Palestine Journal of Mathematics

Vol. 6(1)(2017) , 257–270

ON THE HERMITE-HADAMARD-TYPE AND OSTROWSKI-TYPE INEQUALITIES FOR THE CO-ORDINATED CONVEX FUNCTIONS

Samet Erden and Mehmet Z. Sarikaya

Communicated by Ayman Badawi

MSC 2010 Classifications: Primary 26D07; Secondary 26D15.

Keywords and phrases: Convex function, co-ordinated convex mapping, Ostrowski inequality, Hölder's inequality.

Abstract. In this paper, we give new Hermite-Hadamard-type and Ostrowski-type inequalities of convex functions of 2-variables on the co-ordinates by using Hölder's inequality. Our established results generalize some recent results for functions whose partial derivatives in qth power of absolute value are convex on the co-ordinates on the rectangle from the plane.

1 Introduction

In 1938, the classical integral inequality established by Ostrowski [15] as follows:

Theorem 1.1. *Let $f : [a,b] \to \mathbb{R}$ be a differentiable mapping on (a,b) whose derivative $f^{'} : (a,b) \to \mathbb{R}$ is bounded on (a,b), i.e., $\|f'\|_\infty = \sup_{t\in(a,b)} |f'(t)| < \infty$. Then, the inequality holds:*

$$\left| f(x) - \frac{1}{b-a}\int_a^b f(t)dt \right| \leq \left[\frac{1}{4} + \frac{(x-\frac{a+b}{2})^2}{(b-a)^2}\right](b-a)\|f'\|_\infty \tag{1.1}$$

for all $x \in [a,b]$. The constant $\frac{1}{4}$ is the best possible.

Inequality (1.1) has wide applications in numerical analysis and in the theory of some special means; estimating error bounds for some special means, some mid-point, trapezoid and Simpson rules and quadrature rules, etc. In addition, the current approach of obtaining the bounds, for a particular quadrature rule, have depended on the use of Peano kernel. The general approach in the past has involved the assumption of bounded derivatives of degree greater than one.

In a recent paper [3], Barnett and Dragomir proved the following Ostrowski type inequality for double integrals:

Theorem 1.2. *Let $f : [a,b] \times [c,d] \to \mathbb{R}$ be continuous on $[a,b] \times [c,d]$, $f''_{x,y} = \frac{\partial^2 f}{\partial x \partial y}$ exists on $(a,b) \times (c,d)$ and is bounded, i.e.,*

$$\left\|f''_{x,y}\right\|_\infty = \sup_{(x,y)\in(a,b)\times(c,d)} \left|\frac{\partial^2 f(x,y)}{\partial x \partial y}\right| < \infty.$$

Then, we have the inequality:

$$\begin{aligned}&\left|\int_a^b\int_c^d f(s,t)dtds - (d-c)(b-a)f(x,y)\right.\\ &\left. \qquad - \left[(b-a)\int_c^d f(x,t)dt + (d-c)\int_a^b f(s,y)ds\right]\right| \qquad (1.2)\\ \leq\ & \left[\frac{1}{4}(b-a)^2 + (x-\frac{a+b}{2})^2\right]\left[\frac{1}{4}(d-c)^2 + (y-\frac{d+c}{2})^2\right]\left\|f''_{x,y}\right\|_\infty\end{aligned}$$

for all $(x, y) \in [a, b] \times [c, d]$.

In [3], the inequality (1.2) is established by the use of integral identity involving Peano kernels. In [16], Pachpatte obtained an inequality in the view (1.2) by using elementary analysis. The interested reader is also refered to ([3], [5], [16]) for Ostrowski type inequalities in several independent variables and for recent weighted version of these type inequalities see [1], [2], [20], [21] and [22].

Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be a convex mapping defined on the interval I of real numbers and $a, b \in I$, with $a < b$. The following double inequality is well known in the literature as the Hermite-Hadamard inequality [9]:

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}. \tag{1.3}$$

The inequalities (1.3) have grown into a significant pillar for mathematical analysis and optimization, besides, by looking into a variety of settings, these inequalities are found to have a number of uses. What is more, for a specific choice of the function f, many inequalities with special means are obtainable. Hermite Hadamard's inequality (1.3), for example, is significant in its rich geometry and hence there are many studies on it to demonstrate its new proofs, refinements, extensions and generalizations. You can check ([6], [7], [10], [11], [19]) and the references included there.

Let us now consider a bidimensional interval $\Delta =: [a, b] \times [c, d]$ in $\mathbb{R}^2$ with $a < b$ and $c < d$. A mapping $f : \Delta \to \mathbb{R}$ is said to be convex on Δ if the following inequality:

$$f(tx + (1-t)z, ty + (1-t)w) \leq tf(x, y) + (1-t)f(z, w)$$

holds, for all $(x, y), (z, w) \in \Delta$ and $t \in [0, 1]$. A function $f : \Delta \to \mathbb{R}$ is said to be on the co-ordinates on Δ if the partial mappings $f_y : [a, b] \to \mathbb{R}$, $f_y(u) = f(u, y)$ and $f_x : [c, d] \to \mathbb{R}$, $f_x(v) = f(x, v)$ are convex where defined for all $x \in [a, b]$ and $y \in [c, d]$ (see,[5])

A formal definition for co-ordinated convex function may be stated as follows:

Definition 1.3. A function $f : \Delta \to \mathbb{R}$ will be called co-ordinated canvex on Δ, for all $t, s \in [0, 1]$ and $(x, y), (u, v) \in \Delta$, if the following inequality holds:

$$\begin{aligned} & f(tx + (1-t)y, su + (1-s)v) \\ \leq \quad & tsf(x, u) + s(1-t)f(y, u) + t(1-s)f(x, v) + (1-t)(1-s)f(y, v). \end{aligned}$$

Clearly, every convex function is co-ordinated convex. Furthermore, there exist co-ordinated convex function which is not convex, (see, [5]). For several recent results concerning Hermite-Hadamard's inequality for some convex function on the co-ordinates on a rectangle from the plane $\mathbb{R}^2$, we refer the reader to ([12], [17], [18], [8], [14], [23]-[26]).

Also, in [5], Dragomir establish the following similar inequality of Hadamard's type for co-ordinated convex mapping on a rectangle from the plane $\mathbb{R}^2$.

Theorem 1.4. *Suppose that $f : \Delta \to \mathbb{R}$ is co-ordinated convex on Δ. Then one has the inequali-*

ties:

$$
\begin{aligned}
& f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) && (1.4)\\
\leq\ & \frac{1}{2}\left[\frac{1}{b-a} \int_a^b f\left(x, \frac{c+d}{2}\right) dx+\frac{1}{d-c} \int_c^d f\left(\frac{a+b}{2}, y\right) dy\right] \\
\leq\ & \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(x, y)\, dy dx \\
\leq\ & \frac{1}{4}\left[\frac{1}{b-a} \int_a^b f(x, c)\, dx+\frac{1}{b-a} \int_a^b f(x, d)\, dx\right. \\
& \left.+\frac{1}{d-c} \int_c^d f(a, y)\, dy+\frac{1}{d-c} \int_c^d f(b, y)\, dy\right] \\
\leq\ & \frac{f(a, c)+f(a, d)+f(b, c)+f(b, d)}{4} .
\end{aligned}
$$

The above inequalities are sharp.

In [23], Sarikaya et al. proved some new inequalities that give estimate of the deference between the middle and the right most terms in (1.4) for differentiable co-ordinated convex functions on rectangele from the plane $\mathbb{R}^2$. For several recent results concerning Hermite-Hadamard's inequality for some convex function on the co-ordinates on a rectangle from the plane $\mathbb{R}^2$; we refer the reader to ([4], [13]).

In [26], Sarikaya proved the following Lemma for double integrals. The lemma is necessary and plays an important role in establishing our main results. Firstly, the $S_\lambda(f; g, h)$ operator that we will use throughout the article may be defined as follow:

$$
\begin{aligned}
& S_\lambda(f ; g, h):=\left(\int_a^b g(u) du\right)\left(\int_c^d h(u) du\right) && (1.5)\\
& \times\left[(1-\lambda)^2 f(x, y)+\lambda(1-\lambda) f(b, y)+\lambda(1-\lambda) f(x, d)+\lambda^2 f(b, d)\right] \\
& -\left(\int_a^x g(u) du\right) \int_c^d h(s)[(1-\lambda) f(x, s)+\lambda f(a, s)]\, ds
\end{aligned}
$$

$$
\begin{aligned}
& -\left(\int_x^b g(u) du\right) \int_c^d h(s)[(1-\lambda) f(x, s)+\lambda f(b, s)]\, ds \\
& -\left(\int_c^y h(u) du\right) \int_a^b g(t)[(1-\lambda) f(t, y)+\lambda f(t, c)]\, dt \\
& -\left(\int_y^d h(u) du\right) \int_a^b g(t)[(1-\lambda) f(t, y)+\lambda f(t, d)]\, dt \\
& +\int_a^b \int_c^d g(t) h(s) f(t, s)\, ds dt .
\end{aligned}
$$

Lemma 1.5. *Let $f:[a,b]\times[c,d]\subset\mathbb{R}^2\to\mathbb{R}$ be an absolutely continuous function such that the partial derivative of order 2 exists for all $(t,s)\in[a,b]\times[c,d]=:\Delta$, and the functions*

$g : [a,b] \to [0,\infty)$ *and* $h : [c,d] \to [0,\infty)$ *are integrable. If* $f_{ts}(t,s) \in L(\Delta)$, *then the following equality holds:*

$$\int_a^b \int_c^d P_\lambda(x,t)\, Q_\lambda(y,s)\, f_{ts}(t,s)\, dsdt = S_\lambda(f;g,h)$$

where

$$P_\lambda(x,t) := \begin{cases} (1-\lambda)\int_a^t g(u)\,du + \lambda \int_x^t g(u)\,du & ,a \le t < x \\ \\ (1-\lambda)\int_b^t g(u)\,du + \lambda \int_x^t g(u)\,du & ,x \le t \le b. \end{cases}$$

and

$$Q_\lambda(y,s) := \begin{cases} (1-\lambda)\int_c^s h(u)\,du + \lambda \int_y^s h(u)\,du & ,a \le t < x \\ \\ (1-\lambda)\int_d^s h(u)\,du + \lambda \int_y^s h(u)\,du & ,x \le t \le b. \end{cases}$$

for $\lambda \in [0,1]$.

The main purpose of this paper is to establish new Hadamard-type inequalities of convex functions of 2-variables on the co-ordinates by using Lemma 1.5, Holder's inequality and elementary analysis.

2 Hadamard's Type Inequalities and Results

For convenience, we give the following notations used to simplify the details of presentation,

$$A_\lambda(x) = (1-\lambda)(3b-a-2x) + \lambda(3b-2a-x),$$

$$B_\lambda(x) = (1-\lambda)(b-3a+2x) + \lambda(2b-3a+x),$$

$$C_\lambda(y) = (1-\lambda)(3d-c-2y) + \lambda(3d-2c-y),$$

$$D_\lambda(y) = (1-\lambda)(d-3c+2y) + \lambda(2d-3c+y),$$

$$\begin{aligned} A_\lambda(x,y) &= A_\lambda(x)\, C_\lambda(y)\, |f_{ts}(a,c)|^q + (2-\lambda)(y-c)\, A_\lambda(x)\, |f_{ts}(a,d)|^q \\ &\quad + (2-\lambda)(x-a)\, C_\lambda(y)\, |f_{ts}(b,c)|^q + (2-\lambda)^2 (x-a)(y-c)\, |f_{ts}(b,d)|^q, \end{aligned}$$

$$\begin{aligned} B_\lambda(x,y) &= (2-\lambda)(d-y)\, A_\lambda(x)\, |f_{ts}(a,c)|^q + A_\lambda(x)\, D_\lambda(y)\, |f_{ts}(a,d)|^q \\ &\quad + (2-\lambda)^2 (x-a)(d-y)\, |f_{ts}(b,c)|^q + (2-\lambda)(x-a)\, D_\lambda(y)\, |f_{ts}(b,d)|^q, \end{aligned}$$

$$\begin{aligned} C_\lambda(x,y) &= (2-\lambda)(b-x)\, C_\lambda(y)\, |f_{ts}(a,c)|^q + (2-\lambda)^2 (b-x)(y-c)\, |f_{ts}(a,d)|^q \\ &\quad + B_\lambda(x)\, C_\lambda(y)\, |f_{ts}(b,c)|^q + (2-\lambda)(y-c)\, B_\lambda(x)\, |f_{ts}(b,d)|^q, \end{aligned}$$

$$\begin{aligned} D_\lambda(x,y) &= (2-\lambda)^2 (b-x)(d-y)\, |f_{ts}(a,c)|^q + + (2-\lambda)(b-x)\, D_\lambda(y)\, |f_{ts}(a,d)|^q \\ &\quad + (2-\lambda)(d-y)\, B_\lambda(x)\, |f_{ts}(b,c)|^q + B_\lambda(x)\, D_\lambda(y)\, |f_{ts}(b,d)|^q, \end{aligned}$$

$$\begin{aligned} A(x,y) &= \frac{\left[(b-a)^2-(b-x)^2\right]\left[(d-c)^2-(d-y)^2\right]}{4}\left|f_{ts}(a,c)\right|^q \\ &+\frac{\left[(b-a)^2-(b-x)^2\right](y-c)^2}{4}\left|f_{ts}(a,d)\right|^q \\ &+\frac{\left[(d-c)^2-(d-y)^2\right](x-a)^2}{4}\left|f_{ts}(b,c)\right|^q+\frac{(x-a)^2(y-c)^2}{4}\left|f_{ts}(b,d)\right|^q, \end{aligned}$$

$$\begin{aligned} B(x,y) &= \frac{\left[(b-a)^2-(b-x)^2\right](d-y)^2}{4}\left|f_{ts}(a,c)\right|^q \\ &+\frac{\left[(b-a)^2-(b-x)^2\right]\left[(d-c)^2-(y-c)^2\right]}{4}\left|f_{ts}(a,d)\right|^q \\ &+\frac{(x-a)^2(d-y)^2}{4}\left|f_{ts}(b,c)\right|^q+\frac{\left[(d-c)^2-(y-c)^2\right](x-a)^2}{4}\left|f_{ts}(b,d)\right|^q, \end{aligned}$$

$$\begin{aligned} C(x,y) &= \frac{\left[(d-c)^2-(d-y)^2\right](b-x)^2}{4}\left|f_{ts}(a,c)\right|^q+\frac{(b-x)^2(y-c)^2}{4}\left|f_{ts}(a,d)\right|^q \\ &+\frac{\left[(b-a)^2-(x-a)^2\right]\left[(d-c)^2-(d-y)^2\right]}{4}\left|f_{ts}(b,c)\right|^q \\ &+\frac{\left[(b-a)^2-(x-a)^2\right](y-c)^2}{4}\left|f_{ts}(b,c)\right|^q, \end{aligned}$$

and

$$\begin{aligned} D(x,y) &= \frac{(b-x)^2(d-y)^2}{4}\left|f_{ts}(a,c)\right|^q+\frac{\left[(d-c)^2-(y-c)^2\right](b-x)^2}{4}\left|f_{ts}(a,d)\right|^q \\ &+\frac{\left[(b-a)^2-(x-a)^2\right](d-y)^2}{4}\left|f_{ts}(b,c)\right|^q \\ &+\frac{\left[(b-a)^2-(x-a)^2\right]\left[(d-c)^2-(y-c)^2\right]}{4}\left|f_{ts}(b,d)\right|^q. \end{aligned}$$

Using the Lemma 1.5, we can obtain the following general integral inequalities. We prove following theorem by using Hölder's inequality. However, the right side of acquired inquality will be independent from p which is $p>1$ and $\frac{1}{p}+\frac{1}{q}=1$.

Theorem 2.1. *Let $f:\Delta\subset\mathbb{R}^2\to\mathbb{R}$ be an absolutely continuous function such that the partial derivative of order 2 exists for all $(t,s)\in\Delta$ and the functions $g:[a,b]\to[0,\infty)$ and $h:[c,d]\to[0,\infty)$ are integrable on Δ. If $\left|f_{ts}(t,s)\right|^q$, $q\geq 1$, is a convex function on the co-ordinates on Δ, then the following inequality holds:*

$$\begin{aligned} &\left|S_\lambda(f;g,h)\right| \qquad (2.1) \\ &\leq \frac{\|g\|_{[a,b],\infty}\|h\|_{[c,d],\infty}}{4\cdot 3^{\frac{2}{q}}\left[(b-a)(d-c)\right]^{\frac{1}{q}}} \\ &\times\Big\{(x-a)^2(y-c)^2\left(A_\lambda(x,y)\right)^{\frac{1}{q}}+(x-a)^2(d-y)^2\left(B_\lambda(x,y)\right)^{\frac{1}{q}} \\ &+(b-x)^2(y-c)^2\left(C_\lambda(x,y)\right)^{\frac{1}{q}}+(b-x)^2(d-y)^2\left(D_\lambda(x,y)\right)^{\frac{1}{q}}\Big\} \end{aligned}$$

where $\frac{1}{p}+\frac{1}{q}=1,\ \lambda\in[0,1]\,,\ \|g\|_{[a,b],\infty}=\sup\limits_{u\in[a,b]}|g(u)|$, *and* $\|h\|_{[c,d]\infty}=\sup\limits_{u\in[c,d]}|h(u)|$.

Proof. We take absolute value of (1.5). Using bounded of the mappings g and h, we find that

$$\begin{aligned}
&|S_\lambda(f;g,h)| \qquad (2.2)\\
\leq\ & \int_a^b\int_c^d |P_\lambda(x,t)|\,|Q_\lambda(y,s)|\,|f_{ts}(t,s)|\,dsdt\\
\leq\ & \|g\|_{[a,x],\infty}\|h\|_{[c,y],\infty}\\
&\times\int_a^x\int_c^y [(1-\lambda)(t-a)+\lambda(x-t)]\,[(1-\lambda)(s-c)+\lambda(y-s)]\,|f_{ts}(t,s)|\,dsdt\\
&+\|g\|_{[a,x],\infty}\|h\|_{[y,d],\infty}\\
&\times\int_a^x\int_y^d [(1-\lambda)(t-a)+\lambda(x-t)]\,[(1-\lambda)(d-s)+\lambda(s-y)]\,|f_{ts}(t,s)|\,dsdt\\
&+\|g\|_{[x,b],\infty}\|h\|_{[c,y],\infty}\\
&\times\int_x^b\int_c^y [(1-\lambda)(b-t)+\lambda(t-x)]\,[(1-\lambda)(s-c)+\lambda(y-s)]\,|f_{ts}(t,s)|\,dsdt\\
&+\|g\|_{[x,b],\infty}\|h\|_{[y,d],\infty}\\
&\times\int_x^b\int_y^d [(1-\lambda)(b-t)+\lambda(t-x)]\,[(1-\lambda)(d-s)+\lambda(s-y)]\,|f_{ts}(t,s)|\,dsdt\\
=\ & I_1\|g\|_{[a,x],\infty}\|h\|_{[c,y],\infty}+I_2\|g\|_{[a,x],\infty}\|h\|_{[y,d],\infty}\\
&+I_3\|g\|_{[x,b],\infty}\|h\|_{[c,y],\infty}+I_4\|g\|_{[x,b],\infty}\|h\|_{[y,d],\infty}\,.
\end{aligned}$$

Firstly, we calculate integral I_1. Because of $\frac{1}{p}+\frac{1}{q}=1$, $\frac{1}{p}+\frac{1}{q}$ can be written instead of 1. Using Holder's inequality, we find that

$$\begin{aligned}
&\int_a^x\int_c^y [(1-\lambda)(t-a)+\lambda(x-t)]\,[(1-\lambda)(s-c)+\lambda(y-s)]\,|f_{ts}(t,s)|\,dsdt\\
\leq\ & \left(\int_a^x\int_c^y [(1-\lambda)(t-a)+\lambda(x-t)]\,[(1-\lambda)(s-c)+\lambda(y-s)]\,dsdt\right)^{\frac{1}{p}}\\
&\times\left(\int_a^x\int_c^y [(1-\lambda)(t-a)+\lambda(x-t)]\,[(1-\lambda)(s-c)+\lambda(y-s)]\,|f_{ts}(t,s)|^q\,dsdt\right)^{\frac{1}{q}}.
\end{aligned}$$

Since $|f_{ts}(t,s)|^q$ is convex function on the co-ordinates on Δ, we have

$$\begin{aligned}
& \left| f_{ts}\left(\frac{b-t}{b-a}a + \frac{t-a}{b-a}b, \frac{d-s}{d-c}c + \frac{s-c}{d-c}d \right) \right|^q \\
\leq \; & \frac{(b-t)(d-s)}{(b-a)(d-c)} |f_{ts}(a,c)|^q + \frac{(b-t)(s-c)}{(b-a)(d-c)} |f_{ts}(a,d)|^q \\
& + \frac{(t-a)(d-s)}{(b-a)(d-c)} |f_{ts}(b,c)|^q + \frac{(t-a)(s-c)}{(b-a)(d-c)} |f_{ts}(b,d)|^q .
\end{aligned} \tag{2.3}$$

From (2.3), we get

$$\begin{aligned}
I_1 \leq \; & \left(\frac{(x-a)^2 (y-c)^2}{4} \right)^{\frac{1}{p}} \frac{1}{[(b-a)(d-c)]^{\frac{1}{q}}} \\
& \left(\int_a^x \int_c^y [(1-\lambda)(t-a) + \lambda(x-t)] [(1-\lambda)(s-c) + \lambda(y-s)] \right. \\
& \times [(b-t)(d-s)|f_{ts}(a,c)|^q + (b-t)(s-c)|f_{ts}(a,d)|^q \\
& \left. + (t-a)(d-s)|f_{ts}(b,c)|^q + (t-a)(s-c)|f_{ts}(b,d)|^q] \, dsdt \right)^{\frac{1}{q}} \\
= \; & \frac{(x-a)^2 (y-c)^2}{4 \cdot 3^{\frac{2}{q}} [(b-a)(d-c)]^{\frac{1}{q}}} (A_\lambda(x,y))^{\frac{1}{q}}
\end{aligned}$$

If we calculate the other integrals in a similar way, then we obtain

$$I_2 \leq \frac{(x-a)^2 (d-y)^2}{4 \cdot 3^{\frac{2}{q}} [(b-a)(d-c)]^{\frac{1}{q}}} (B_\lambda(x,y))^{\frac{1}{q}}$$

$$I_3 \leq \frac{(b-x)^2 (y-c)^2}{4 \cdot 3^{\frac{2}{q}} [(b-a)(d-c)]^{\frac{1}{q}}} (C_\lambda(x,y))^{\frac{1}{q}}$$

$$I_4 \leq \frac{(b-x)^2 (d-y)^2}{4 \cdot 3^{\frac{2}{q}} [(b-a)(d-c)]^{\frac{1}{q}}} (D_\lambda(x,y))^{\frac{1}{q}}$$

Substituting integrals I_1, I_2, I_3 and I_4 in (2.2) and using triangle inequality for moduls, we have

$$\begin{aligned}
|S_\lambda(f;g,h)| \leq \; & \frac{1}{4 \cdot 3^{\frac{2}{q}} [(b-a)(d-c)]^{\frac{1}{q}}} \\
& \times \left\{ \|g\|_{[a,x],\infty} \|h\|_{[c,y],\infty} (x-a)^2 (y-c)^2 (A_\lambda(x,y))^{\frac{1}{q}} \right. \\
& + \|g\|_{[a,x],\infty} \|h\|_{[y,d],\infty} (x-a)^2 (d-y)^2 (B_\lambda(x,y))^{\frac{1}{q}} \\
& + \|g\|_{[x,b],\infty} \|h\|_{[c,y],\infty} (b-x)^2 (y-c)^2 (C_\lambda(x,y))^{\frac{1}{q}} \\
& \left. + \|g\|_{[x,b],\infty} \|h\|_{[y,d],\infty} (b-x)^2 (d-y)^2 (D_\lambda(x,y))^{\frac{1}{q}} \right\} .
\end{aligned}$$

Because of $\|g\|_{[a,x],\infty}, \|g\|_{[x,b],\infty} \leq \|g\|_{[a,b],\infty}$ and $\|h\|_{[c,y],\infty}, \|h\|_{[y,d],\infty} \leq \|h\|_{[c,d],\infty}$, we easily deduce required inequality (2.1) which completes the proof. □

Corollary 2.2. *If we choose $g(u) = h(u) = 1$ and $\lambda = 1$ in Theorem 2.1, then the following inequality holds:*

$$\begin{aligned}
&\left|(b-a)(d-c)f(b,d) - (x-a)\int_c^d f(a,s)\,ds - (b-x)\int_c^d f(b,s)\,ds\right. \qquad (2.4)\\
&\left.-(y-c)\int_a^b f(t,c)\,dt - (d-y)\int_a^b f(t,d)\,dt + \int_a^b\int_c^d f(t,s)\,dsdt\right|\\
\leq\ & \frac{1}{4\cdot 3^{\frac{2}{q}}\left[(b-a)(d-c)\right]^{\frac{1}{q}}}\\
&\times\left\{(x-a)^2(y-c)^2(A_1(x,y))^{\frac{1}{q}} + (x-a)^2(d-y)^2(B_1(x,y))^{\frac{1}{q}}\right.\\
&\left.+(b-x)^2(y-c)^2(C_1(x,y))^{\frac{1}{q}} + (b-x)^2(d-y)^2(D_1(x,y))^{\frac{1}{q}}\right\}.
\end{aligned}$$

Remark 2.3. If we take $x = \frac{a+b}{2}$ and $y = \frac{c+d}{2}$ in (2.4), we get

$$\begin{aligned}
&\left|f(b,d) - \frac{1}{2(d-c)}\left[\int_c^d f(a,s)\,ds + \int_c^d f(b,s)\,ds\right]\right.\\
&\left.-\frac{1}{2(b-a)}\left[\int_a^b f(t,c)\,dt + \int_a^b f(t,d)\,dt\right] + \frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(t,s)\,dsdt\right|\\
\leq\ & \frac{(b-a)(d-c)}{64}\\
&\times\left\{\frac{\left(25\left|f_{ts}(a,c)\right|^q + 5\left|f_{ts}(a,d)\right|^q + 5\left|f_{ts}(b,c)\right|^q + \left|f_{ts}(b,d)\right|^q\right)^{\frac{1}{q}}}{36}\right.\\
&+\left(\frac{5\left|f_{ts}(a,c)\right|^q + 25\left|f_{ts}(a,d)\right|^q + \left|f_{ts}(b,c)\right|^q + 5\left|f_{ts}(b,d)\right|^q}{36}\right)^{\frac{1}{q}}\\
&+\left(\frac{5\left|f_{ts}(a,c)\right|^q + \left|f_{ts}(a,d)\right|^q + 25\left|f_{ts}(b,c)\right|^q + 5\left|f_{ts}(b,d)\right|^q}{36}\right)^{\frac{1}{q}}\\
&\left.+\left(\frac{\left|f_{ts}(a,c)\right|^q + 5\left|f_{ts}(a,d)\right|^q + 5\left|f_{ts}(b,c)\right|^q + 25\left|f_{ts}(b,d)\right|^q}{36}\right)^{\frac{1}{q}}\right\}.
\end{aligned}$$

In [13], Latif et. al proved an Ostrowski type inequality by accepting bounded of $\left|f_{ts}(t,s)\right|$ and co-ordinated convex of $\left|f_{ts}(t,s)\right|^q$, and we give a new Ostrowski type inequality whose left hand side is same as the left hand side of the inequality of Latif et. al by accepting condition of the Theorem 2.1 in the following corollary.

Corollary 2.4. *If we choose $g(u) = h(u) = 1$ and $\lambda = 0$ in Theorem 2.6, then the following* *inequality holds:*

$$\begin{aligned}
&\left|f(x,y)-\frac{1}{(d-c)}\int_c^d f(x,s)\,ds\right. \qquad (2.5)\\
&\left.-\frac{1}{(b-a)}\int_a^b f(t,y)\,dt+\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(t,s)\,dsdt\right|\\
\leq\ &\frac{1}{4\cdot 3^{\frac{2}{q}}\left[(b-a)(d-c)\right]^{\frac{1}{q}}}\\
&\times\left\{(x-a)^2(y-c)^2\left(A_0(x,y)\right)^{\frac{1}{q}}+(x-a)^2(d-y)^2\left(B_0(x,y)\right)^{\frac{1}{q}}\right.\\
&\left.+(b-x)^2(y-c)^2\left(C_0(x,y)\right)^{\frac{1}{q}}+(b-x)^2(d-y)^2\left(D_0(x,y)\right)^{\frac{1}{q}}\right\}.
\end{aligned}$$

Remark 2.5. If we take $x=\frac{a+b}{2}$ and $y=\frac{c+d}{2}$ in (2.5), we obtain

$$\begin{aligned}
&\left|f\left(\frac{a+b}{2},\frac{c+d}{2}\right)-\frac{1}{(d-c)}\int_c^d f\left(\frac{a+b}{2},s\right)ds\right. \qquad (2.6)\\
&\left.-\frac{1}{(b-a)}\int_a^b f\left(t,\frac{c+d}{2}\right)dt+\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(t,s)\,dsdt\right|\\
\leq\ &\frac{(b-a)(d-c)}{64}\\
&\times\left\{\left(\frac{16\left|f_{ts}(a,c)\right|^q+8\left|f_{ts}(a,d)\right|^q+8\left|f_{ts}(b,c)\right|^q+4\left|f_{ts}(b,d)\right|^q}{36}\right)^{\frac{1}{q}}\right.\\
&+\left(\frac{8\left|f_{ts}(a,c)\right|^q+16\left|f_{ts}(a,d)\right|^q+4\left|f_{ts}(b,c)\right|^q+8\left|f_{ts}(b,d)\right|^q}{36}\right)^{\frac{1}{q}}\\
&+\left(\frac{8\left|f_{ts}(a,c)\right|^q+4\left|f_{ts}(a,d)\right|^q+16\left|f_{ts}(b,c)\right|^q+8\left|f_{ts}(b,d)\right|^q}{36}\right)^{\frac{1}{q}}\\
&\left.+\left(\frac{4\left|f_{ts}(a,c)\right|^q+8\left|f_{ts}(a,d)\right|^q+8\left|f_{ts}(b,c)\right|^q+16\left|f_{ts}(b,d)\right|^q}{36}\right)^{\frac{1}{q}}\right\}.
\end{aligned}$$

In Theorem 2.1, we acquired an inequality by using Hölder's inequality and Lemma 1. In following theorem, we will use again Hölder's inequality and Lemma 1, but we will obtain a new inequality whose left side is independent of λ by calculating operations in a different way.

Theorem 2.6. *Let $f:\Delta\subset\mathbb{R}^2\rightarrow\mathbb{R}$ be an absolutely continuous function such that the partial derivative of order 2 exists for all $(t,s)\in\Delta$ and the functions $g:[a,b]\rightarrow[0,\infty)$ and $h:[c,d]\rightarrow[0,\infty)$ are integrable on Δ. If $\left|f_{ts}(t,s)\right|^q$, $q>1$, is a convex function on the co-ordinates on Δ,* *then the following inequality holds:*

$$
\begin{aligned}
& |S_\lambda(f;g,h)| \qquad (2.7)\\
\leq\ & \frac{\|g\|_{[a,b],\infty}\|h\|_{[c,d],\infty}}{[(b-a)(d-c)]^{\frac{1}{q}}(p+1)^{\frac{2}{p}}}\\
& \times\Big\{(x-a)^{1+\frac{1}{p}}(y-c)^{1+\frac{1}{p}}(A(x,y))^{\frac{1}{q}}+(x-a)^{1+\frac{1}{p}}(d-y)^{1+\frac{1}{p}}(B(x,y))^{\frac{1}{q}}\\
& +(b-x)^{1+\frac{1}{p}}(y-c)^{1+\frac{1}{p}}(C(x,y))^{\frac{1}{q}}+(b-x)^{1+\frac{1}{p}}(d-y)^{1+\frac{1}{p}}(D(x,y))^{\frac{1}{q}}\Big\}
\end{aligned}
$$

where $\frac{1}{p}+\frac{1}{q}=1,\ \lambda\in[0,1]$, $\|g\|_{[a,b],\infty}=\sup_{u\in[a,b]}|g(u)|$, *and* $\|h\|_{[c,d]\infty}=\sup_{u\in[c,d]}|h(u)|$.

Proof. We take absolute value of (1.5). Using bounded of the mappings g and h, we find the inequality (2.2). Now, we calculate in a different way the integrals I_1, I_2, I_3 and I_4 in (2.2), Firstly, we calculate integral I_1, we find that

$$
\begin{aligned}
I_1 &= \int_a^x\int_c^y[(1-\lambda)(t-a)+\lambda(x-t)][(1-\lambda)(s-c)+\lambda(y-s)]|f_{ts}(t,s)|\,dsdt\\
&= (1-\lambda)^2\int_a^x\int_c^y(t-a)(s-c)|f_{ts}(t,s)|\,dsdt+\lambda(1-\lambda)\int_a^x\int_c^y(t-a)(y-s)|f_{ts}(t,s)|\,dsdt\\
&\quad +\lambda(1-\lambda)\int_a^x\int_c^y(x-t)(s-c)|f_{ts}(t,s)|\,dsdt+\lambda^2\int_a^x\int_c^y(x-t)(y-s)|f_{ts}(t,s)|\,dsdt.
\end{aligned}
$$

Using Hölder's inequality for double integrals and the inequality (2.3), we get

$$
\begin{aligned}
I_1 &\leq \left(\int_a^x\int_c^y|f_{ts}(t,s)|^q\,dsdt\right)^{\frac{1}{q}}\\
&\quad\times\Bigg\{(1-\lambda)^2\left(\int_a^x\int_c^y(t-a)^p(s-c)^p\,dsdt\right)^{\frac{1}{p}}\\
&\quad+\lambda(1-\lambda)\left(\int_a^x\int_c^y(t-a)^p(y-s)^p\,dsdt\right)^{\frac{1}{p}}\\
&\quad+\lambda^2\left(\int_a^x\int_c^y(x-t)^p(y-s)^p\,dsdt\right)^{\frac{1}{p}}\\
&\quad+\lambda(1-\lambda)\left(\int_a^x\int_c^y(x-t)^p(s-c)^p\,dsdt\right)^{\frac{1}{p}}\Bigg\}\\
&= \frac{(A(x,y))^{\frac{1}{q}}}{(b-a)^{\frac{1}{q}}(d-c)^{\frac{1}{q}}}\frac{(x-a)^{1+\frac{1}{p}}(y-c)^{1+\frac{1}{p}}}{(p+1)^{\frac{2}{p}}}.
\end{aligned}
$$

If we calculate the other integrals in a similar way, then we obtain

$$I_2 \leq \frac{(B(x,y))^{\frac{1}{q}}}{(b-a)^{\frac{1}{q}}(d-c)^{\frac{1}{q}}} \frac{(x-a)^{1+\frac{1}{p}}(d-y)^{1+\frac{1}{p}}}{(p+1)^{\frac{2}{p}}},$$

$$I_3 \leq \frac{(C(x,y))^{\frac{1}{q}}}{(b-a)^{\frac{1}{q}}(d-c)^{\frac{1}{q}}} \frac{(b-x)^{1+\frac{1}{p}}(y-c)^{1+\frac{1}{p}}}{(p+1)^{\frac{2}{p}}},$$

and

$$I_4 \leq \frac{(D(x,y))^{\frac{1}{q}}}{(b-a)^{\frac{1}{q}}(d-c)^{\frac{1}{q}}} \frac{(b-x)^{1+\frac{1}{p}}(d-y)^{1+\frac{1}{p}}}{(p+1)^{\frac{2}{p}}}.$$

Substituting integrals I_1, I_2, I_3 and I_4 in (2.2) and using triangle inequality for moduls, then we obtain

$$\begin{aligned}
&|S_\lambda(f;g,h)| \\
\leq\; & \frac{1}{[(b-a)(d-c)]^{\frac{1}{q}}(p+1)^{\frac{2}{p}}} \\
&\times \Big\{ \|g\|_{[a,x],\infty}\|h\|_{[c,y],\infty}(x-a)^{1+\frac{1}{p}}(y-c)^{1+\frac{1}{p}}(A(x,y))^{\frac{1}{q}} \\
&+\|g\|_{[a,x],\infty}\|h\|_{[y,d],\infty}(x-a)^{1+\frac{1}{p}}(d-y)^{1+\frac{1}{p}}(B(x,y))^{\frac{1}{q}} \\
&+\|g\|_{[x,b],\infty}\|h\|_{[c,y],\infty}(b-x)^{1+\frac{1}{p}}(y-c)^{1+\frac{1}{p}}(C(x,y))^{\frac{1}{q}} \\
&+\|g\|_{[x,b],\infty}\|h\|_{[y,d],\infty}(b-x)^{1+\frac{1}{p}}(d-y)^{1+\frac{1}{p}}(D(x,y))^{\frac{1}{q}} \Big\}.
\end{aligned}$$

Because of $\|g\|_{[a,x],\infty}, \|g\|_{[x,b],\infty} \leq \|g\|_{[a,b],\infty}$ and $\|h\|_{[c,y],\infty}, \|h\|_{[y,d],\infty} \leq \|h\|_{[c,d],\infty}$, we easily deduce required inequality (2.7) which completes the proof. □

Corollary 2.7. *If we choose $g(u) = h(u) = 1$ and $\lambda = 1$ in Theorem 2.6; then the following inequality holds:*

$$\begin{aligned}
&\Bigg|(b-a)(d-c)f(b,d)-(x-a)\int_c^d f(a,s)\,ds-(b-x)\int_c^d f(b,s)\,ds \\
&-(y-c)\int_a^b f(t,c)\,dt-(d-y)\int_a^b f(t,d)\,dt+\int_a^b\int_c^d f(t,s)\,dsdt\Bigg| \\
\leq\; & \frac{1}{[(b-a)(d-c)]^{\frac{1}{q}}(p+1)^{\frac{2}{p}}} \\
&\times\Big\{(x-a)^{1+\frac{1}{p}}(y-c)^{1+\frac{1}{p}}(A(x,y))^{\frac{1}{q}}+(x-a)^{1+\frac{1}{p}}(d-y)^{1+\frac{1}{p}}(B(x,y))^{\frac{1}{q}} \\
&+(b-x)^{1+\frac{1}{p}}(y-c)^{1+\frac{1}{p}}(C(x,y))^{\frac{1}{q}}+(b-x)^{1+\frac{1}{p}}(d-y)^{1+\frac{1}{p}}(D(x,y))^{\frac{1}{q}}\Big\}
\end{aligned} \tag{2.8}$$

Remark 2.8. If we take $f(a,c)=f(a,d)=f(b,c)=f(b,d)$ in (2.8), then we get

$$\begin{aligned}
& \Big|(b-x)(d-y)f(b,d)+(b-x)(y-c)f(b,c)+(x-a)(d-y)f(a,d) \\
& +(x-a)(y-c)f(a,c)-\int_a^b\left[(d-y)f(t,d)+(y-c)f(t,c)\right]dt \\
& -\int_c^d\left[(b-x)f(b,s)+(x-a)f(a,s)\right]ds+\int_a^b\int_c^d f(t,s)\,dsdt\Big| \\
\leq\ & \frac{1}{\left[(b-a)(d-c)\right]^{\frac{1}{q}}(p+1)^{\frac{2}{p}}} \\
& \times\Big\{(x-a)^{1+\frac{1}{p}}(y-c)^{1+\frac{1}{p}}(A(x,y))^{\frac{1}{q}}+(x-a)^{1+\frac{1}{p}}(d-y)^{1+\frac{1}{p}}(B(x,y))^{\frac{1}{q}} \\
& +(b-x)^{1+\frac{1}{p}}(y-c)^{1+\frac{1}{p}}(C(x,y))^{\frac{1}{q}}+(b-x)^{1+\frac{1}{p}}(d-y)^{1+\frac{1}{p}}(D(x,y))^{\frac{1}{q}}\Big\}
\end{aligned}$$

which is proved by Sarikaya in [25].

Corollary 2.9. *If we take $x=\frac{a+b}{2}$ and $y=\frac{c+d}{2}$ in (2.8), then we get*

$$\begin{aligned}
& \Bigg|f(b,d)-\frac{1}{2(d-c)}\left[\int_c^d f(a,s)\,ds+\int_c^d f(b,s)\,ds\right] \qquad (2.9)\\
& -\frac{1}{2(b-a)}\left[\int_a^b f(t,c)\,dt+\int_a^b f(t,d)\,dt\right]+\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(t,s)\,dsdt\Bigg| \\
\leq\ & \frac{(b-a)(d-c)}{16} \\
& \times\left\{\left(\frac{9|f_{ts}(a,c)|^q+3|f_{ts}(a,d)|^q+3|f_{ts}(b,c)|^q+|f_{ts}(b,d)|^q}{16}\right)^{\frac{1}{q}}\right. \\
& +\left(\frac{3|f_{ts}(a,c)|^q+9|f_{ts}(a,d)|^q+|f_{ts}(b,c)|^q+3|f_{ts}(b,d)|^q}{16}\right)^{\frac{1}{q}} \\
& +\left(\frac{3|f_{ts}(a,c)|^q+|f_{ts}(a,d)|^q+9|f_{ts}(b,c)|^q+3|f_{ts}(b,d)|^q}{16}\right)^{\frac{1}{q}} \\
& \left.+\left(\frac{|f_{ts}(a,c)|^q+3|f_{ts}(a,d)|^q+3|f_{ts}(b,c)|^q+9|f_{ts}(b,d)|^q}{16}\right)^{\frac{1}{q}}\right\}.
\end{aligned}$$

Remark 2.10. If we take $f(a,c) = f(a,d) = f(b,c) = f(b,d)$ in (2.9), then we have

$$\begin{aligned}
&\left| \frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4} - \frac{1}{2(d-c)}\left[\int_c^d f(a,s)\,ds + \int_c^d f(b,s)\,ds\right]\right. \\
&\left. -\frac{1}{2(b-a)}\left[\int_a^b f(t,c)\,dt + \int_a^b f(t,d)\,dt\right] + \frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(t,s)\,dsdt\right| \\
\leq\ & \frac{(b-a)(d-c)}{16} \\
&\times\left\{\frac{\left(9|f_{ts}(a,c)|^q + 3|f_{ts}(a,d)|^q + 3|f_{ts}(b,c)|^q + |f_{ts}(b,d)|^q\right)^{\frac{1}{q}}}{16}\right. \\
&+\left(\frac{3|f_{ts}(a,c)|^q + 9|f_{ts}(a,d)|^q + |f_{ts}(b,c)|^q + 3|f_{ts}(b,d)|^q}{16}\right)^{\frac{1}{q}} \\
&+\left(\frac{3|f_{ts}(a,c)|^q + |f_{ts}(a,d)|^q + 9|f_{ts}(b,c)|^q + 3|f_{ts}(b,d)|^q}{16}\right)^{\frac{1}{q}} \\
&\left.+\left(\frac{|f_{ts}(a,c)|^q + 3|f_{ts}(a,d)|^q + 3|f_{ts}(b,c)|^q + 9|f_{ts}(b,d)|^q}{16}\right)^{\frac{1}{q}}\right\}.
\end{aligned} \tag{2.10}$$

which is proved by Sarikaya et al. in [23].

Corollary 2.11. *If we choose $g(u) = h(u) = 1$ and $\lambda = 0$ in Theorem 2.6, then we obtain*

$$\begin{aligned}
&\left| f(x,y) - \frac{1}{(d-c)}\int_c^d f(x,s)\,ds\right. \\
&\left. -\frac{1}{(b-a)}\int_a^b f(t,y)\,dt + \frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(t,s)\,dsdt\right| \\
\leq\ & \frac{1}{[(b-a)(d-c)]^{\frac{1}{q}}(p+1)^{\frac{2}{p}}} \\
&\times\left\{(x-a)^{1+\frac{1}{p}}(y-c)^{1+\frac{1}{p}}(A(x,y))^{\frac{1}{q}} + (x-a)^{1+\frac{1}{p}}(d-y)^{1+\frac{1}{p}}(B(x,y))^{\frac{1}{q}}\right. \\
&\left.+(b-x)^{1+\frac{1}{p}}(y-c)^{1+\frac{1}{p}}(C(x,y))^{\frac{1}{q}} + (b-x)^{1+\frac{1}{p}}(d-y)^{1+\frac{1}{p}}(D(x,y))^{\frac{1}{q}}\right\}.
\end{aligned} \tag{2.11}$$

Remark 2.12. If we take $x = \frac{a+b}{2}$ and $y = \frac{c+d}{2}$ in (2.11), then we get a new Hermite-Hadamard-type inequality whose left hand side is same as left hand side of the inequality (2.6) and right hand side is same as right hand side of the inequality (2.9) either.

References

[1] F. Ahmad, N. S. Barnett and S. S. Dragomir, *New Weighted Ostrowski and Cebysev Type Inequalities*, Nonlinear Analysis: Theory, Methods & Appl., 71 (12), (2009), 1408-1412.

[2] F. Ahmad, A. Rafiq, N. A. Mir, *Weighted Ostrowski type inequality for twice differentiable mappings*, Global Journal of Research in Pure and Applied Math., 2 (2) (2006), 147-154.

[3] N. S. Barnett and S. S. Dragomir, *An Ostrowski type inequality for double integrals and applications for cubature formulae*, Soochow J. Math., 27 (1), (2001), 109-114.

[4] F. Chen, *On Hermite-Hadamard type inequalities for s-convex functions on the coordinates via Riemann-Liouville fractional integrals*, Journal of Applied Mathematics, Volume 2014, Article ID 248710, 8 pages.

[5] S. S. Dragomir, *On Hadamard's inequality for convex functions on the co-ordinates in a rectangle from the plane*, Taiwanese Journal of Mathematics, 4 (2001), 775-788.

[6] S. S. Dragomir and R.P. Agarwal, *Two inequalities for differentiable mappings and applications to special means of real numbers and to trapezoidal formula*, Appl. Math. lett., 11(5) (1998), 91-95.

[7] S. S. Dragomir and C. E. M. Pearce, *Selected Topics on Hermite-Hadamard Inequalities and Applications*, RGMIA Monographs, Victoria University, 2000.

[8] S. Erden and M. Z. Sarikaya, *On the Hermite- Hadamard's and Ostrowski's inequalities for the co-ordinated convex functions*, RGMIA Research Report Collection, Vol. 18, 2015. Art. 57.

[9] J. Hadamard, *Etude sur les proprietes des fonctions entieres et en particulier d'une fonction considree par,* Riemann, J. Math. Pures. et Appl. 58 (1893), 171–215.

[10] M. A. Latif and M. Alomari, *Hadamard-type inequalities for product two convex functions on the co-ordinetes*, Int. Math. Forum, 4(47), 2009, 2327-2338.

[11] M. A. Latif and M. Alomari, *On the Hadamard-type inequalities for h-convex functions on the co-ordinetes*, Int. J. of Math. Analysis, 3(33), 2009, 1645-1656.

[12] M. A. Latif and S.S. Dragomir, *On some new inequalities for differentiable co-ordinated convex functions*, Journal of Inequalities and Applications 2012, 2012:28.

[13] M. A. Latif, S. Hussain and S. S. Dragomir, *New Ostrowski type inequalities for co-ordinated convex functions*, TJMM, 4 (2012), No. 2, 125-136.

[14] M.E. Ozdemir, E. Set and M.Z. Sarikaya, *New some Hadamard's type inequalities for coordinated m-convex and (α,m)-convex functions*, Hacettepe Journal of Mathematics and Statistics , 40(2), 2011, 219-229.

[15] A. M. Ostrowski, *Über die absolutabweichung einer differentiebaren funktion von ihrem integralmitelwert*, Comment. Math. Helv. 10(1938), 226-227.

[16] B. G. Pachpatte, *On a new Ostrowski type inequality in two independent variables*, Tamkang J. Math., 32(1), (2001), 45-49

[17] J. Park, *Generalizations of the Simpson-like type inequalities for co-ordinated s-convex mappings in the second sense*, International Journal of Mathematics and Mathematical Sciences Volume 2012, Article ID 715751, 16 pages.

[18] J. Park, *Some Hadamard's type inequalities for co-ordinated (s;m)-convex mappings in the second sense*, Far East Journal of Mathematical Sciences, vol. 51, no. 2, pp. 205–216, 2011.

[19] J. Pečarić, F. Proschan and Y.L. Tong, *Convex functions, partial ordering and statistical applications*, Academic Press, New York, 1991.

[20] A. Rafiq and F. Ahmad, *Another weighted Ostrowski-Grüss type inequality for twice differentiable mappings*, Kragujevac Journal of Mathematics, 31 (2008), 43-51.

[21] M. Z. Sarikaya and H. Ogunmez, *On the weighted Ostrowski type integral inequality for double integrals*, The Arabian Journal for Science and Engineering (AJSE)-Mathematics, (2011) 36: 1153-1160.

[22] M. Z. Sarikaya, *On the Ostrowski type integral inequality*, Acta Math. Univ. Comenianae, Vol. LXXIX, 1(2010), pp. 129-134.

[23] M. Z. Sarikaya, E. Set, M. E. Özdemir and S. S. Dragomir, *New some Hadamard's type inequalities for co-ordineted convex functions*, Tamsui Oxford Journal of Information and Mathematical Sciences, 28(2) (2012) 137-152.

[24] M. Z. Sarikaya, *On the Hermite-Hadamard-type inequalities for co-ordinated convex function via fractional integrals*, Integral Transforms and Special Functions, Vol. 25, No. 2, 134-147, 2014.

[25] M. Z. Sarikaya, *Some inequalities for differentiable co-ordinated convex mappings*, Asian-European Journal of Mathematics, Vol: 08, 1550058 (2015), 21 pages.

[26] M. Z. Sarikaya, *On the generalized Montgomery identity for double integrals*, Submited, 2015.

Author information

Samet Erden, Department of Mathematics, Bartın University, Bartın, TURKEY.
E-mail: `erdensmt@gmail.com`

Mehmet Z. Sarikaya, Department of Mathematics, Düzce University, Düzce, TURKEY.
E-mail: `sarikayamz@gmail.com`

Received: November 24, 2015.

Accepted: March 25, 2016.