv 1 \pmod 4$ and $m \equiv 0 \pmod 2$.
Similar to case 1.
Case 5. $n \equiv 2 \pmod 4$ and $m \equiv 1 \pmod 2$.
Similar to case 1.
Case 6. $n \equiv 2 \pmod 4$ and $m \equiv 0 \pmod 2$.
Similar to case 1.
Case 7. $n \equiv 3 \pmod 4$ and $m \equiv 1 \pmod 2$.
Similar to case 2.
Case 8. $n \equiv 3 \pmod 4$ and $m \equiv 0 \pmod 2$.
Similar to case 1.
The table 4 shows that f is a parity combination cordial labeling of $C_n \widetilde{\circ} K_{1,m}$.

□

Example 3.10. The graph $C_7 \widetilde{\circ} K_{1,9}$ is given in FIGURE 2.

Values of m and n	$e_f(0)$	$e_f(1)$
$n \equiv 0 \pmod 4$ and $m \equiv 1 \pmod 2$	$\frac{m+n+1}{2}$	$\frac{m+n-1}{2}$
$n \equiv 0 \pmod 4$ and $m \equiv 0 \pmod 2$	$\frac{m+n}{2}$	$\frac{m+n}{2}$
$n \equiv 1 \pmod 4$ and $m \equiv 1 \pmod 2$	$\frac{m+n}{2}$	$\frac{m+n}{2}$
$n \equiv 1 \pmod 4$ and $m \equiv 0 \pmod 2$	$\frac{m+n-1}{2}$	$\frac{m+n+1}{2}$
$n \equiv 2 \pmod 4$ and $m \equiv 1 \pmod 2$	$\frac{m+n-1}{2}$	$\frac{m+n+1}{2}$
$n \equiv 2 \pmod 4$ and $m \equiv 0 \pmod 2$	$\frac{m+n}{2}$	$\frac{m+n}{2}$
$n \equiv 3 \pmod 4$ and $m \equiv 1 \pmod 2$	$\frac{m+n}{2}$	$\frac{m+n}{2}$
$n \equiv 3 \pmod 4$ and $m \equiv 0 \pmod 2$	$\frac{m+n+1}{2}$	$\frac{m+n-1}{2}$

Table 4.

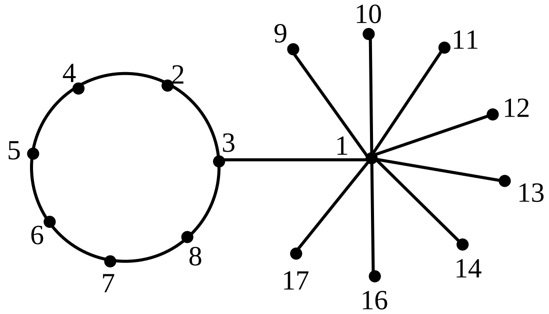

Figure 2.

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Received: October 7, 2015.

Accepted: March 22, 2016.