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MULTIPLICATIVE GENERALIZED DERIVATIONS ON LIE IDEALS IN SEMIPRIME RINGS I

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Abstract. Let R be a semiprime ring and U is an Lie ideal of R such that $U \not\subseteq Z(R)$. A map $F : R \to R$ is called a multiplicative generalized derivation if there exists a map $d : R \to R$ such that $F(xy) = F(x)y + xd(y)$, for all $x, y \in R$. In the present paper, we shall prove that d is commuting map on U if any one of the following holds: i) $F([x, y]) = 0$, ii) $F([x, y]) = \pm[x, y]$, iii) $F([x, y]) = \pm(xoy) = 0$, iv) $F(xoy) = 0$, v) $F(xoy) = \pm(xoy)$, vi) $F(xoy) = \pm[x, y]$, vii) $F([x, y]) = \pm[F(x), y]$, viii) $F(xoy) = \pm(F(x)oy)$, ix) $F(xy) \pm xy \in Z$, x) $F(xy) \pm yx \in Z$, xi) $F(xy) \pm [x, y] \in Z$, xii) $F(xy) \pm (xoy) \in Z$, for all $x, y \in U$.

1 Introduction

Let R will be an associative ring with center Z. For any $x, y \in R$, as usual $[x, y] = xy - yx$ and $xoy = xy + yx$ will denote the well-known Lie and Jordan products respectively. Recall that a ring R is prime if for $x, y \in R$, $xRy = 0$ implies either $x = 0$ or $y = 0$ and R is semiprime if for $x \in R$, $xRx = 0$ implies $x = 0$. An additive subgroup U of R is said to be a Lie ideal of R if $[u, r] \in U$, for all $u \in U, r \in R$. An additive mapping $d : R \to R$ is called a derivation if $d(xy) = d(x)y + xd(y)$ holds for all $x, y \in R$. For a fixed $a \in R$, the mapping $I_a : R \to R$ given by $I_a(x) = [a, x]$ is a derivation which is said to be an inner derivation. An additive function $F : R \to R$ is called a generalized inner derivation if $F(x) = ax + xb$ for fixed $a, b \in R$. For such a mapping F, it is easy to see that

$$F(xy) = F(x)y + x[y, b] = f(x)y + xI_b(y) \text{ for all } x, y \in R.$$

This observation leads to the following definition given by M. Bresar in [5]: An additive mapping $F : R \to R$ is called a generalized derivation if there exists a derivation $d : R \to R$ such that

$$F(xy) = F(x)y + xd(y), \quad \text{for all } x, y \in R.$$

Familiar examples of generalized derivations are derivations and generalized inner derivations, and the later include left multipliers and right multipliers (i.e., $F(xy) = F(x)y$ for all $x, y \in R$).

The commutativity of prime or semiprime rings with derivation was initiated by Posner in [13]. Thereafter, several authors have proved commutativity theorems of prime or semiprime rings with derivations. In [6], the notion of multiplicative derivation was introduced by Daif motivated by Martindale in [12]. $d : R \to R$ is called a multiplicative derivation if $d(xy) = d(x)y + xd(y)$ holds for all $x, y \in R$. These maps are not additive. In [11], Goldman and Semrl gave the complete description of these maps. We have $R = C[0, 1]$, the ring of all continuous (real or complex valued) functions and define a map $d : R \to R$ such as

$$d(f)(x) = \left\{ \begin{array}{ll} f(x)\log|f(x)|, & f(x) \neq 0 \\ 0, & \text{otherwise} \end{array} \right\}.$$

It is clear that d is multiplicative derivation, but d is not additive. Inspired by the definition multiplicative derivation, the notion of multiplicative generalized derivation was extended by Daif and Tamman El-Sayiad in [8] as follows:

$F : R \to R$ is called a multiplicative generalized derivation if there exists a derivation $d : R \to R$ such that $F(xy) = F(x)y + xd(y)$, for all $x, y \in R$. Dhara and Ali gave a slight generalization of this definition taking g is any map (not necessarily an additive map or a derivation) in [9]. Every generalized derivation is a multiplicative generalized derivation. But the converse is not ture in general (see example [9, Example 1.1]). Hence, one may observe that the concept of multiplicative generalized derivations includes the concept of derivations, multiplicative derivation and the left multipliers. So, it should be interesting to extend some results concerning these notions to multiplicative generalized derivations. But there are only few papers about this subject. (see [8], [9], [10] for a partial bibliography).

In [7], Daif and Bell proved that R is semiprime ring, U is a nonzero ideal of R and d is a derivation of R such that $d([x, y]) = \pm[x, y]$, for all $x, y \in U$, then $U \subseteq Z$. This theorem considered for generalized derivations by Quadri et al. in [15] and extended by Dhara proving $F([x, y]) \pm [x, y] \in Z$, for all $x, y \in U$, when F is a generalized derivation of R in [9].

On the other hand, in [2], Ashraf and Rehman showed that R is prime ring with a nonzero ideal U of R and d is a derivation of R such that $d(xy) \pm xy \in Z$, for all $x, y \in U$, then R is commutative. Ashraf et al. proved this result for a generalized derivation of R in [1].

In the present paper, we shall extend above results for Lie ideals of semiprime rings with multiplicative generalized derivation of R. Also, we will investigate these results for Jordan product.

2 Results

We will make some extensive use of the basic commutator identities:

$[x, yz] = y[x, z] + [x, y]z$

$[xy, z] = [x, z]y + x[y, z]$

$xo(yz) = (xoy)z - y[x, z] = y(xoz) + [x, y]z$

$(xy)oz = x(yoz) - [x, z]y = (xoz)y + x[y, z]$.

Moreover, we shall require the following lemmas.

Lemma 2.1. *[4, Lemma 4] If $U \not\subseteq Z(R)$ is a Lie ideal of a 2-torsion free prime ring R and $a, b \in R$ such that $aUb = \{0\}$, then $a = 0$ or $b = 0$.*

Lemma 2.2. *[4, Lemma 2]Let R be a prime ring with characteristic not two. If U a noncentral Lie ideal of R , then $C_R(U) = Z$.*

Lemma 2.3. *[4, Lemma 5]Let R be a prime ring with characteristic not two and U a nonzero Lie ideal of R. If d is a nonzero derivation of R such that $d(U) = 0$, then $U \subseteq Z$.*

Lemma 2.4. *[3, Theorem 7]Let R be a prime ring with characteristic not two and U a nonzero Lie ideal of R. If d is a nonzero derivation of R such that $[d(u), u] \in Z$ for all $u \in U$, then $U \subseteq Z$.*

Lemma 2.5. *[2, Lemma 2]Let R be a 2-torsion free semiprime ring, U is a Lie ideal of R such that $U \not\subseteq Z(R)$ and $a \in U$. If $aUa = 0$, then $a^2 = 0$ and there exists a nonzero ideal $K = R[U, U]R$ of R generated by $[U, U]$ such that $[K, R] \subseteq U$ and $Ka = aK = 0$.*

Corollary 2.6. *Let R be a 2-torsion free semiprime ring, U a Lie ideal of R such that $U \not\subseteq Z(R)$ and $a, b \in U$.*

(i) If $aUa = 0$, then $a = 0$.

(ii) If $aU = 0$(or $Ua = 0$), then $a = 0$

(iii) If U is square-closed, and $aUb = 0$, then $ab = 0$ and $ba = 0$.

Theorem 2.7. *Let R be a 2-torsion free semiprime ring and U a square-closed Lie ideal of R such that $U \not\subseteq Z(R)$. Suppose that R admits a multiplicative generalized derivation F associated with a nonzero map d such that $d(x) \in U$, for all $x \in U$. If*

(i) $F([x, y]) = 0$, for all $x, y \in U$, or

(ii) $F([x, y]) = \pm[x, y]$, for all $x, y \in U$, or

(iii) $F([x, y]) = \pm(xoy)$, for all $x, y \in U$,

then d is commuting map on U.

Proof. (i) By the hypothesis, we have

$$F([x,y]) = 0, \text{ for all } x,y \in U.$$

Replacing yx by y in the above equation and using this equation, we get

$$[x,y]d(x) = 0, \quad \text{for all } x,y \in U. \tag{2.1}$$

Writing $d(x)y$ for y in (2.1) and using (2.1), we obtain that

$$[x,d(x)]yd(x) = 0, \text{ for all } x,y \in U. \tag{2.2}$$

Replacing y by yx in (2.2), we find that

$$[x,d\,(x)]yxd\,(x) = 0, \text{ for all } x,y \in U. \tag{2.3}$$

Multiplying (2.2) on the right by x, we have

$$[x,d\,(x)]yd\,(x)\,x = 0, \text{ for all } x,y \in U. \tag{2.4}$$

Subtracting (2.4) from (2.3), we arrive at

$$[x,d\,(x)]y[x,d\,(x)] = 0, \text{ for all } x,y \in U.$$

By Corollary 1, we conclude that $[x,d\,(x)] = 0$, for all $x \in R$, and so d is commuting map on U.

(ii) Suppose that

$$F([x,y]) = \pm[x,y], \quad \text{for all } x,y \in U. \tag{2.5}$$

Replacing y by yx in (2.5) and using this equation, we arrive that

$$[x,y]d(x) = 0, \quad \text{for all } x,y \in U.$$

Using the same arguments after (2.1) in the proof of Theorem 1 (i), we get the required result.

(iii) Assume that

$$F([x,y]) = \pm(xoy), \quad \text{for all } x,y \in U. \tag{2.6}$$

Replacing yx by y in (2.6) and using this equation, we get

$$[x,y]d(x) = 0, \quad \text{for all } x,y \in U.$$

This equation is same as (2.1) in the proof of Theorem 1 (i). Hence, using the same arguments in there, we get the required result. □

Corollary 2.8. *Let R be a 2-torsion free prime ring and U a square-closed Lie ideal of R. Suppose that R admits a multiplicative generalized derivation F associated with a nonzero derivation d. If*

(i) $F([x,y]) = 0$, for all $x,y \in U$, or
(ii) $F([x,y]) = \pm[x,y]$, for all $x,y \in U$, or
(iii) $F([x,y]) = \pm(xoy)$, for all $x,y \in U$, then $U \subseteq Z$.

Proof. By the same techniques in the proof of Theorem 1, we obtain that

$$[x,y]d(x) = 0, \text{ for all } x,y \in U.$$

Replacing y by yz, $z \in U$ in the above equation, we have

$$[x,y]zd(x) = 0, \text{ for all } x,y,z \in U. \tag{2.7}$$

By Lemma 1, we get either $[x,y] = 0$ or $d(x) = 0$, for each $x \in U$. We set $K = \{x \in U \mid [x,y] = 0, \text{ for all } y \in U\}$ and $L = \{x \in U \mid d(x) = 0\}$. Clearly each of K and L is additive subgroup of U. Morever, U is the set-theoretic union of K and L. But a group can not be the set-theoretic union of two proper subgroups, hence $K = U$ or $L = U$. In the first case, we have $U \subseteq Z$ by Lemma 2. In the latter case, we have $U \subseteq Z$ by Lemma 3. This completes the proof. □

Theorem 2.9. *Let R be a 2-torsion free semiprime ring and U a square-closed Lie ideal of R such that $U \nsubseteq Z(R)$. Suppose that R admits a multiplicative generalized derivation F associated with a nonzero map d such that $d(x) \in U$, for all $x \in U$. If*
(i) $F(xoy) = 0$, for all $x, y \in U$, or
(ii) $F(xoy) = \pm(xoy)$, for all $x, y \in U$, or
(iii)$F(xoy) = \pm[x, y]$, for all $x, y \in U$,
then d is commuting map on U.

Proof. (i) Assume that
$$F(xoy) = 0, \text{ for all } x, y \in U. \tag{2.8}$$
Writing yx for y in (2.8) and using (2.8), we have
$$(xoy)d(x) = 0, \text{ for all } x, y \in U. \tag{2.9}$$
Taking $d(x)y$ for y in (2.9) and using (2.9), we obtain that
$$[x, d(x)]yd(x) = 0, \text{ for all } x, y \in U.$$
Using the same arguments after (2.2) in the proof of Theorem 1 (i), we get the required result.

(ii) We have
$$F(xoy) = \pm(xoy), \text{ for all } x, y \in U.$$
Replacing y by yx in this equation and using this, we arrive that
$$(xoy)d(x) = 0, \text{ for all } x, y \in U.$$
Using the same arguments after (2.9) in the proof of Theorem 2 (i), we conclude the required result.

(iii) Suppose that
$$F(xoy) = \pm[x, y], \text{ for all } x, y \in U. \tag{2.10}$$
Replacing yx by y in (2.10) and using this, we get
$$(xoy)d(x) = 0, \text{ for all } x, y \in U.$$
This equation is same as (2.9) in the proof of Theorem 2 (i). Hence, using the same arguments in there, we get the required result. □

Corollary 2.10. *Let R be a 2-torsion free prime ring and U a square-closed Lie ideal of R. Suppose that R admits a multiplicative generalized derivation F associated with a nonzero derivation d. If*
(i) $F(xoy) = 0$, for all $x, y \in U$, or
(ii) $F(xoy) = \pm(xoy)$, for all $x, y \in U$, or
(iii) $F(xoy) = \pm[x, y]$, for all $x, y \in U$, then $U \subseteq Z$.

Proof. Using the same methods in the proof of Theorem 2, we have
$$(xoy)d(x) = 0, \text{ for all } x, y \in U.$$
Taking y by yz, we get
$$[x, y]zd(x) = 0, \text{ for all } x, y, z \in U.$$
This equation is same as equation (2.7) in the proof of Corollary 2. Hence, using the same arguments in there, we get the required result. □

Theorem 2.11. *Let R be a 2-torsion free semiprime ring and U a square-closed Lie ideal of R such that $U \nsubseteq Z(R)$. Suppose that R admits a multiplicative generalized derivation F associated with a nonzero map d such that $d(x) \in U$, for all $x \in U$. If*
(i) $F([x, y]) = \pm[F(x), y]$, for all $x, y \in U$, or
(ii) $F(xoy) = \pm(F(x)oy)$, for all $x, y \in U$,
then d is commuting map on U.

Proof. (i) By our hypothesis, we get

$$F([x,y]) = \pm[F(x),y], \text{ for all } x,y \in U. \tag{2.11}$$

Replacing y by yx in (2.11) and using this equation, we arrive that

$$[x,y]d(x) = \pm y[F(x),x], \text{ for all } x,y \in U. \tag{2.12}$$

Writing $d(x)y$ instead of y in (2.12) and using (2.12), we have

$$[x,d(x)]yd(x) = 0, \text{ for all } x,y \in U. \tag{2.13}$$

Using the same arguments after equation (2.2) in the proof of Theorem 1 (i), we get the required results.

(ii) We get

$$F(xoy) = \pm(F(x)oy), \text{ for all } x,y \in U. \tag{2.14}$$

Writting yx for y in (2.14) and using (2.14), we obtain that

$$(xoy)d(x) = \mp y[F(x),x], \text{ for all } x,y \in U. \tag{2.15}$$

Substituting $d(x)y$ for y in (2.15) and using this equation, we find that

$$[x,d(x)]yd(x) = 0, \text{ for all } x,y \in U.$$

Using the same arguments after equation (2.2) in the proof of Theorem 1 (i), we complete the proof. □

Corollary 2.12. *Let R be a 2-torsion free prime ring and U a square-closed Lie ideal of R. Suppose that R admits a multiplicative generalized derivation F associated with a nonzero derivation d. If*

(i) $F([x,y]) = \pm[F(x),y]$, for all $x,y \in U$, or

(ii) $F(xoy) = \pm(F(x)oy)$, for all $x,y \in U$, then $U \subseteq Z$.

Proof. i) Using the same methods in the proof of Theorem 3 (i), we obtain that

$$[x,y]d(x) = \pm y[F(x),x], \text{ for all } x,y \in U.$$

Replacing y by yz, $z \in U$ in this equation and using this, we obtain

$$[x,y]zd(x) = 0, \text{ for all } x,y,z \in U.$$

This equation is same as equation (2.7) in the proof of Corollary 2. Hence, using the same arguments in there, we get the required result.

ii) By the same methods in the proof of Theorem 3 (ii), we get

$$(xoy)d(x) = \mp y[F(x),x], \text{ for all } x,y \in U.$$

Taking y by yz in the last equation and using this equation, we find

$$[x,y]zd(x) = 0, \text{ for all } x,y,z \in U.$$

This equation is same as equation (2.7) in the proof of Corollary 2. We hade done in there. The result is obtained. □

Theorem 2.13. *Let R be a 2-torsion free semiprime ring, U a square-closed Lie ideal of R such that $U \not\subseteq Z(R)$. Suppose that R admits a multiplicative generalized derivation F associated with a nonzero map d such that $d(x) \in U$, for all $x \in U$. If $F(xy) \pm xy \in Z$, for all $x,y \in U$, then d is commuting map on U.*

Proof. By the hypothesis, we get

$$F(xy) \pm xy \in Z, \quad \text{for all } x, y \in U. \tag{2.16}$$

Replacing yz by y in (2.16) and using this, we arrive that

$$(F(xy) \pm xy)z + xyd(z) \in Z, \quad \text{for all } x, y, z \in U.$$

Commuting this equation with z and using $F(xy) \pm xy \in Z$, we obtain that

$$[xyd(z), z] = 0, \quad \text{for all } x, y, z \in U. \tag{2.17}$$

Taking $xd(z)$ for x in this equation and using (2.17), we get

$$[x, z]d(z)yd(z) = 0, \quad \text{for all } x, y, z \in U.$$

Substituting $y[x, z]$ for y, we get

$$[x, z]d(z)y[x, z]d(z) = 0, \quad \text{for all } x, y, z \in U.$$

By Corollary 1, we have

$$[x, z]d(z) = 0, \quad \text{for all } x, y, z \in U.$$

Using the same arguments after (2.1) in the proof of Theorem 1 (i), we get the required result. □

Theorem 2.14. *Let R be a 2-torsion free semiprime ring, U a square-closed Lie ideal of R such that $U \nsubseteq Z(R)$. Suppose that R admits a multiplicative generalized derivation F associated with a nonzero map d such that $d(x) \in U$, for all $x \in U$. If $F(xy) \pm yx \in Z$, for all $x, y \in U$, then d is commuting map on U.*

Proof. We consider that

$$F(xy) - yx \in Z, \text{ for all } x, y \in U. \tag{2.18}$$

Substituting yz for y in the hypothesis, where $z \in U$, we get

$$F(xyz) - (yz)x = F(xy)z + xyd(z) - yzx = (F(xy) - yx)z + y[x, z] + xyd(z) \in Z. \tag{2.19}$$

Commuting both sides of (2.19) with z and using equation (2.18), we obtain that

$$[y[x, z], z] + [xyd(z), z] = 0, \text{ for all } x, y, z \in U. \tag{2.20}$$

Replacing x by xz in (2.20), we get

$$[y[x, z], z]z + [xzyd(z), z] = 0 \tag{2.21}$$

Right multiplying (2.20) by z and subtracting it from (2.21), we get

$$[x[yd(z), z], z] = 0, \text{ for all } x, y, z \in U. \tag{2.22}$$

Taking x by $tx, t \in U$ in the above relation and using (2.22), we have

$$\begin{aligned} 0 &= [tx[yd(z), z], z] = t[x[yd(z), z], z] + [t, z]x[yd(z), z] \\ &= [t, z]x[yd(z), z]. \end{aligned} \tag{2.23}$$

Replacing t by $yd(z)$, we have

$$[yd(z), z]x[yd(z), z] = 0 \text{ for all } x, y, z \in U.$$

By Corollary 1, we get

$$[yd(z), z] = 0, \text{ for all } y, z \in U.$$

Replacing y with $d(z)y$, we get

$$[d(z)yd(z),z]=0.$$

That is,

$$d(z)yd(z)z-zd(z)yd(z)=0,\ \text{for all } y,z\in U. \tag{2.24}$$

Taking y by $yd(z)u, u\in U$ in this equation, we have

$$d(z)yd(z)ud(z)z-zd(z)yd(z)ud(z)=0.$$

Using (2.24), we obtain

$$d(z)yzd(z)ud(z)-d(z)yd(z)zud(z)=0.$$

That is,

$$d(z)y[d(z),z]ud(z)=0,\ \text{for all } y,z\in U.$$

This implies that

$$[d(z),z]y[d(z),z]u[d(z),z]=0,\ \text{for all } y,z,u\in U.$$

Right multiplying this equation by $y[d(z),z]$, we get

$$[d(z),z]y[d(z),z]u[d(z),z]y[d(z),z]=0,\ \text{for all } y,z,u\in U.$$

By Corollary 1, we obtain

$$[d(z),z]y[d(z),z]=0,\ \text{for all } y,z\in U.$$

That is, $[d(z),z]=0$, for all $z\in U$ by Corollary 1. Hence, d is commuting on U. In a similar manner, we can prove that the same conclusion holds for $F(xy)+yx\in Z$, for all $x,y\in U$. □

Corollary 2.15. *Let R be a 2-torsion free prime ring, U a square-closed Lie ideal of R. Suppose that R admits a multiplicative generalized derivation F associated with a nonzero derivation d. If*

i) If $F(xy)\pm xy\in Z$, for all $x,y\in U$,

ii) If $F(xy)\pm yx\in Z$, for all $x,y\in U$, then $U\subseteq Z$.

Proof. i) By the same methods in the proof of Theorem 4, we obtain

$$[xyd(z),z]=0,\quad \text{for all } x,y,z\in U.$$

Replacing x by $xd(z)$, $z\in[U,U]$, we get

$$[x,z]d(z)yd(z)=0,\quad \text{for all } x,y\in U,\ z\in[U,U].$$

Thus, we get either $[x,z]d(z)=0$ or $d(z)=0$, for all $x\in U$ by Lemma 1. Now, we assume that $[x,z]d(z)=0$, for all $x\in U$. Replacing x by $xy, y\in U$ in this equation and using this, we have $[x,z]yd(z)=0$ for all $x,y\in U$. Hence, we get either $[x,z]=0$ or $d(z)=0$, for each $z\in[U,U]$ by Lemma 1. Thus, we conclude that

$$[x,z]=0 \text{ or } d(z)=0,\ \text{for all } x\in U, z\in[U,U].$$

We set $K=\{z\in[U,U]\mid [x,z]=0, \text{ for all } x\in U\}$ and $L=\{z\in[U,U]\mid d(z)=0\}$. Clearly each of K and L is additive subgroup of $[U,U]$. Morever, $[U,U]$ is the set-theoretic union of K and L. But a group can not be the set-theoretic union of two proper subgroups, hence $K=[U,U]$ or $L=[U,U]$. In the former case, using Lemma 2, we have $U\subseteq Z$. In the latter case, $d([U,U])=0$. That is $[U,U]\subset Z$ by Lemma 3, and so again using Lemma 2, we get $U\subseteq Z$. This completes the proof.

ii) Using the same techniques in the proof of Theorem 5, we get

$$[t,z]x[yd(z),z]=0,\ \text{for all } x,y,z,t\in U.$$

Taking t by $yd(z)$, $z \in [U,U]$, we get

$$[yd(z),z]x[yd(z),z] = 0, \text{ for all } x,y \in U, \ z \in [U,U].$$

Using Lemma 1, we have

$$[yd(z),z] = 0, \text{ for all } y \in U, \ z \in [U,U].$$

We conclude that

$$y[d(z),z] + [y,z]d(z) = 0, \text{ for all } y \in U, \ z \in [U,U].$$

Replacing y by xy in the last equation and this equation, we have

$$[x,z]yd(z) = 0, \text{ for all } x,y \in U, \ z \in [U,U].$$

By Lemma 1, we get either $[x,z] = 0$ or $d(z) = 0$, for each $z \in [U,U]$. We had done this situation in the last paragraph in the proof of Corollary 5. The proof is completed. □

Theorem 2.16. *Let R be a 2-torsion free semiprime ring and U a square-closed Lie ideal of R such that $U \nsubseteq Z(R)$. Suppose that R admits a multiplicative generalized derivation F associated with a nonzero map d such that $d(x) \in U$, for all $x \in U$. If*
(i) $F(xy) \pm [x,y] \in Z$, for all $x,y \in U$, or
(ii) $F(xy) \pm (xoy) \in Z$, for all $x,y \in U$,
then d is commuting map on U.

Proof. (i) Assume that

$$F(xy) \pm [x,y] \in Z, \ \text{ for all } x,y \in U.$$

Define the map $G : R \to R, G(r) = F(r) \pm r$, for all $r \in R$. G is a multiplicative generalized derivation associated with a nonzero map d of R. By the hypothesis, we have $G(xy) \pm yx \in Z$, for all $x,y \in U$. Hence, the conclusion is obtained by Theorem 5. Thus, d is commuting map on U.

(ii) By the hypothesis, we get

$$F(xy) \pm (xoy) \in Z, \ \text{ for all } x,y \in U.$$

Define the map $G : R \to R, G(r) = F(r) \pm r$, for all $r \in R$. G is a multiplicative generalized derivation associated with a nonzero map d of R such that $G(xy) \pm yx \in Z$, for all $x,y \in U$. By Theorem 5, we get d is commuting map on U. □

Corollary 2.17. *Let R be a 2-torsion free prime ring, U a square-closed Lie ideal of R and F be a multiplicative generalized derivation associated with a nonzero derivation d of R. If*
(i) $F(xy) \pm [x,y] \in Z$, for all $x,y \in U$, or
(ii) $F(xy) \pm (xoy) \in Z$, for all $x,y \in U$, then $U \subseteq Z$.

Proof. Using the same methods in the beginning of the proof in Theorem 6, we get $G(xy) \pm yx \in Z$, for all $x,y \in U$ such that $G : R \to R, G(r) = F(r) \pm r$, for all $r \in R$. Also, G is a multiplicative generalized derivation associated with a nonzero map d of R. Hence, we apply the same techniques

$$[t,z]x[yd(z),z] = 0, \text{ for all } x,y,z,t \in U.$$

This equation is the same Corollary 5 (ii). Also, we have proved in there. The proof is completed. □

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