Palestine Journal of Mathematics

Vol. 6(1)(2017) , 247–256

THE MINIMUM HUB ENERGY OF A GRAPH

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Communicated by Ayman Badawi

MSC 2010 Classifications: Primary 05C50; Secondary 05C99.

Keywords and phrases: minimum hub set, minimum hub matrix, minimum hub eigenvalue, minimum hub energy of a graph.

Abstract. In this paper, we introduce minimum hub energy $E_H(G)$ of a graph G and compute minimum hub energies of some standard graphs and a number of well-known families of graphs. Upper and lower bounds for $E_H(G)$ are established.

1 Introduction

Throughout the paper, we consider a simple graph $G = (V, E)$, that is nonempty, finite, having no loops, no multiple and directed edges. Let p and q be the number of its vertices and edges, respectively. The symbols $\Delta(G)$ and $\delta(G)$ denote the maximum degree and the minimum degree of G, respectively. For graph theoretic terminology, we refer to [11].

M. Walsh [16] introduced the theory of hub numbers in the year 2006. Suppose that $H \subseteq V(G)$ and let $x, y \in V(G)$. An H-path between x and y is a path where all intermediate vertices are from H. (This includes the degenerate cases where the path consists of the single edge xy or a single vertex x if $x = y$, call such an H-path trivial). A set $H \subseteq V(G)$ is a hub set of G if it has the property that, for any $x, y \in V(G) - H$, there is an H-path in G between x and y. The smallest size of a hub set in G is called the hub number of G, and is denoted by $h(G)$ [16]. For more details on the hub number see [5]. A set $S \subseteq V(G)$ is called a dominating set of G if each vertex of $V - S$ is adjacent to at least one vertex of S. The domination number of a graph G denoted as $\gamma(G)$ is the minimum cardinality of a dominating set in G [12].

Eigenvalues and Eigenvectors provide insight into the geometry associated with the linear transformation. The concept of energy $E(G)$ of a graph G was introduced by I. Gutman [7] in the year 1978, and is defined as the sum of the absolute values of the eigenvalues of its adjacency matrix $A(G)$. i.e. $E(G) = \Sigma_{i=1}^{p} |\lambda_i|$. Let G be a graph with p vertices and q edges and let $A = (a_{ij})$ be the adjacency matrix of G. The eigenvalues $\lambda_1, \lambda_2, ..., \lambda_p$ of A, assumed in nonincreasing order, are the eigenvalues of the graph G. As A is real symmetric, let $\lambda_1, \lambda_2, ..., \lambda_s$ be the distinct eigenvalues of G with multiplicity $m_1, m_2, ..., m_s$, respectively. The multiset

$$Spec(G) = \begin{pmatrix} \lambda_1 & \lambda_2 & \cdots & \lambda_s \\ m_1 & m_2 & \cdots & m_s \end{pmatrix}$$

of eigenvalues of $A(G)$ is called the adjacency spectrum of G, the eigenvalues of G are real with sum equal to zero. The work of Coulson [4] shows that there is a continuous interest towards the general mathematical properties of the total π-electron energy as calculated within the framework of the Huckel molecular orbital (HMO)model. The properties of this energy are discussed in detail in [2, 8, 9, 10, 15].

We introduce minimum hub energy $E_H(G)$ of a graph G and compute minimum hub energies of some standard graphs and well-known families of graphs. Upper and lower bounds for $E_H(G)$ are established.

2 The minimum hub energy

Let G be a graph of order p with vertex set $V = \{v_1, v_2, ..., v_p\}$ and edge set E. Any hub set H of a graph G with minimum cardinality is called a minimum hub set. Let H be a minimum hub set of G. The minimum hub matrix of G is the $p \times p$ matrix $A_H(G) = (a_{ij})$, where

$$a_{ij} = \begin{cases} 1, & \text{if } v_iv_j \in E; \\ 1, & \text{if } i = j \text{ and } v_i \in H; \\ 0, & \text{otherwise.} \end{cases}$$

The characteristic polynomial of $A_H(G)$ denoted by $f_p(G, \lambda)$ is defined as

$$f_p(G, \lambda) := det(\lambda I - A_H(G)).$$

The minimum hub eigenvalues of the graph G are the eigenvalues of $A_H(G)$. Since $A_H(G)$ is real and symmetric, its eigenvalues are real numbers and we label them in non-increasing order $\lambda_1 \geq \lambda_2 \geq ... \geq \lambda_p$. The minimum hub energy of G is defined as:

$$E_H(G) = \Sigma_{i=1}^{p} |\lambda_i|.$$

Example 2.1. Let $G = P_4$ with vertices v_1, v_2, v_3, v_4 and let its minimum hub set be $H_1 = \{v_1, v_2\}$.
Then the minimum hub matrix of G is

$$A_{H_1}(G) = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

The characteristic polynomial of $A_{H_1}(G)$ is $f_p(G, \lambda) = \lambda^4 - 2\lambda^3 - 2\lambda^2 + 3\lambda$, the minimum hub eigenvalues are $\lambda_1 = 2.3028, \lambda_2 = 1, \lambda_3 = 0, \lambda_4 = -1.3028$, and therefore the minimum hub energy of G is

$$E_{H_1}(G) = 4.6056.$$

If we take another minimum hub set of G, namely $H_2 = \{v_2, v_3\}$, then

$$A_{H_2}(G) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

The characteristic polynomial of $A_{H_2}(G)$ is $f_p(G, \lambda) = \lambda^4 - 2\lambda^3 - 2\lambda^2 + 2\lambda + 1$, the minimum hub eigenvalues are $\lambda_1 = 2.4142, \lambda_2 = 1, \lambda_3 = -0.4142, \lambda_4 = -1$, and therefore the minimum hub energy of G is

$$E_{H_2}(G) = 4.8284.$$

The above example illustrates that the minimum hub energy of a graph G depends on the choice of the minimum hub set. i.e., the minimum hub energy is not a graph invariant. We need the following to prove main results.

Theorem 2.2. *[14] For any (p, q) graph G, $p - q \leq \gamma(G)$. Furthermore, $\gamma(G) = p - q$ if and only if each component of G is a star.*

Lemma 2.3. *[16] For any graph G, $\gamma(G) \leq h(G) + 1$.*

Theorem 2.4. *[16] If G is a connected graph then $h(G) \leq |V(G)| - \Delta(G)$, and the inequality is sharp.*

3 Minimum hub energy of some standard graphs

In this section, we investigate the exact values of the minimum hub energy of some standard graphs.

Theorem 3.1. *For the complete graph $K_p,\ p \geq 2$,*

$$E_H(K_p) = 2p - 2.$$

Proof. Let K_p be the complete graph with vertex set $V = \{v_1, v_2, \cdots, v_p\}$. Then the minimum hub number is $h(K_p) = 0$. Then

$$A_H(K_p) = \begin{pmatrix} 0 & 1 & 1 & \cdots & 1 & 1 & \cdots & 1 & 1 \\ 1 & 0 & 1 & \cdots & 1 & 1 & \cdots & 1 & 1 \\ 1 & 1 & 0 & \cdots & 1 & 1 & \cdots & 1 & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 1 & 1 & \cdots & 0 & 1 & \cdots & 1 & 1 \\ 1 & 1 & 1 & \cdots & 1 & 0 & \cdots & 1 & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 1 & 1 & \cdots & 1 & 1 & \cdots & 0 & 1 \\ 1 & 1 & 1 & \cdots & 1 & 1 & \cdots & 1 & 0 \end{pmatrix}_{p \times p}$$

The respective characteristic polynomial is

$$f_p(K_p, \lambda) = \begin{vmatrix} \lambda & -1 & -1 & \cdots & -1 & -1 & \cdots & -1 & -1 \\ -1 & \lambda & -1 & \cdots & -1 & -1 & \cdots & -1 & -1 \\ -1 & -1 & \lambda & \cdots & -1 & -1 & \cdots & -1 & -1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\ -1 & -1 & -1 & \cdots & \lambda & -1 & \cdots & -1 & -1 \\ -1 & -1 & -1 & \cdots & -1 & \lambda & \cdots & -1 & -1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\ -1 & -1 & -1 & \cdots & -1 & -1 & \cdots & \lambda & -1 \\ -1 & -1 & -1 & \cdots & -1 & -1 & \cdots & -1 & \lambda \end{vmatrix}$$

$$= (\lambda - (p-1))(\lambda + 1)^{p-1}.$$

The spectrum of K_p will be written as

$$MH\ Spec(K_p) = \begin{pmatrix} p-1 & -1 \\ 1 & p-1 \end{pmatrix}$$

Hence, the minimum hub energy of a complete graph is $E_H(K_p) = 2p - 2$. □

Theorem 3.2. *For the complete bipartite graph $K_{n,n},\ n \geq 3$, the minimum hub energy is $n + 1 + (n-1)\sqrt{n}$.*

Proof. For the complete bipartite graph $K_{n,n},\ n \geq 3$ with vertex set $V = \{u_1, u_2, \cdots, u_n, v_1, v_2, \cdots, v_n\}$. The minimum hub set is $H = \{u_1, v_1\}$. Then

$$A_H(K_{n,n}) = \begin{pmatrix} 1 & 0 & \cdots & 0 & 1 & 1 & \cdots & 1 \\ 0 & 0 & \cdots & 0 & 1 & 1 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 1 & 1 & \cdots & 1 \\ 1 & 1 & \cdots & 1 & 1 & 0 & \cdots & 0 \\ 1 & 1 & \cdots & 1 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 1 & 0 & 0 & \cdots & 0 \end{pmatrix}_{(2n) \times (2n)}$$

The characteristic polynomial of $A_H(K_{n,n})$ is

$$f_{2n}(K_{n,n},\lambda)=\begin{vmatrix} \lambda-1 & 0 & \cdots & 0 & -1 & -1 & \cdots & -1 \\ 0 & 0 & \cdots & 0 & -1 & -1 & \cdots & -1 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & -1 & -1 & \cdots & -1 \\ -1 & -1 & \cdots & -1 & \lambda-1 & 0 & \cdots & 0 \\ -1 & -1 & \cdots & -1 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ -1 & -1 & \cdots & -1 & 0 & 0 & \cdots & 0 \end{vmatrix}$$

$$=\lambda^{2n-4}(\lambda^2-(n+1)\lambda+(n-1))(\lambda^2+(n-1)\lambda-(n-1)),$$

and

$$MH\ Spec(K_{n,n})=\begin{pmatrix} 0 & \frac{n+1}{2}+\frac{\sqrt{n^2-2n+5}}{2} & \frac{n+1}{2}-\frac{\sqrt{n^2-2n+5}}{2} & \frac{1-n}{2}+\frac{(n-1)\sqrt{n}}{2} & \frac{1-n}{2}-\frac{(n-1)\sqrt{n}}{2} \\ 2n-4 & 1 & 1 & 1 & 1 \end{pmatrix}$$

Hence, $E_H(K_{n,n})=n+1+(n-1)\sqrt{n}$. □

Theorem 3.3. *For $p\geq 2$, the minimum hub energy of a star graph $K_{1,p-1}$ is equal to $\sqrt{4p-3}$.*

Proof. Let $K_{1,p-1}$ be a star graph with vertex set $V=\{v_0,v_1,v_2,\cdots,v_{p-1}\}$, v_0 is the center, and the minimum hub set is $H=\{v_0\}$. Then

$$A_H(K_{1,p-1})=\begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & 0 & 0 & \cdots & 0 \\ 1 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & 0 & \cdots & 0 \end{pmatrix}_{p\times p}$$

The characteristic polynomial of $A_H(K_{1,p-1})$ is

$$f_p(K_{1,p-1},\lambda)=\begin{vmatrix} \lambda-1 & -1 & -1 & \cdots & -1 \\ -1 & \lambda & 0 & \cdots & 0 \\ -1 & 0 & \lambda & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -1 & 0 & 0 & \cdots & \lambda \end{vmatrix}$$

$$=\lambda^{p-2}(\lambda^2-\lambda-(p-1)).$$

and

$$MH\ Spec(K_{1,p-1})=\begin{pmatrix} 0 & \frac{1+\sqrt{4p-3}}{2} & \frac{1-\sqrt{4p-3}}{2} \\ p-2 & 1 & 1 \end{pmatrix}$$

Therefore , $E_H(K_{1,p-1})=\sqrt{4p-3}$. □

Definition 3.4. [6] The double star graph $S_{n,m}$ (see Figure 1) is the graph constructed from $K_{1,n-1}$ and $K_{1,m-1}$ by joining their centers v_0 and u_0. A vertex set $V(S_{n,m})=V(K_{1,n-1})\cup V(K_{1,m-1})=\{v_0,v_1,...,v_{n-1},u_0,u_1,...,u_{m-1}\}$ and edge set $E(S_{n,m})=\{v_0u_0,v_0v_i,u_0u_j|1\leq i\leq n-1,1\leq j\leq m-1\}$.

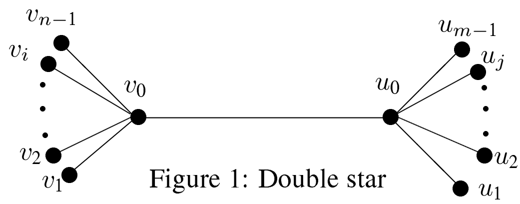

Figure 1: Double star

Theorem 3.5. *For $n \geq 3$, the minimum hub energy of the double star $S_{n,n}$ is equal to $2(\sqrt{n-1}+\sqrt{n})$.*

Proof. For the double star graph $S_{n,n}$ with vertex set $V = \{v_0, v_1, ..., v_{n-1}, u_0, u_1, ..., u_{n-1}\}$, the minimum hub set is $H = \{v_0, u_0\}$. Then

$$A_H(S_{n,n}) = \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 & 1 & 0 & \cdots & 0 \\ 1 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ 1 & 0 & 0 & \cdots & 0 & 1 & 1 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 \end{pmatrix}_{2n \times 2n}$$

The characteristic polynomial of $A_H(S_{n,n})$ is

$$f_{2n}(S_{n,n}, \lambda) = \begin{vmatrix} \lambda-1 & -1 & -1 & \cdots & -1 & -1 & 0 & \cdots & 0 \\ -1 & \lambda & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ -1 & 0 & 0 & \cdots & \lambda & 0 & 0 & \cdots & 0 \\ -1 & 0 & 0 & \cdots & 0 & \lambda-1 & -1 & \cdots & -1 \\ 0 & 0 & 0 & \cdots & 0 & -1 & \lambda & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & -1 & 0 & \cdots & \lambda \end{vmatrix}$$

$$= \lambda^{\frac{n-4}{2}}(\lambda^2 - (n-1))(\lambda^2 - 2\lambda - (n-1)).$$

and

$$MH\ Spec(S_{n,n}) = \begin{pmatrix} 0 & \sqrt{n-1} & -\sqrt{n-1} & 1+\sqrt{n} & 1-\sqrt{n} \\ 2n-4 & 1 & 1 & 1 & 1 \end{pmatrix}$$

Hence, $E_H(S_{n,n}) = 2(\sqrt{n-1} + \sqrt{n})$. □

Definition 3.6. [1] The cocktail party graph, denoted by $K_{2\times p}$, is a graph having vertex set $V(G) = \bigcup_{i=1}^{p}\{u_i, v_i\}$ and edge set $E(G) = \{u_iu_j, v_iv_j, u_iv_j, v_iu_j : 1 \leq i < j \leq p\}$. i.e. $|V(G)| = 2p$, $|E(G)| = \frac{p^2-3p}{2}$.

Theorem 3.7. *For the cocktail party graph $K_{2\times p}$, the minimum hub energy is*

$$E_H(K_{2\times p}) \geq (4p-7) + 2\sqrt{2p}.$$

Proof. Let $K_{2\times p}$ be the cocktail party graph, having vertex set $V(K_{2\times p}) = \bigcup_{i=1}^{p}\{u_i, v_i\}$. Then the hub number of $K_{2\times p}$ is

$$h(K_{2\times p}) = 1.$$

Therefore, $H = \{u_1\}$. Then

$$A_H(K_{2\times p}) = \begin{pmatrix} 1 & 0 & 1 & 1 & \cdots & 1 & 1 \\ 0 & 0 & 1 & 1 & \cdots & 1 & 1 \\ 1 & 1 & 0 & 0 & \cdots & 1 & 1 \\ 1 & 1 & 0 & 0 & \cdots & 1 & 1 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 1 & 1 & 1 & \cdots & 0 & 0 \\ 1 & 1 & 1 & 1 & \cdots & 0 & 0 \end{pmatrix}_{2p\times 2p}$$

The characteristic polynomial of $A_H(K_{2\times p})$ is

$$f_{2p}(K_{2\times p},\lambda) = \begin{bmatrix} \lambda-1 & 0 & -1 & -1 & \cdots & -1 & -1 \\ 0 & \lambda & -1 & -1 & \cdots & -1 & -1 \\ -1 & -1 & \lambda & 0 & \cdots & -1 & -1 \\ -1 & -1 & 0 & \lambda & \cdots & -1 & -1 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ -1 & -1 & -1 & -1 & \cdots & \lambda & 0 \\ -1 & -1 & -1 & -1 & \cdots & 0 & \lambda \end{bmatrix}$$

$$\begin{aligned} &= \lambda^{p-1}(\lambda+2)^{p-2}(\lambda^3-(2p-3)\lambda^2-2p\lambda+(2p-2)) \\ &= \lambda^{p-1}(\lambda+2)^{p-2}\left[(\lambda^3-(2p-3)\lambda^2-2p\lambda+(2p-2))-(4p^2-4p-2)\right] \\ &\geq \lambda^{p-1}(\lambda+2)^{p-2}\left[\lambda^2(\lambda-(2p-3))-2p(\lambda-(2p-3))\right] \\ &= \lambda^{p-1}(\lambda+2)^{p-2}\left[(\lambda-(2p-3))(\lambda^2-2p)\right] \end{aligned}$$

Therefore,

$$MH\ Spec(K_{2\times p}) \approxeq \begin{pmatrix} -2 & 0 & 2p-3 & \sqrt{2p} & -\sqrt{2p} \\ p-2 & p-1 & 1 & 1 & 1 \end{pmatrix},$$

where $\approxeq$ represents approximately equal. Hence, $E_H(K_{2\times p}) \geq (4p-7)+2\sqrt{2p}$. □

4 Some properties of minimum hub energy of graphs

In this section, we introduce some properties of characteristic polynomials of minimum hub matrix of a graph G and some properties of minimum hub eigenvalues.

Theorem 4.1. *Let G be a graph of order p, size q , and hub number $h(G)$. Let $f_p(G,\lambda) = c_0\lambda^p + c_1\lambda^{p-1} + c_2\lambda^{p-2} + ... + c_p$ be the characteristic polynomial of minimum hub matrix of G. Then*

(i) $c_0 = 1$.

(ii) $c_1 = -h(G)$.

(iii) $c_2 = \begin{pmatrix} h(G) \\ 2 \end{pmatrix} - q$.

Proof. (i) Follows by the definition of $f_p(G,\lambda)$.

(ii) Since the sum of diagonal elements of $A_H(G)$ is equal to $|H| = h(G)$, the sum of determinants of all 1×1 principal submatrices of $A_H(G)$ is the trace of $A_H(G)$, which evidently is equal to $h(G)$. Thus, $(-1)^1 c_1 = h(G)$.

(iii) $(-1)^2 c_2$ is equal to the sum of determinants of all 2×2 principal submatrices of $A_H(G)$, that is

$$\begin{aligned} c_2 &= \sum_{1 \le i < j \le p} \begin{vmatrix} a_{ii} & a_{ij} \\ a_{ji} & a_{jj} \end{vmatrix} \\ &= \sum_{1 \le i < j \le p} (a_{ii}a_{jj} - a_{ij}a_{ji}) \\ &= \sum_{1 \le i < j \le p} a_{ii}a_{jj} - \sum_{1 \le i < j \le p} a_{ij}^2 \\ &= \binom{h(G)}{2} - q. \end{aligned}$$

□

Theorem 4.2. *Let $\lambda_1, \lambda_2, ..., \lambda_p$ be the eigenvalues of $A_H(G)$. Then*

(i) $\sum_{i=1}^{p} \lambda_i = h(G)$.

(ii) $\sum_{i=1}^{p} \lambda_i^2 = h(G) + 2q$.

Proof. (i) Since the sum of eigenvalues of $A_H(G)$ is the trace of $A_H(G)$, we have

$$\sum_{i=1}^{p} \lambda_i = \sum_{i=1}^{p} a_{ii} = |H| = h(G).$$

(ii) Similarly, the sum of squares of the eigenvalues of $A_H(G)$ is the trace of $(A_H(G))^2$. Then

$$\begin{aligned} \sum_{i=1}^{p} \lambda_i^2 &= \sum_{i=1}^{p} \sum_{j=1}^{p} a_{ij}a_{ji} \\ &= \sum_{i=1}^{p} a_{ii}^2 + \sum_{i \ne j}^{p} a_{ij}a_{ji} \\ &= \sum_{i=1}^{p} a_{ii}^2 + 2\sum_{i<j}^{p} a_{ij}^2 \\ &= |H| + 2q \\ &= h(G) + 2q. \end{aligned}$$

□

Theorem 4.3. *Let G be a graph of order p, size q, and let $\lambda_1(G)$ be the largest minimum hub eigenvalue of $A_H(G)$. Then*

$$\lambda_1(G) \ge \frac{2q + h(G)}{p}.$$

Proof. Let G be a graph of order p and let λ_1 be the largest minimum hub eigenvalue of $A_H(G)$. Then from [2] we have $\lambda_1 = \max_{X \ne 0} \left\{ \frac{X^t A X}{X^t X} \right\}$, where X is any nonzero vector and X^t is its transpose and A is a matrix . If we take $X = J = \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}$ then we get

$$\lambda_1 \ge \frac{J^t A_H(G) J}{J^t J} = \frac{2q + h(G)}{p}.$$

□

5 Bounds on minimum hub energy of graphs

In this section, we shall investigate some bounds for minimum hub energy of graphs.

Theorem 5.1. *Let G be a connected graph of order p and size q. Then*

$$\sqrt{2q+h(G)} \leq E_H(G) \leq \sqrt{p(2q+h(G))}$$

Proof. Consider the Cauchy-Schwartz inequality

$$\left(\sum_{i=1}^{p} a_i b_i\right)^2 \leq \left(\sum_{i=1}^{p} a_i^2\right)\left(\sum_{i=1}^{p} b_i^2\right).$$

By choosing $a_i = 1$ and $b_i = |\lambda_i|$, we get

$$(E_H(G))^2 = \left(\sum_{i=1}^{p} |\lambda_i|\right)^2 \leq \left(\sum_{i=1}^{p} 1\right)\left(\sum_{i=1}^{p} \lambda_i^2\right) = p\,(2q+h(G)).$$

Therefore, the upper bound holds.
Now, since

$$\left(\sum_{i=1}^{p} |\lambda_i|\right)^2 \geq \sum_{i=1}^{p} \lambda_i^2,$$

we have $(E_H(G))^2 \geq \sum_{i=1}^{p} \lambda_i^2 = 2q + h(G)$. Therefore, $E_H(G) \geq \sqrt{2q+h(G)}$. □

Theorem 5.2. *For a connected graph G of order p and size q,*

$$\sqrt{2p-q-1} \leq E_H(G) \leq p\sqrt{p-\frac{\Delta}{p}}.$$

Proof. By Lemma 2.3, and Theorem 2.4, we have

$$\gamma(G) - 1 \leq h(G) \leq p - \Delta \tag{5.1}$$

Since for any graph, $2q \leq p^2 - p$, it follow by Theorem 5.1, that

$$E_H(G) \leq \sqrt{p(2q+h(G))} \leq \sqrt{p[(p^2-p)+p-\Delta]} = p\sqrt{p-\frac{\Delta}{p}}.$$

For the lower bound, since for any connected graph $p \leq 2q$, by Theorem 5.1, Equation 5.1, and Theorem 2.2, we get

$$E_H(G) \geq \sqrt{2q+h(G)} \geq \sqrt{p+\gamma(G)-1} \geq \sqrt{p+p-q-1} = \sqrt{2p-q-1}.$$

□

Theorem 5.3. *Let G be a graph with p vertices and q edges. Then*

$$E_H(G) \leq \frac{2q+h(G)}{p} + \sqrt{(p-1)\left[2q+h(G)-\left(\frac{2q+h(G)}{p}\right)^2\right]}$$

Proof. Consider the Cauchy-Schwartz inequality

$$\left(\sum_{i=1}^{p} a_i b_i\right)^2 \leq \left(\sum_{i=1}^{p} a_i^2\right)\left(\sum_{i=1}^{p} b_i^2\right).$$

By choosing $a_i = 1$ and $b_i = |\lambda_i|$, we get

$$\left(\sum_{i=2}^{p} |\lambda_i|\right)^2 \leq \left(\sum_{i=2}^{p} 1\right)\left(\sum_{i=2}^{p} \lambda_i^2\right).$$

By Theorem 4.2, we have

$$(E_H(G) - |\lambda_1|)^2 \leq (p-1)(2q + h(G) - \lambda_1^2).$$

Therefore,

$$E_H(G) \leq \lambda_1 + \sqrt{(p-1)(2q + h(G) - \lambda_1^2)}.$$

From Theorem 4.3, we have $\lambda_1 \geq \frac{2q+h(G)}{p}$.
Since $f(x) = x + \sqrt{(p-1)(2q + h(G) - x^2)}$ is a decreasing function, it follows that

$$f(\lambda_1) \leq f(\frac{2q + h(G)}{p}).$$

Thus,

$$E_H(G) \leq f(\lambda_1) \leq f(\frac{2q + h(G)}{p}).$$

Therefore,

$$E_H(G) \leq \frac{2q + h(G)}{p} + \sqrt{(p-1)\left[2q + h(G) - \left(\frac{2q + h(G)}{p}\right)^2\right]}.$$

□

Theorem 5.4. *Let G be a connected graph of order and size p and q, respectively. If $K = det(A_H(G))$, then*

$$E_H(G) \geq \sqrt{2q + h(G) + p(p-1)K^{2/p}}.$$

Proof. Since

$$(E_H(G))^2 = \left(\sum_{i=1}^{p} |\lambda_i|\right)^2 = \left(\sum_{i=1}^{p} |\lambda_i|\right)\left(\sum_{i=1}^{p} |\lambda_i|\right) = \sum_{i=1}^{p} |\lambda_i|^2 + \sum_{i \neq j} |\lambda_i||\lambda_j|.$$

using the inequality between the arithmetic and geometric means, we get

$$\frac{1}{p(p-1)} \sum_{i \neq j} |\lambda_i||\lambda_j| \geq \left(\prod_{i \neq j} |\lambda_i||\lambda_j|\right)^{1/[p(p-1)]}.$$

Thus

$$\begin{aligned}
(E_H(G))^2 &\geq \sum_{i=1}^{p} |\lambda_i|^2 + p(p-1)\left(\prod_{i \neq j} |\lambda_i||\lambda_j|\right)^{1/[p(p-1)]} \\
&\geq \sum_{i=1}^{p} |\lambda_i|^2 + p(p-1)\left(\prod_{i=1}^{p} |\lambda_i|^{2(p-1)}\right)^{1/[p(p-1)]} \\
&= \sum_{i=1}^{p} |\lambda_i|^2 + p(p-1)\left|\prod_{i=1}^{p} \lambda_i\right|^{2/p} \\
&= 2q + h(G) + p(p-1)K^{2/p}.
\end{aligned}$$

□

Theorem 5.5. *Let G be a graph with a minimum hub set H. If the minimum hub energy $E_H(G)$ of G is a rational number, then*

$$E_H(G) \equiv |H| (mod\, 2).$$

Proof. Let $\lambda_1, \lambda_2, ..., \lambda_p$ be minimum hub eigenvalues of a graph G of which $\lambda_1, \lambda_2, ..., \lambda_s$ are positive and the remaining are non-positive, then

$$\sum_{i=1}^{p} |\lambda_i| = (\lambda_1 + \lambda_2 + ... + \lambda_s) - (\lambda_{s+1} + ... + \lambda_p)$$
$$= 2(\lambda_1 + \lambda_2 + ... + \lambda_s) - (\lambda_1 + \lambda_2 + ... + \lambda_p)$$

i.e. $E_H(G) = 2(\lambda_1 + \lambda_2 + ... + \lambda_s) - |H|$. Since $\lambda_1, \lambda_2, ..., \lambda_s$ are algebraic integers, so is their sum. Therefore $(\lambda_1 + \lambda_2 + ... + \lambda_s)$ must be an integer if $E_H(G)$ is rational. Hence the theorem. □

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Received: July 4, 2015.

Accepted: February 20, 2016.