Palestine Journal of Mathematics

Vol. 6(1)(2017) , 238–246

SOME ALGEBRAIC IDENTITIES IN RINGS AND RINGS WITH INVOLUTION

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Communicated by Ayman Badawi

MSC 2010 Classifications: $16W25$, $16N60$, $16U80$.

Keywords and phrases: Generalized Jordan (θ, ϕ)-derivation, Generalized Jordan left $(\theta, \phi)^*$-derivation, Generalized Jordan $(\theta, \phi)^*$-derivation.

Abstract. In this paper, we study algebraic identities which are (i) $2F(x^{n+1}) = F(x)\theta(x)^n + \phi(x)D(x^n) + F(x^n)\theta(x) + \phi(x)^n D(x)$ (ii) $F(x^{n+1}) = F(x)(\theta(x^*))^n + \sum_{i=1}^{n}(\phi(x))^i D(x)(\theta(x^*))^{n-i}$ (iii) $F(x^{n+1}) = (\theta(x^*))^n F(x) + \sum_{i=1}^{n}(\theta(x^*))^{n-i}(\phi(x))^i D(x)$, where F and D are additive mappings on ring and ring with involution.

1 Introduction

Throughout this paper R denotes an associative ring with identity e and $Z(R)$ denotes the center of R. An additive mapping $x \mapsto x^*$ satisfying $(x^*)^* = x$ and $(xy)^* = y^*x^*$ is called an involution. A ring equipped with an involution is called *-ring or ring with involution. A ring R is said to be prime if for any $a, b \in R$, $aRb = \{0\}$ implies either $a = 0$ or $b = 0$ and and is said to be semiprime if for any $a \in R$, $aRa = 0$ implies $a = 0$. Given an integer $n > 1$, a ring R is said to be n-torsion free if for any $x \in R$, $nx = 0$ implies $x = 0$. An additive mapping D from R to R is said to be a derivation if $D(xy) = D(x)y + xD(y)$ for all $x, y \in R$ and is said to be a Jordan derivation if $D(x^2) = D(x)x + xD(x)$ for all $x \in R$. We notice that every derivation is a Jordan derivation but the converse need not be true. Herstein [9] proved a mile stone result which states that a Jordan derivation on a prime ring R with characteristic different from two is a derivation. A brief proof can be found in Cusack [6]. Cusack [6] generalized Herstein's result and proved that if R is a semi prime ring which is 2-torsion free then every Jordan derivation on R is a derivation. We have divided this paper in two sections. In Section 1, R is any associative ring where as in Section 2, R is any associative ring with involution.

Brešǎr [5] introduced the concept of generalized derivation mapping. An additive mapping F on R is said to be generalized derivation if there exists a derivation D on R such that $F(xy) = F(x)y + xD(y)$ for all $x, y \in R$. An additive mapping F on R is said to be a generalized Jordan derivation if there exists a Jordan derivation D on R such that $F(x^2) = F(x)x + xD(x)$ for all $x \in R$. Vukman [11] proved that if R is a 2-torsion free semi prime ring, then every generalized Jordan derivation on R is a generalized derivation.

An additive mapping $D : R \to R$ is called (θ, ϕ)-derivation (resp. Jordan (θ, ϕ)-derivation) if $D(xy) = D(x)\theta(y) + \phi(x)D(y)$ (resp. $D(x^2) = D(x)\theta(x) + \phi(x)D(x)$) holds for all $x, y \in R$. An additive mapping $F : R \to R$ is said to be generalized (θ, ϕ)-derivation (resp. generalized Jordan (θ, ϕ)-derivation) if there exists an (θ, ϕ)-derivation(resp. Jordan (θ, ϕ)-deviation) $D : R \to R$ such that $F(xy) = F(x)\theta(y) + \phi(x)D(y)$ (resp. $F(x^2) = F(x)\theta(x) + \phi(x)D(x)$) for all $x, y \in R$.

Recently, Dhara and Sharma [7] proved an additive map satisfying an identity to be derivation. In 2013, Ashraf et al. [3] worked on additive mappings satisfying some algebraic identities. In Section 1, we will prove an additive mapping satisfying an algebraic identity to be generalized Jordan (θ, ϕ)-derivation.

In Section 2, we will study the results in rings with involution. Bresar and Vukman [4] studied the notions of a *-derivation and a Jordan*-derivation. Let R be a *- ring. An additive mapping $D : R \to R$ is said to be a *-derivation (resp. Jordan *-derivation) if $D(xy) = D(x)y^* + xD(y)$ (resp. $D(x^2) = D(x)x^* + xD(x)$) holds for all $x, y \in R$. Further, let θ, ϕ be the automorphisms on R. An additive mapping $D : R \to R$ is said to be $(\theta,\phi)^*$-derivation if $D(xy) = D(x)\theta(y^*) + \phi(x)D(y)$ and D is said to be a left $(\theta,\phi)^*$-derivation if $D(xy) = \theta(y^*)D(x) + \phi(x)D(y)$ holds for all $x, y \in R$.

An additive mapping $F : R \to R$ is said to be a generalized *-derivation associated with *-derivation D if $F(xy) = F(x)y^* + xD(y)$ holds for all $x, y \in R$. Further, let θ, ϕ be automorphisms of R. An additive mapping $F : R \to R$ is said to be a generalized $(\theta,\phi)^*$-derivation (resp. generalized Jordan $(\theta,\phi)^*$-derivation) with associated $(\theta,\phi)^*$-derivation D (resp. Jordan $(\theta,\phi)^*$-derivation) if $F(xy) = F(x)\theta(y^*) + \phi(x)D(y)$ (resp. $F(x^2) = F(x)\theta(x^*) + \phi(x)D(x)$) and F is said to be a left generalized $(\theta,\phi)^*$-derivation (resp. Jordan left generalized $(\theta,\phi)^*$-derivation) with associated left $(\theta,\phi)^*$-derivation D (resp. Jordan left $(\theta,\phi)^*$-derivation) if $F(xy) = \theta(y^*)F(x) + \phi(x)D(y)$ (resp. $F(x^2) = \theta(x^*)F(x) + \phi(x)D(x)$) holds for all $x, y \in R$.

Vukman [12] proved the following result: Let R be a 6-torsion free semiprime *-ring. Let $D : R \to R$ be an additive mapping satisfying the relation $D(xyx) = D(x)y^*x^* + xD(y)x^* + xyD(x)$ for all $x, y \in R$. Then D is a Jordan *-derivation. Ali [1] extended this result to Jordan triple $(\theta,\phi)^*$-derivation.

Very recently, N.Rehman et al. [10] considered additive mappings satisfying some algebraic identities on ring with involution. In Section 2, we will define some algebraic identities on ring with involution.

2 Algebraic Identity on Ring

Dhara and Sharma [8] proved an additive map satisfying an identity to be generalized Jordan derivation. Motivated by [8], we define an identity on a ring R and prove the following:

Theorem 2.1. *Let $n \geq 1$ be any fixed integer, R be an $(n+1)!$-torsion free any ring with identity element and θ, ϕ be two automorphisms on R. If $F : R \to R$ and $D : R \to R$ are additive mappings such that $2F(x^{n+1}) = F(x)(\theta(x))^n + \phi(x)D(x^n) + F(x^n)\theta(x) + (\phi(x))^n D(x)$ for all $x \in R$, then D is a Jordan (θ,ϕ)-derivation and F is a generalized Jordan (θ,ϕ)-derivation.*

Proof. We have the identity

$$2F(x^{n+1}) = F(x)(\theta(x))^n + \phi(x)D(x^n) + F(x^n)\theta(x) + (\phi(x))^n D(x) \tag{2.1}$$

holds for all $x \in R$. Replacing x by e in (2.1), where e is an identity of R, we get $2F(e) = 2F(e) + 2D(e)$ which implies $2D(e) = 0$. Since R is $(n+1)!$-torsion free, we get $D(e) = 0$. Now replacing x by $x + le$ in (5), where l is any positive integer, we get

$$\begin{aligned} 2F\left\{(x+le)^{n+1}\right\} &= F(x+le)(\theta(x)+le)^n + (\phi(x)+le)D\left\{(x+le)^n\right\} \\ &\quad + F\left\{(x+le)^n\right\}(\theta(x)+le) + (\phi(x)+le)^n D(x+le) \end{aligned} \tag{2.2}$$

Expanding the powers of $(x + le)$ and using $D(e) = 0$, we get

$$\begin{aligned} &2F\left\{x^{n+1} + \cdots + \binom{n+1}{n-1} l^{n-1}x^2 + \binom{n+1}{n} l^n x + l^{n+1}e\right\} \\ &= F(x+le)\left\{(\theta(x))^n + \cdots + \binom{n}{n-2} l^{n-2}(\theta(x))^2 + \binom{n}{n-1} l^{n-1}\theta(x) + l^n e\right\} \\ &+ (\phi(x)+le)D\left\{x^n + \cdots + \binom{n}{n-2} l^{n-2}x^2 + \binom{n}{n-1} l^{n-1}x + l^n e\right\} \\ &+ F\left\{x^n + \cdots + \binom{n}{n-2} l^{n-2}x^2 + \binom{n}{n-1} l^{n-1}x + l^n e\right\}(\theta(x)+le) \\ &+ \left\{(\phi(x))^n + \cdots + \binom{n}{n-2} l^{n-2}(\phi(x))^2 + \binom{n}{n-1} l^{n-1}\phi(x) + l^n e\right\} D(x) \end{aligned} \tag{2.3}$$

Using (2.1), the above relation can be written as

$$lf_1(\theta(x),\phi(x),e)+l^2f_2(\theta(x),\phi(x),e)+...l^nf_n(\theta(x),\phi(x),e)=0 \tag{2.4}$$

for all $x\in R$. Now, replacing l by $1,2,...,n$ in (2.4) and considering the resulting system of n homogenous equations, we get that the resulting matrix of the system is a Van der Monde matrix

$$\begin{bmatrix} 1 & 1 & \dots & 1 \\ 2 & 2^2 & \dots & 2^n \\ \vdots & \vdots & \ddots & \vdots \\ n & n^2 & \dots & n^n \end{bmatrix}$$

Since the determinent of the matrix is equal to a product of positive integers, each of which is less than n and R is $(n+1)!$ torsion free. It follows that the system has only a zero solution. Thus $f_i(\theta(x),\phi(x),e)=0$ for all $x\in R$ and $i=1,2,...,n$. Now, $f_n(\theta(x),\phi(x),e)=0$ implies that

$$(n+1)F(x)=(n+1)F(e)\theta(x)+(n+1)D(x) \tag{2.5}$$

Again since R is $(n+1)!$-torsion free, we get $F(x)=F(e)\theta(x)+D(x)$ for all $x\in R$. Now, $f_{n-1}(\theta(x),\phi(x),e)=0$ gives

$$\begin{aligned} 2\frac{n(n+1)}{2!}F(x^2) &= nF(x)\theta(x)+\frac{n(n-1)}{2!}F(e)(\theta(x))^2+n\phi(x)D(x)+\frac{n(n-1)}{2!}D(x^2) \\ &\quad +nF(x)\theta(x)+\frac{n(n-1)}{2!}F(x^2)+n\phi(x)D(x) \end{aligned} \tag{2.6}$$

Multiplying both sides by 2 in above equation, we get

$$\begin{aligned} 2n(n+1)F(x^2) &= 4nF(x)\theta(x)+4n\phi(x)D(x)+n(n-1)F(x^2) \\ &\quad +n(n-1)F(e)(\theta(x))^2+n(n-1)D(x^2) \end{aligned} \tag{2.7}$$

Using $(n+1)!$ torsion freeness of R and $F(x)=F(e)\theta(x)+D(x)$, we get $D(x^2)=D(x)\theta(x)+\phi(x)D(x)$, $\forall x\in R$, hence D is a Jordan (θ,ϕ)-derivation in R. Again using $F(x)=F(e)\theta(x)+D(x)$, we get $F(x^2)=F(e)(\theta(x))^2+D(x)\theta(x)+\phi(x)D(x)=F(x)\theta(x)+\phi(x)D(x)$, $\forall x\in R$ which implies that F is a generalized Jordan (θ,ϕ)-derivation in R. Thus the proof of theorem is completed. □

3 Algebraic Identities on Ring with Involution

In 2014, N.Rehman et al. [10] considered the additive mappings $F:R\to R$ and $D:R\to R$ satisfying the condition $F(x^{n+1})=(F(x))(x^*)^n+\sum_{i=1}^n x^iD(x)(x^*)^{n-i}$ for all $x\in R$ and proved that if R is an $(n+1)!$-torsion free *-ring with identity, then D is a Jordan *-derivation and F is a generalized Jordan *-derivation on R. We will extend the results of A. Ansari et al. [2] to ring with involution as follows:

Theorem 3.1. *Let $n\geq 1$ be any fixed integer, R be an $(n+1)!$-torsion free any ring with identity element and θ, ϕ be two automorphisms on R. If $F:R\to R$ and $D:R\to R$ are additive mappings such that $F(x^{n+1})=F(x)(\theta(x^*))^n+\sum_{i=1}^n(\phi(x))^iD(x)(\theta(x^*))^{n-i}$ for all $x\in R$, then D is a Jordan $(\theta,\phi)^*$-derivation and F is a generalized Jordan $(\theta,\phi)^*$-derivation.*

Proof. We have the identity

$$F(x^{n+1})=F(x)(\theta(x^*))^n+\sum_{i=1}^n(\phi(x))^iD(x)(\theta(x^*))^{n-i} \tag{3.1}$$

for all $x \in R$. We replace x by e in (3.1). Clearly $e^* = e$ so that $\theta(e^*) = \phi(e) = e$. Hence, by n-torsion freeness of R, $nD(e) = 0$ implies $D(e) = 0$. Again replacing x by $x + le$ in (3.1), where l is any positive integer, we obtain

$$\begin{aligned} F((x+le)^{n+1}) &= F(x+le)(\theta((x+le)^*))^n \\ &\quad + \sum_{i=1}^{n}(\phi(x+le))^i D(x)(\theta((x+le)^*))^{n-i} \\ &= (F(x)+lF(e))(\theta(x^*)+le)^n \\ &\quad + \sum_{i=1}^{n}(\phi(x)+le)^i D(x)(\theta(x^*)+le)^{n-i} \end{aligned} \tag{3.2}$$

for all $x \in R$. By expanding the powers of $(x + le)$, we get

$$\begin{aligned} &F\left(x^{n+1} + \binom{n+1}{1}x^n l + \binom{n+1}{2}x^{n-1}l^2 + \dots + l^{n+1}e\right) \\ &= (F(x)+lF(e))\left((\theta(x^*))^n + \binom{n}{1}(\theta(x^*))^{n-1}l + \binom{n}{2}(\theta(x^*))^{n-2}l^2 + \dots + l^n e\right) \\ &+ \sum_{i=1}^{n}\left((\phi(x))^i + \binom{i}{1}(\phi(x))^{i-1}l + \binom{i}{2}(\phi(x))^{i-2}l^2 + \dots + l^i e\right)D(x)\left((\theta(x^*))^{n-i}\right. \\ &\left. + \binom{n-i}{1}(\theta(x^*))^{n-i-1}l + \binom{n-i}{2}(\theta(x^*))^{n-i-2}l^2 + \dots + l^{n-i}e\right) \end{aligned} \tag{3.3}$$

for all $x \in R$. (3.3) can be rewritten as

$$\begin{aligned} &F(x^{n+1}) + F\left(\binom{n+1}{1}x^n l + \binom{n+1}{2}x^{n-1}l^2 + \dots + l^{n+1}e\right) \\ &= (F(x)+lF(e))(\theta(x^*))^n \\ &+ (F(x)+lF(e))\left(\binom{n}{1}(\theta(x^*))^{n-1}l + \binom{n}{2}(\theta(x^*))^{n-2}l^2 + \dots + l^n e\right) \\ &+ \sum_{i=1}^{n}(\phi(x))^i D(x)\left((\theta(x^*))^{n-i} + \binom{n-i}{1}(\theta(x^*))^{n-i-1}l + \binom{n-i}{2}(\theta(x^*))^{n-i-2}l^2\right. \\ &\left. + \dots + l^{n-i}e\right) + \sum_{i=1}^{n}\left(\binom{i}{1}(\phi(x))^{i-1}l + \binom{i}{2}(\phi(x))^{i-2}l^2 + \dots + l^i e\right)D(x)\left((\theta(x^*))^{n-i}\right. \\ &\left. + \binom{n-i}{1}(\theta(x^*))^{n-i-1}l + \binom{n-i}{2}(\theta(x^*))^{n-i-2}l^2 + \dots + l^{n-i}e\right) \end{aligned} \tag{3.4}$$

for all $x \in R$. Using (3.1), we have

$$\begin{aligned}
&F\left(\binom{n+1}{1}x^n l + \binom{n+1}{2}x^{n-1}l^2 + ... + l^{n+1}e\right) \\
&= lF(e)(\theta(x^*))^n + (F(x)+lF(e))\left(\binom{n}{1}(\theta(x^*))^{n-1}l + \binom{n}{2}(\theta(x^*))^{n-2}l^2 + ... + l^n e\right) \\
&+ \sum_{i=1}^{n}(\phi(x))^i D(x)\left(\binom{n-i}{1}(\theta(x^*))^{n-i-1}l + \binom{n-i}{2}(\theta(x^*))^{n-i-2}l^2 + ... + l^{n-i}e\right) \\
&+ \sum_{i=1}^{n}\left(\binom{i}{1}(\phi(x))^{i-1}l + \binom{i}{2}(\phi(x))^{i-2}l^2 + ... + l^i e\right)D(x)\Big((\theta(x^*))^{n-i} \\
&+ \binom{n-i}{1}(\theta(x^*))^{n-i-1}l + \binom{n-i}{2}(\theta(x^*))^{n-i-2}l^2 + ... + l^{n-i}e\Big)
\end{aligned} \tag{3.5}$$

for all $x \in R$, where we denote $\binom{n}{k} = 0$ for $k < 0$ and for $k > n$. The above relation can be written as

$$lf_1(\theta(x^*), \phi(x), e) + l^2 f_2(\theta(x^*), \phi(x), e) + ...l^n f_n(\theta(x^*), \phi(x), e) = 0 \tag{3.6}$$

for all $x \in R$. We proceed in similar way as in the proof of Theorem (2.1), we get $f_i(\theta(x^*), \phi(x), e) = 0$, $i = 1, 2, ..., n$. Now, $f_n(\theta(x^*), \phi(x), e) = 0$ implies that

$$\binom{n+1}{n}F(x) = F(x) + \binom{n}{n-1}F(e)\theta(x^*) + nD(x) \tag{3.7}$$

(3.7) implies that

$$(n+1)F(x) = F(x) + nF(e)\theta(x^*) + nD(x) \tag{3.8}$$

Since R is n-torsion free, we obtain

$$F(x) = F(e)\theta(x^*) + D(x) \tag{3.9}$$

Again $f_{n-1}(\theta(x^*), \phi(x), e) = 0$ implies that

$$\begin{aligned}
\binom{n+1}{n-1}F(x^2) = \binom{n}{n-1}F(x)\theta(x^*) + \binom{n}{n-2}F(e)(\theta(x^*))^2 + \frac{n(n+1)}{2}\phi(x)D(x) \\
+ \frac{n(n-1)}{2}D(x)\theta(x^*)
\end{aligned} \tag{3.10}$$

for all $x \in R$. (3.10) can be rewritten as

$$n(n+1)F(x^2) = 2nF(x)\theta(x^*) + n(n-1)F(e)(\theta(x^*))^2 + n(n+1)\phi(x)D(x) + n(n-1)D(x)\theta(x^*) \tag{3.11}$$

for all $x \in R$. Since R is n-torsion free, we get

$$(n+1)F(x^2) = 2F(x)\theta(x^*) + (n-1)F(e)(\theta(x^*))^2 + (n+1)\phi(x)D(x) + (n-1)D(x)\theta(x^*) \tag{3.12}$$

Using (3.9) in (3.12), we find

$$(n+1)F(x^2) = (n+1)F(e)(\theta(x^*))^2 + (n+1)\phi(x)D(x) + (n+1)D(x)\theta(x^*) \tag{3.13}$$

for all $x \in R$. Since R is $(n+1)$-torsion free, so we have

$$F(x^2) = F(e)(\theta(x^*))^2 + \phi(x)D(x) + D(x)\theta(x^*) \tag{3.14}$$

Replacing x by x^2 in (3.9), we obtain

$$F(x^2) = F(e)(\theta(x^*))^2 + D(x^2) \tag{3.15}$$

Comparing (3.14) and (3.15), we find that

$$D(x^2) = D(x)\theta(x^*) + \phi(x)D(x) \tag{3.16}$$

for all $x \in R$. Using (3.16) in (3.15), we get

$$F(x^2) = F(e)(\theta(x^*))^2 + D(x)\theta(x^*) + \phi(x)D(x) = \{F(e)\theta(x^*) + D(x)\}\theta(x^*) + \phi(x)D(x) \tag{3.17}$$

for all $x \in R$. Again, using (3.9) in (3.17), we conclude $F(x^2) = F(x)\theta(x^*) + \phi(x)D(x)$. Thereby the proof of the theorem is completed. □

Corollary 3.2 ([10], Theorem 2.1). *Let $n \geq 1$ be any fixed integer and R be an $(n+1)!$-torsion free any ring with identity element. If $F : R \to R$ and $D : R \to R$ are additive mappings such that $F(x^{n+1}) = (F(x))(x^*)^n + \sum_{i=1}^{n} x^i D(x)(x^*)^{n-i}$ for all $x \in R$, then D is a Jordan *-derivation and F is a generalized Jordan *-derivation.*

Proof. Take $\theta = \phi = I$, where I is the identity map on R. □

Theorem 3.3. *Let $n \geq 1$ be any fixed integer, R be an $(n+1)!$-torsion free any ring with identity element and θ, ϕ be two automorphisms on R. If $F : R \to R$ and $D : R \to R$ are additive mappings such that $F(x^{n+1}) = (\theta(x^*))^n F(x) + \sum_{i=1}^{n} (\theta(x^*))^{n-i}(\phi(x))^i D(x)$ for all $x \in R$, then D is a Jordan left $(\theta, \phi)^*$-derivation and F is a generalized Jordan left $(\theta, \phi)^*$-derivation.*

Proof. We have the identity

$$F(x^{n+1}) = (\theta(x^*))^n F(x) + \sum_{i=1}^{n} (\theta(x^*))^{n-i}(\phi(x))^i D(x) \tag{3.18}$$

for all $x \in R$. We replace x by e in (3.1). Clearly $e^* = e$ so that $\theta(e^*) = \phi(e) = e$. Hence, by n-torsion freeness of R, $nD(e) = 0$ implies $D(e) = 0$. Again replacing x by $(x + le)$ in (3.18), where l is any positive integer, we obtain

$$\begin{aligned} F((x+le)^{n+1}) &= (\theta((x+le)^*))^n F(x+le) \\ &\quad + \sum_{i=1}^{n} (\theta((x+le)^*))^{n-i}(\phi(x+le))^i D(x) \\ &= (\theta(x^*) + le)^n (F(x) + lF(e)) \\ &\quad + \sum_{i=1}^{n} (\theta(x^*) + le)^{n-i}(\phi(x) + le)^i D(x) \end{aligned} \tag{3.19}$$

for all $x \in R$. By expanding the powers of $x + le$, we get

$$\begin{aligned} & F\left(x^{n+1} + \binom{n+1}{1} x^n l + \binom{n+1}{2} x^{n-1} l^2 + \ldots + l^{n+1} e\right) \\ & = \left((\theta(x^*))^n + \binom{n}{1}(\theta(x^*))^{n-1} l + \binom{n}{2}(\theta(x^*))^{n-2} l^2 + \ldots + l^n e\right)(F(x) + lF(e)) \\ & + \sum_{i=1}^{n} \left((\theta(x^*))^{n-i} + \binom{n-i}{1}(\theta(x^*))^{n-i-1} l + \binom{n-i}{2}(\theta(x^*))^{n-i-2} l^2 \right. \\ & \left. + \ldots + l^{n-i} e\right)\left((\phi(x))^i + \binom{i}{1}(\phi(x))^{i-1} l + \binom{i}{2}(\phi(x))^{i-2} l^2 + \ldots + l^i e\right) D(x) \end{aligned} \tag{3.20}$$

for all $x \in R$. (3.20) can be rewritten as

$$\begin{aligned}
&F(x^{n+1}) + F\left(\binom{n+1}{1}x^n l + \binom{n+1}{2}x^{n-1}l^2 + ... + l^{n+1}e\right) \\
&\quad = (\theta(x^*))^n(F(x) + lF(e)) \\
&\quad + \left(\binom{n}{1}(\theta(x^*))^{n-1}l + \binom{n}{2}(\theta(x^*))^{n-2}l^2 + ... + l^n e\right)(F(x) + lF(e)) \\
&\quad + \sum_{i=1}^{n}\left((\theta(x^*))^{n-i} + \binom{n-i}{1}(\theta(x^*))^{n-i-1}l + \binom{n-i}{2}(\theta(x^*))^{n-i-2}l^2\right. \\
&\quad \left. + ... + l^{n-i}e\right)(\phi(x))^i D(x) + \sum_{i=1}^{n}\left((\theta(x^*))^{n-i} + \binom{n-i}{1}(\theta(x^*))^{n-i-1}l\right. \\
&\quad + \left.\binom{n-i}{2}(\theta(x^*))^{n-i-2}l^2 + ... + l^{n-i}e\right)\left(\binom{i}{1}(\phi(x))^{i-1}l + \binom{i}{2}(\phi(x))^{i-2}l^2\right. \\
&\quad \left. + ... + l^i e\right)D(x)
\end{aligned} \tag{3.21}$$

for all $x \in R$. Using (3.18), we have

$$\begin{aligned}
&F\left(\binom{n+1}{1}x^n l + \binom{n+1}{2}x^{n-1}l^2 + ... + l^{n+1}e\right) \\
&= (\theta(x^*))^n lF(e) + \left(\binom{n}{1}(\theta(x^*))^{n-1}l + \binom{n}{2}(\theta(x^*))^{n-2}l^2 + ... + l^n e\right)(F(x) + lF(e)) \\
&+ \sum_{i=1}^{n}\left(\binom{n-i}{1}(\theta(x^*))^{n-i-1}l + \binom{n-i}{2}(\theta(x^*))^{n-i-2}l^2 + ... + l^{n-i}e\right)(\phi(x))^i D(x) \\
&+ \sum_{i=1}^{n}\left((\theta(x^*))^{n-i} + \binom{n-i}{1}(\theta(x^*))^{n-i-1}l + \binom{n-i}{2}(\theta(x^*))^{n-i-2}l^2\right. \\
&+ \left. ... + l^{n-i}e\right)\left(\binom{i}{1}(\phi(x))^{i-1}l + \binom{i}{2}(\phi(x))^{i-2}l^2 + ... + l^i e\right)D(x)
\end{aligned} \tag{3.22}$$

for all $x \in R$, where we denote $\binom{n}{k} = 0$ for $k < 0$ and for $k > n$. The above relation can be written as

$$lf_1(\theta(x^*), \phi(x), e) + l^2 f_2(\theta(x^*), \phi(x), e) + ...l^n f_n(\theta(x^*), \phi(x), e) = 0 \tag{3.23}$$

for all $x \in R$. We proceed in the similar way as in the proof of Theorem (2.1), we get $f_i(\theta(x^*), \phi(x), e) = 0$, $i = 1, 2, ..., n$. Now, $f_n(\theta(x^*), \phi(x), e) = 0$ implies that

$$\binom{n+1}{n}F(x) = F(x) + \binom{n}{n-1}\theta(x^*)F(e) + nD(x) \tag{3.24}$$

(3.24) implies that

$$(n+1)F(x) = F(x) + n\theta(x^*)F(e) + nD(x) \tag{3.25}$$

Since R is n-torsion free, we obtain

$$F(x) = \theta(x^*)F(e) + D(x) \tag{3.26}$$

Again $f_{n-1}(\theta(x^*), \phi(x), e) = 0$ implies that

$$\begin{aligned}
\binom{n+1}{n-1}F(x^2) &= \binom{n}{n-1}\theta(x^*)F(x) + \binom{n}{n-2}(\theta(x^*))^2F(e) + \frac{n(n+1)}{2}\phi(x)D(x) \\
&+ \frac{n(n-1)}{2}\theta(x^*)D(x)
\end{aligned} \tag{3.27}$$

for all $x \in R$. (3.27) can be rewritten as

$$n(n+1)F(x^2) = 2n\theta(x^*)F(x)+n(n-1)(\theta(x^*))^2F(e)+n(n+1)\phi(x)D(x)+n(n-1)\theta(x^*)D(x) \tag{3.28}$$

for all $x \in R$. Since R is n-torsion free, we get

$$(n+1)F(x^2) = 2\theta(x^*)F(x) + (n-1)(\theta(x^*))^2F(e) + (n+1)\phi(x)D(x) + (n-1)\theta(x^*)D(x) \tag{3.29}$$

Using (3.26), (3.29) becomes

$$(n+1)F(x^2) = (n+1)(\theta(x^*))^2F(e) + (n+1)\phi(x)D(x) + (n+1)\theta(x^*)D(x) \tag{3.30}$$

for all $x \in R$. Since R is $(n+1)$-torsion free, so we have

$$F(x^2) = (\theta(x^*))^2F(e) + \phi(x)D(x) + \theta(x^*)D(x) \tag{3.31}$$

Replacing x by x^2 in (3.26), we obtain

$$F(x^2) = (\theta(x^*))^2F(e) + D(x^2) \tag{3.32}$$

Comparing (3.31) and (3.32), we find that

$$D(x^2) = \theta(x^*)D(x) + \phi(x)D(x) \tag{3.33}$$

for all $x \in R$. Using (3.33) in (3.32), we get

$$F(x^2) = (\theta(x^*))^2F(e) + \theta(x^*)D(x) + \phi(x)D(x) = \theta(x^*)\{\theta(x^*)F(e) + D(x)\} + \phi(x)D(x) \tag{3.34}$$

for all $x \in R$. Again, using (3.26) in (3.34), we conclude $F(x^2) = \theta(x^*)F(x) + \phi(x)D(x)$. Thereby the proof of the theorem is completed. □

Corollary 3.4. *Let $n \geq 1$ be any fixed integer, R be an $(n+1)!$-torsion free any ring with identity element and ϕ be an automorphism on R. If $F : R \to R$ and $D : R \to R$ are additive mappings such that $F(x^{n+1}) = (x^*)^nF(x) + \sum_{i=1}^{n}(x^*)^{n-i}(\phi(x))^iD(x)$ for all $x \in R$, then D is a Jordan skew left *-derivation and F is a generalized Jordan skew left *-derivation.*

Proof. Take $\theta = I$, where I is the identity map on R. □

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Received: June 29, 2015.

Accepted: February 26, 2016.