Palestine Journal of Mathematics

Vol. 6(1)(2017) , 228–237

Jacobi-Sohncke Type Mixed Modular Equations and their applications to overpartitions

M. S. Mahadeva Naika, K. Sushan Bairy, D. S. Gireesh and N. P. Suman

Communicated by P. K. Banerji

MSC 2010 Classifications: 33D10, 11A55, 11F27

Keywords and phrases: Modular equation, theta-function, continued fraction.

This research is supported by DST, Govt. of India under the project grant SR/S4/MS:739/11.
The authors would like to thank Prof. P. K. Banerji for his useful comments.

Abstract. The modular relations between the Ramanujan-Selberg continued fractions $H(q)$, $H(q^5)$, $H(q^n)$, and $H(q^{5n})$ for $n = 2$, 3, 4, 5, 7, 9, 11, and 13 are established as an application to several new Jacobi-Sohncke type mixed modular equations for composite degrees 1, 5, n, and $5n$. We also obtain congruence properties for color overpartition of n with odd parts.

1 Introduction

The theory of modular equations commenced when A. M. Legendre [12], in his paper derived a modular equation of degree 3 in 1825. In fact prior to this, Landen records Landen's transformation in his papers [10], [11] in 1771 and 1775. After Legendre, Jacobi established modular equations of degree 3 and 5 in his famous book [8]. Sohncke [22, 23] established modular equations of degrees 7, 11, 13, 17, and 19. Subsequently many mathematicians have contributed to the theory of modular equations. Some of them are Guetzlaff, Schröter, Schläfli, Klein, Cayley, Russell, Weber. Ramanujan's contributions in the area of modular equations are immense. Perhaps, Ramanujan found more modular equations than all of his predecessors discovered together. For more on modular equations one can refer [7, 9, 13, 14, 18].

The complete elliptic integral of first kind is defined as

$$K(k) := \int_0^{\frac{\pi}{2}} \frac{d\phi}{\sqrt{1-k^2\sin^2\phi}} = \frac{\pi}{2}\sum_{n=0}^{\infty}\frac{\left(\frac{1}{2}\right)_n^2}{(n!)^2}k^{2n} = \frac{\pi}{2}\,{}_2F_1\left(\frac{1}{2},\frac{1}{2};1;k^2\right), \tag{1.1}$$

where $0 < k < 1$ and ${}_2F_1$ is the ordinary or Gaussian hypergeometric function defined by

$${}_2F_1(a,b;c;z) := \sum_{n=0}^{\infty}\frac{(a)_n\,(b)_n}{(c)_n\,n!}z^n, \quad 0 \le |z| < 1,$$

$$(a)_0 = 1, \ \ (a)_n = a(a+1)\cdots(a+n-1) \ \text{ for } \ n \text{ a positive integer}$$

and a, b, c are complex numbers such that $c \neq 0, -1, -2, \ldots$. The number k is called the modulus of K, and $k' := \sqrt{1-k^2}$ is called the complementary modulus.

Now we define modular equation in brief. Let K, K', L, and L' denote the complete elliptic integrals of the first kind associated with the moduli k, k', l, and l', respectively. Suppose that the equality

$$n\frac{K'}{K} = \frac{L'}{L} \tag{1.2}$$

holds for some positive integer n. Then a modular equation of degree n is a relation between the moduli k and l which is induced by (1.2). Following Ramanujan, set $\alpha = k^2$ and $\beta = l^2$. Then we say β is of degree n over α. The multiplier m is defined by

$$m = \frac{K}{L}. \tag{1.3}$$

However, if we set

$$q = \exp\{-\pi K'/K\} \quad \text{and} \quad q' = \exp\{-\pi L'/L\} \tag{1.4}$$

we see that (1.2) is equivalent to the relation $q^n = q'$. Thus a modular equation can be viewed as an identity involving theta-functions at the arguments q and q^n.

Let K, K', L_1, L_1', L_2, L_2', L_3, and L_3' denote complete elliptic integrals of the first kind corresponding, in pairs, to the moduli $\sqrt{\alpha}$, $\sqrt{\beta}$, $\sqrt{\gamma}$, and $\sqrt{\delta}$, and their complementary moduli, respectively. Let n_1, n_2, and n_3 be positive integers such that $n_3 = n_1 n_2$. Suppose that the equalities

$$n_1 \frac{K'}{K} = \frac{L_1'}{L_1}, \quad n_2 \frac{K'}{K} = \frac{L_2'}{L_2} \quad \text{and} \quad n_3 \frac{K'}{K} = \frac{L_3'}{L_3} \tag{1.5}$$

hold. Then a "mixed" modular equation is a relation between the moduli $\sqrt{\alpha}$, $\sqrt{\beta}$, $\sqrt{\gamma}$, and $\sqrt{\delta}$ that is induced by (1.5). We say that β, γ, and δ are of degrees n_1, n_2, and n_3, respectively over α. The multipliers m and m' are associated with α, β and γ, δ.

Remark 1.1. The equations involving $\alpha^{1/8}$ and $\beta^{1/8}$ are referred to them as Jacobi-Sohncke type modular equations.

Recently, M. S. Mahadeva Naika, K. Sushan Bairy and C. Shivashankar [17] have obtained Jacobi-Sohncke type mixed modular equations of degrees 1, 3, n and $3n$.
In [21, eq.(54)], Selberg first proved the continued fraction identity

$$\begin{aligned} H(q) \;&:=\; \frac{q^{\frac{1}{8}}}{1} \, \underset{+}{} \, \frac{q}{1} \, \underset{+}{} \, \frac{q^2+q}{1} \, \underset{+}{} \, \frac{q^3}{1} \, \underset{+}{} \, \frac{q^4+q^2}{1} \, \underset{+}{} \, \cdots \\ &= \frac{q^{\frac{1}{8}}(-q^2;q^2)_\infty}{(-q;q^2)_\infty}, \qquad |q|<1, \end{aligned} \tag{1.6}$$

which appears as Formula 5 [20, p. 290]. Other proofs have been given by Ramanathan [19] and Andrews, Berndt, Jacobsen and Lamphere [2]. Recently , two integral representations for $H(q)$, a relation between $H(q)$ and $H(q^n)$ and some explicit evaluations of $H(q)$ have been obtained by Adiga, Mahadeva Naika and Ramya Rao [1]. Mahadeva Naika et al. [15, 16], have obtained several integral representations and also Rogers-Ramanujan type functions for $H(q)$.

In section 2, we collect the identities which helps us to prove our main results. In sections 3, we establish several new mixed modular equations. In section 4, we establish modular relations for Ramanujan-Selberg continued fraction with four moduli. In section 5, we establish some congruence properties for overpartitions of n in $p-$colors with odd parts.

2 Preliminary Results

In this section, we collect the results which are useful in obtaining our main results.

Lemma 2.1. *[3, Entry 17.3.1, p. 385] If β is of degree 2 over α, then*

$$(1-\sqrt{1-\alpha})(1-\sqrt{\beta}) = 2\sqrt{\beta(1-\alpha)}. \tag{2.1}$$

Lemma 2.2. *[4, Entry 5 (ii), p. 230] If β has degree 3 over α, then*

$$(\alpha\beta)^{1/4} + \{(1-\alpha)(1-\beta)\}^{1/4} = 1. \tag{2.2}$$

Lemma 2.3. *[3, Entry 17.3.2, p. 385] If β has degree 4 over α, then*

$$(1-\sqrt[4]{1-\alpha})(1-\sqrt[4]{\beta}) = 2\sqrt[4]{\beta(1-\alpha)}. \tag{2.3}$$

Lemma 2.4. *[4, Entry 13 (i), p. 280] If β has degree 5 over α, then*

$$(\alpha\beta)^{1/2} + \{(1-\alpha)(1-\beta)\}^{1/2} + 2\{16\alpha\beta(1-\alpha)(1-\beta)\}^{1/6} = 1. \tag{2.4}$$

Lemma 2.5. *[4, Entry 19 (i), p. 314] If β has degree 7 over α, then*

$$(\alpha\beta)^{1/8} + \{(1-\alpha)(1-\beta)\}^{1/8} = 1. \tag{2.5}$$

Lemma 2.6. *[4, Entry 3 (x), (xi), p. 352] If β has degree 9 over α, then*

$$\left(\frac{\beta}{\alpha}\right)^{1/8} + \left(\frac{1-\beta}{1-\alpha}\right)^{1/8} - \left(\frac{\beta(1-\beta)}{\alpha(1-\alpha)}\right)^{1/8} = \sqrt{m}, \tag{2.6}$$

$$\left(\frac{\alpha}{\beta}\right)^{1/8} + \left(\frac{1-\alpha}{1-\beta}\right)^{1/8} - \left(\frac{\alpha(1-\alpha)}{\beta(1-\beta)}\right)^{1/8} = \frac{3}{\sqrt{m}}. \tag{2.7}$$

Lemma 2.7. *[4, Entry 7, p. 363] If β has degree 11 over α, then*

$$(\alpha\beta)^{1/4} + \{(1-\alpha)(1-\beta)\}^{1/4} + 2\{16\alpha\beta(1-\alpha)(1-\beta)\}^{1/12} = 1. \tag{2.8}$$

Lemma 2.8. *[5, pp. 387–388] Let*

$$U_1 = 1 - \sqrt{\alpha\beta} - \sqrt{(1-\alpha)(1-\beta)}, \tag{2.9}$$

$$V_1 = 64\left(\sqrt{\alpha\beta} + \sqrt{(1-\alpha)(1-\beta)} - \sqrt{\alpha\beta(1-\alpha)(1-\beta)}\right) \tag{2.10}$$

and

$$W_1 = 32\sqrt{\alpha\beta(1-\alpha)(1-\beta)}, \tag{2.11}$$

then

$$\sqrt{U_1}(U_1^3 + 8W_1) = \sqrt{W_1}(11U_1^2 + V_1), \tag{2.12}$$

where β is of degree 13 over α.

Lemma 2.9. *[4, Entry 13. (xv), p. 382] If $M = (\alpha\beta)^{1/4}$ and $N = (\beta/\alpha)^{1/8}$, then*

$$\left(N - \frac{1}{N}\right)^3 + 8\left(N - \frac{1}{N}\right) = 4\left(M - \frac{1}{M}\right). \tag{2.13}$$

Lemma 2.10. *We have*

$$\chi(q) = 2^{1/6}(\alpha(1-\alpha)/q)^{-1/24}, \tag{2.14}$$

$$\chi(-q) = 2^{1/6}(1-\alpha)^{1/12}(\alpha/q)^{-1/24}, \tag{2.15}$$

where $\alpha = k^2$, k is called the modulus of K.

Proof. For proofs of (2.14) and (2.15), see [4, Entry 12 (v), (vi), Ch.17, p. 124]. □

Lemma 2.11. *[4, Ch.18, Entry 24(v), p. 216] If we replace α by $1-\beta$, β by $1-\alpha$, and m by n/m, where n is the degree of the modular equation, we obtain a modular equation of the same degree.*

3 Mixed Modular Equations

In this section, we derive several Jacobi-Sohncke type mixed modular equations for the degrees 1, 5, n and $5n$ for $n = 2, 3, 4, 5, 7, 9, 11$, and 13.
Throughout this section, we set

$$R := \left\{\frac{\alpha\gamma}{\beta\delta}\right\}^{1/16} \tag{3.1}$$

and

$$T := \left\{\frac{\alpha\delta}{\beta\gamma}\right\}^{1/16}. \tag{3.2}$$

We use the following notations

$$\mathbb{R}_\kappa := \left(R^\kappa + \frac{1}{R^\kappa}\right) \tag{3.3}$$

and

$$\mathbb{T}_\kappa := \left(T^\kappa + \frac{1}{T^\kappa}\right). \tag{3.4}$$

Lemma 3.1. *If α, β, γ and δ are of degrees 1, 5, n and $5n$ respectively, then*

$$\alpha = \left(\frac{u+s}{2N^2}\right)^2, \tag{3.5}$$

$$\gamma = \left(\frac{v+t}{2N'^2}\right)^2, \tag{3.6}$$

where $u = \frac{1}{4}\left(\left(N - \frac{1}{N}\right)^3 + 8\left(N - \frac{1}{N}\right)\right)$, $N = (\beta/\alpha)^{1/8}$, $s^2 = u^2 + 4$,
$v = \frac{1}{4}\left(\left(N' - \frac{1}{N'}\right)^3 + 8\left(N' - \frac{1}{N'}\right)\right)$, $N' = (\delta/\gamma)^{1/8}$, *and* $t^2 = v^2 + 4$.

Proof. The proof of equations (3.5) and (3.6) directly follows from the equation (2.13). □

Theorem 3.2. *If α, β, γ, and δ are of degrees 1, 5, 3, and 15 respectively, then*

$$\mathbb{R}_2 + 3\mathbb{T}_2 - 6 = \mathbb{T}_4. \tag{3.7}$$

Proof. From equations (3.1) and (3.2) we see that $N = \dfrac{1}{RT}$ and $N' = \dfrac{T}{R}$. Now using equations

(3.5) and (3.6) in the equation (2.2), we deduce

$$\begin{aligned}
&R^{12}T^{20} + 10R^{10}T^{18} - 4R^9T^{17}s + 8R^9T^{13}mt + 15R^8T^{16} - 2R^{12}T^{10}m\\
&- 10R^{10}T^{12}m - 12R^9T^{13}t + 10R^8T^{14}m - 20R^7T^{15}s + 2R^6T^{16}m - 2T^{10}m\\
&+ 15R^{10}T^{12} - 15R^8T^{14} - 3R^6T^{16} + 40R^7T^{11}mt - 20R^6T^{14} - 10R^{10}T^8m\\
&- 50R^8T^{10}m - 60R^7T^{11}t + 50R^6T^{12}m - 32R^6T^{10}mst + 20R^5T^{13}s\\
&+ 10R^4T^{14}m + 15R^{10}T^8 + 8R^9T^7ms + 75R^8T^{10} + 40R^7T^9ms - 75R^6T^{12}\\
&+ 48R^6T^{10}st - 40R^5T^{11}ms - 15R^4T^{14} - 8R^3T^{13}ms - 12R^9T^7s + T^8\\
&+ 60R^5T^{11}s + 12R^3T^{13}s - 40R^5T^9mt + 15R^4T^{12} + 10R^8T^6m + 50R^6T^8m\\
&+ 60R^5T^9t - 50R^4T^{10}m + 4R^3T^{11}s - 10R^2T^{12}m - 15R^8T^6 - 75R^6T^8\\
&+ 75R^4T^{10} + 15R^2T^{12} - 4R^9T^3t - 20R^7T^5t + 20R^5T^7t + 4R^3T^9t\\
&- 128R^2T^{10}m + R^{12} + 10R^{10}T^2 + 15R^8T^4 - 20R^6T^6 + 15R^4T^8 + 3T^{10}\\
&+ 20R^2T^{10} + T^{12} + 2R^6T^4m + 10R^4T^6m + 12R^3T^7t - 10R^2T^8m\\
&- 3R^6T^4 - 15R^4T^6 + 15R^2T^8 + 3R^{12}T^{10} - 60R^7T^9s - 8R^3T^7mt = 0,
\end{aligned} \tag{3.8}$$

where $m := (\alpha\beta)^{1/4}$.
Now eliminating m, s and t from the above eqaution (3.8), we deduce

$$R^4T^4 - R^2T^8 + 3R^2T^6 - 6R^2T^4 + 3R^2T^2 + T^4 - R^2 = 0. \tag{3.9}$$

By employing equations (3.3) and (3.4) in the above equation (3.9), we arrive at (3.7). □

Theorem 3.3. *If α, β, γ, and δ are of degrees 1, 5, 2, and 10 respectively, then*

$$2\left(X^2 + \frac{1}{X^2}\right) + \left(X^2 - \frac{1}{X^2}\right)\left(Y + \frac{1}{Y}\right) - \left(Y^2 + \frac{1}{Y^2}\right) - 2 = 0. \tag{3.10}$$

Proof. The proof of Theorem (3.3) is similar to the proof of Theorem (3.2), except that in the place of equation (2.2); equation (2.1) is used and where we set $X := TR$ and $Y := \dfrac{T}{R}$ instead of setting $\mathbb{R}_\kappa := \left(R^\kappa + \dfrac{1}{R^\kappa}\right)$ and $\mathbb{T}_\kappa := \left(T^\kappa + \dfrac{1}{T^\kappa}\right)$. □

Theorem 3.4. *If α, β, γ, and δ are of degrees 1, 5, 4, and 20 respectively, then*

$$\begin{aligned}
&\left(X^4 - \frac{1}{X^4}\right)\left\{\left(Y^3 + \frac{1}{Y^3}\right) + 15\left(Y + \frac{1}{Y}\right)\right\} + 4\left(X^2 - \frac{1}{X^2}\right)\\
&\left\{\left(Y^3 + \frac{1}{Y^3}\right) - 5\left(Y + \frac{1}{Y}\right)\right\} - 2\left(X^4 + \frac{1}{X^4}\right)\left\{3\left(Y^2 + \frac{1}{Y^2}\right) + 10\right\}\\
&+ 32\left(X^2 + \frac{1}{X^2}\right) - \left(Y^4 + \frac{1}{Y^4}\right) + 16\left(Y^2 + \frac{1}{Y^2}\right) - 30 = 0.
\end{aligned} \tag{3.11}$$

Proof. The proof of Theorem (3.4) is similar to the proof of Theorem (3.3), except that in the place of equation (2.1); equation (2.3) is used. □

Theorem 3.5. *If α, β, γ, and δ are of degrees 1, 5, 5, and 25 respectively, then*

$$\begin{aligned}
&\mathbb{R}_8 + \mathbb{R}_6\left(5\mathbb{T}_2 + 14\right) - \mathbb{R}_4\left(\mathbb{T}_6 + 5\mathbb{T}_4 + 10\mathbb{T}_2 + 4\right) + \mathbb{R}_2\left(\mathbb{T}_6 - 10\mathbb{T}_2 - 6\right)\\
&- \mathbb{T}_6 + 15\mathbb{T}_2 + 42 = 0.
\end{aligned} \tag{3.12}$$

Proof. The proof of Theorem (3.5) is similar to that of Theorem (3.2), but rather than (2.2), we use (2.4). □

Theorem 3.6. *If α, β, γ, and δ are of degrees 1, 5, 7, and 35 respectively, then*

$$\mathbb{T}_8 + 28\mathbb{T}_4 - 42\mathbb{T}_2 + 126 = \mathbb{R}_6 - 7\mathbb{R}_4(4 - \mathbb{T}_2) + 7\mathbb{R}_2(1 + 4\mathbb{T}_2). \tag{3.13}$$

Proof. The proof of Theorem (3.6) is similar to the proof of the Theorem (3.2), except that in the place of (2.2); equation (2.5) is used. □

Theorem 3.7. *If α, β, γ, and δ are of degrees 1, 5, 9, and 45 respectively, then*

$$\begin{aligned}&\mathbb{R}_{12} - 8\mathbb{R}_{10} - 8\mathbb{R}_8 + 18\mathbb{R}_6 + 693\mathbb{R}_4 + 432\mathbb{R}_2 - \mathbb{T}_8 - 66\mathbb{T}_6 + 330\mathbb{T}_4\\ &- 1020\mathbb{T}_2 - 462\mathbb{R}_2\mathbb{T}_2 + 141\mathbb{R}_4\mathbb{T}_4 - \mathbb{R}_2\mathbb{T}_8 - 54\mathbb{R}_2\mathbb{T}_6 + 15\mathbb{R}_8\mathbb{T}_2 - 12\mathbb{R}_6\mathbb{T}_2\\ &- 471\mathbb{R}_4\mathbb{T}_2 - 9\mathbb{R}_6\mathbb{T}_4 + 285\mathbb{R}_2\mathbb{T}_4 - 9\mathbb{R}_4\mathbb{T}_6 + 1566 = 0.\end{aligned} \tag{3.14}$$

Proof. The proof of Theorem (3.7) is similar to that of Theorem (3.2), but rather than (2.2), we use (2.6) and (2.7). □

Theorem 3.8. *If α, β, γ, and δ are of degrees 1, 5, 11, and 55 respectively, then*

$$\begin{aligned}&\mathbb{R}_{10} + 88\mathbb{R}_8 + 11\left(\mathbb{T}_2 - 8\right) + \mathbb{R}_6\left(22\mathbb{T}_4 + 264\mathbb{T}_2 + 737\right) - \mathbb{R}_4\left(22\mathbb{T}_6 - 473\mathbb{T}_4\right.\\ &\left.-1826\mathbb{T}_2 - 2838\right) + \mathbb{R}_2\left(11\mathbb{T}_8 + 330\mathbb{T}_6 + 1958\mathbb{T}_4 + 1458\mathbb{T}_2 + 6522\right) - \mathbb{T}_{12}\\ &- 11\mathbb{T}_{10} - 99\mathbb{T}_8 - 671\mathbb{T}_6 - 2409\mathbb{T}_4 - 5852\mathbb{T}_2 - 8294 = 0.\end{aligned} \tag{3.15}$$

Proof. The proof of Theorem (3.8) is similar to the proof of Theorem (3.2), except that in the place of (2.2); equation (2.8) is used. □

Theorem 3.9. *If α, β, γ, and δ are of degrees 1, 5, 13, and 65 respectively, then*

$$\begin{aligned}&\mathbb{T}_{12} - 8\mathbb{T}_8 + 18\mathbb{T}_6 + 693\mathbb{T}_4 + 432\mathbb{T}_2 + 1566 = \mathbb{R}_8(1 + \mathbb{T}_2) + 3\mathbb{R}_6(22 + 18\mathbb{T}_2 + 3\mathbb{T}_4)\\ &- 3\mathbb{R}_4(110 + 47\mathbb{T}_4 - 3\mathbb{T}_6 + 95\mathbb{T}_2) + \mathbb{R}_2(1020 + 462\mathbb{T}_2 + 471\mathbb{T}_4 + 12\mathbb{T}_6 - 15\mathbb{T}_8).\end{aligned} \tag{3.16}$$

Proof. The proof of Theorem (3.9) is similar to that of Theorem (3.2), but rather than (2.2), we use (2.12). □

4 Modular identities for Ramanujan-Selberg continued fraction

In this section, we establish modular relations connecting Ramanujan-Selberg continued fractions $H(q)$, $H(q^5)$, $H(q^n)$, and $H(q^{5n})$ for $n = 2, 3, 4, 5, 7, 9, 11$, and 13.

We define

$$x := \frac{H(q)H(q^n)}{H(q^5)H(q^{5n})} \tag{4.1}$$

and

$$y := \frac{H(q)H(q^{5n})}{H(q^5)H(q^n)}. \tag{4.2}$$

Throughout this section, we use the following notations

$$\mathbb{X}_\kappa := \left(x^\kappa + \frac{1}{x^\kappa}\right) \tag{4.3}$$

and

$$\mathbb{Y}_\kappa := \left(y^\kappa + \frac{1}{y^\kappa}\right). \tag{4.4}$$

Lemma 4.1. *[15, Th.3.2] We have*

$$H(q)=\frac{\alpha^{1/8}}{\sqrt{2}}, \tag{4.5}$$

where q is as defined in (1.4) and $\alpha=k^2$, k is called the modulus of K.

Using the above lemma in the corresponding Jacobi-Sohncke type mixed modular equations obtained in the previous section, we deduce the following theorems.

Theorem 4.2. *If α, β, γ and δ are of degrees 1, 5, 2, and 10 respectively, then*

$$2\left(xy+\frac{1}{xy}\right)+\left(xy-\frac{1}{xy}\right)\left(\sqrt{\frac{x}{y}}+\sqrt{\frac{y}{x}}\right)-\left(\frac{x}{y}+\frac{y}{x}\right)-2=0. \tag{4.6}$$

Theorem 4.3. *If α, β, γ and δ are of degrees 1, 5, 3, and 15 respectively, then*

$$\mathbb{X}_1+3\mathbb{Y}_1-6=\mathbb{Y}_2. \tag{4.7}$$

Theorem 4.4. *If α, β, γ and δ are of degrees 1, 5, 4, and 20 respectively, then*

$$\begin{aligned}&\left(x^2y^2-\frac{1}{x^2y^2}\right)\left\{\left(\sqrt{\frac{x^3}{y^3}}+\sqrt{\frac{y^3}{x^3}}\right)+15\left(\sqrt{\frac{x}{y}}+\sqrt{\frac{y}{x}}\right)\right\}+32\left(xy+\frac{1}{xy}\right)\\&+4\left(xy-\frac{1}{xy}\right)\left\{\left(\sqrt{\frac{x^3}{y^3}}+\sqrt{\frac{y^3}{x^3}}\right)-5\left(\sqrt{\frac{x}{y}}+\sqrt{\frac{y}{x}}\right)\right\}-\left(\frac{x^2}{y^2}+\frac{y^2}{x^2}\right)\\&-2\left(x^2y^2+\frac{1}{x^2y^2}\right)\left\{3\left(\frac{x}{y}+\frac{y}{x}\right)+10\right\}+16\left(\frac{x}{y}+\frac{y}{x}\right)-30=0.\end{aligned} \tag{4.8}$$

Theorem 4.5. *If α, β, γ and δ are of degrees 1, 5, 5, and 25 respectively, then*

$$\begin{aligned}&\mathbb{X}_4+\mathbb{X}_3\left(5\mathbb{Y}_1+14\right)-\mathbb{X}_2\left(\mathbb{Y}_3+5\mathbb{Y}_2+10\mathbb{Y}_1+4\right)+\mathbb{X}_1\left(\mathbb{Y}_3-10\mathbb{Y}_1-6\right)\\&-\mathbb{Y}_3+15\mathbb{Y}_1+42=0.\end{aligned} \tag{4.9}$$

Theorem 4.6. *If α, β, γ and δ are of degrees 1, 5, 7, and 35 respectively, then*

$$\mathbb{Y}_4+28\mathbb{Y}_2-42\mathbb{Y}_1+126=\mathbb{X}_3-7\mathbb{X}_2(4-\mathbb{Y}_1)+7\mathbb{X}_1(1+4\mathbb{Y}_1). \tag{4.10}$$

Theorem 4.7. *If α, β, γ and δ are of degrees 1, 5, 9, and 45 respectively, then*

$$\begin{aligned}&\mathbb{X}_6-8\mathbb{X}_5-8\mathbb{X}_4+18\mathbb{X}_3+693\mathbb{X}_2+432\mathbb{X}_1-\mathbb{Y}_4-66\mathbb{Y}_3+330\mathbb{Y}_2-1020\mathbb{Y}_1\\&-462\mathbb{X}_1\mathbb{Y}_1+141\mathbb{X}_2\mathbb{Y}_2-\mathbb{X}_1\mathbb{Y}_4-54\mathbb{X}_1\mathbb{Y}_3+15\mathbb{X}_4\mathbb{Y}_1-12\mathbb{X}_3\mathbb{Y}_1-471\mathbb{X}_2\mathbb{Y}_1\\&-9\mathbb{X}_3\mathbb{Y}_2+285\mathbb{X}_1\mathbb{Y}_2-9\mathbb{X}_2\mathbb{Y}_3+1566=0.\end{aligned} \tag{4.11}$$

Theorem 4.8. *If α, β, γ and δ are of degrees 1, 5, 11, and 55 respectively, then*

$$\begin{aligned}&\mathbb{X}_5+\mathbb{X}_4+11\left(\mathbb{Y}_1-8\right)+\mathbb{X}_3\left(22\mathbb{Y}_2+264\mathbb{Y}_1+737\right)-\mathbb{X}_2\left(22\mathbb{Y}_3-473\mathbb{Y}_2\right.\\&\left.-1826\mathbb{Y}_1-2838\right)+\mathbb{X}_1\left(11\mathbb{Y}_4+330\mathbb{Y}_3+1958\mathbb{Y}_2+1458\mathbb{Y}_1+6522\right)-\mathbb{Y}_6\\&-11\mathbb{Y}_5-99\mathbb{Y}_4-671\mathbb{Y}_3-2409\mathbb{Y}_2-5852\mathbb{Y}_1-8294=0.\end{aligned} \tag{4.12}$$

Theorem 4.9. *If α, β, γ and δ are of degrees 1, 5, 13, and 65 respectively, then*

$$\begin{aligned}&\mathbb{Y}_6-8\mathbb{Y}_4+18\mathbb{Y}_3+693\mathbb{Y}_2+432\mathbb{Y}_1+1566=\mathbb{X}_4(1+\mathbb{Y}_1)+3\mathbb{X}_3(22+18\mathbb{Y}_1+3\mathbb{Y}_2)\\&-3\mathbb{X}_2(110+47\mathbb{Y}_2-3\mathbb{Y}_3+95\mathbb{Y}_1)+\mathbb{X}_1(1020+462\mathbb{Y}_1+471\mathbb{Y}_2+12\mathbb{Y}_3-15\mathbb{Y}_4).\end{aligned} \tag{4.13}$$

5 Color partition identities for overpartition with odd parts

An overpartition of n is a non increasing sequence of natural numbers whose sum is n in which the first occurrence of a number may be overlined. Let $\overline{p}(n)$ denote the number of overpartitions of an integer n. For convenience, we set $\overline{p}(0) = 1$. For example, there are eight overpartitions of 3 are 3, $\overline{3}$, $2+1$, $\overline{2}+1$, $2+\overline{1}$, $\overline{2}+\overline{1}$, $1+1+1$, $\overline{1}+1+1$.

The generating function for $\overline{p}(n)$ is given by Corteel and Lovejoy [6] as

$$\sum_{n=0}^{\infty} \overline{p}(n)q^n = \frac{(-q;q)_\infty}{(q;q)_\infty}. \tag{5.1}$$

Similarly, let $\mathbb{P}(n)$ be the number of overpartitions of n in which only odd parts are considered.

$$\sum_{n=0}^{\infty} \mathbb{P}(n)q^n = \frac{(-q;q^2)_\infty}{(q;q^2)_\infty}. \tag{5.2}$$

Now from (2.14) and (2.15), we can write

$$(1-\alpha)^{-1/8} = \frac{\chi(q)}{\chi(-q)} = \sum_{n=0}^{\infty} \mathbb{P}(n)q^n. \tag{5.3}$$

By Lemma 2.11, (3.1) and (3.2) reduces to

$$C := \left\{ \frac{(1-\beta)(1-\delta)}{(1-\alpha)(1-\gamma)} \right\}^{1/16}, \tag{5.4}$$

$$D := \left\{ \frac{(1-\beta)(1-\gamma)}{(1-\alpha)(1-\delta)} \right\}^{1/16}. \tag{5.5}$$

Let l, m be positive integers with $l \neq m$, we define the following :

$$\sum_{n=0}^{\infty} \mathbb{R}^p_{l,m}(n)q^n = \frac{\chi^p(q)\chi^p(-q^l)\chi^p(q^m)\chi^p(-q^{lm})}{\chi^p(-q)\chi^p(q^l)\chi^p(-q^m)\chi^p(q^{lm})}, \tag{5.6}$$

where $\mathbb{R}^p_{l,m}(n)$ denotes the number of overpartitions of n in $2p$ colors in which p colors appears only in odd parts that are not multiples of l, and another p colors appears only in odd parts that are multiples of m but are not multiples of lm.

For l, m relatively prime, let $\mathbb{T}^p_{l,m}(n)$ denotes the number of overpartitions of n into odd parts with p colors that are not multiples of l or m. The generating function for $\mathbb{T}^p_{l,m}(n)$ is defined as,

$$\sum_{n=0}^{\infty} \mathbb{T}^p_{l,m}(n)q^n = \frac{\chi^p(q)\chi^p(-q^l)\chi^p(-q^m)\chi^p(q^{lm})}{\chi^p(-q)\chi^p(q^l)\chi^p(q^m)\chi^p(-q^{lm})}. \tag{5.7}$$

Theorem 5.1. *We have*

$$\mathbb{R}_{5,3}(2n) \equiv \mathbb{T}^2_{5,3}(2n) \pmod{3}, \tag{5.8}$$

$$\mathbb{R}^4_{5,7}(2n) \equiv \mathbb{T}^3_{5,7}(2n) \pmod{7}, \tag{5.9}$$

$$\mathbb{R}^5_{5,11}(2n) \equiv \mathbb{T}^6_{5,11}(2n) \pmod{11}. \tag{5.10}$$

Proof. Using Lemma 2.11 in equation (3.7), we get

$$\left(C^2 + \frac{1}{C^2}\right) \equiv \left(D^4 + \frac{1}{D^4}\right) \pmod{5}. \tag{5.11}$$

Using (5.3)–(5.5), (5.11) can be expressed as

$$\begin{aligned}&\frac{\chi(q)\chi(-q^5)\chi(-q^3)\chi(q^{15})}{\chi(-q)\chi(q^5)\chi(q^3)\chi(-q^{15})}+\frac{\chi(-q)\chi(q^5)\chi(q^3)\chi(-q^{15})}{\chi(q)\chi(-q^5)\chi(-q^3)\chi(q^{15})}\\&\equiv\frac{\chi^2(q)\chi^2(-q^5)\chi^2(q^3)\chi^2(-q^{15})}{\chi^2(-q)\chi^2(q^5)\chi^2(-q^3)\chi^2(q^{15})}+\frac{\chi^2(-q)\chi^2(q^5)\chi^2(-q^3)\chi^2(q^{15})}{\chi^2(q)\chi^2(-q^5)\chi^2(q^3)\chi^2(-q^{15})}\pmod 3.\end{aligned}\tag{5.12}$$

With the aid of (5.6) and (5.7), we arrive at (5.8). □

Proofs of (5.9) and (5.10) are similar to that of (5.8), but rather than (3.7), we use (3.13) and (3.15).

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Author information

M. S. Mahadeva Naika, Department of Mathematics, Bangalore University, Central College Campus, Bengaluru – 560 001, INDIA.
E-mail: `msmnaika@rediffmail.com`

K. Sushan Bairy, PG Department of Mathematics, Vijaya College, R. V. Road, Basavanagudi, Bengaluru – 560 004, INDIA.
E-mail: `ksbairy@rediffmail.com`

D. S. Gireesh, Department of Mathematics, Bangalore University, Central College Campus, Bengaluru – 560 001, INDIA.
E-mail: `gireeshdap@gmail.com`

N. P. Suman, Department of Mathematics, R. V. College of Engineering, R. V. Vidyaniketan Post, Mysore Road, Bengaluru – 560 059, INDIA.
E-mail: `npsuman.13@gmail.com`

Received: May 3, 2016.

Accepted: May 27, 2016.