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CENTRAL SETS AND RADII OF $\Gamma_I(R)$

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Abstract. Let R be a commutative ring and I be an ideal of R. In this paper, we consider the ideal-based zero-divisor graph $\Gamma_I(R)$ of a commutative ring R. We discuss some graph theoretical properties of $\Gamma_I(R)$. We find radius and central sets of $\Gamma_I(R)$. We find the relationship between median and central sets of $\Gamma_I(R)$. Further the relationship between domination number of $\Gamma_I(R)$ and radius of $\Gamma_I(R)$ is also discussed.

1 Introduction

Let R be a commutative ring with identity 1 and $Z(R)$ be the set of its zero-divisors. The zero-divisor graph of R denoted by $\Gamma(R)$ is an undirected graph whose vertices are the nonzero zero-divisors of R with two distinct vertices x and y joined by an edge if and only if $xy = 0$. Beck [7] introduced the concept of a zero-divisor graph of a commutative ring, but this work was mostly concerned with colorings of rings. The above definition of $\Gamma(R)$ first appeared in Anderson and Livingston [4], where many of the most basic features of $\Gamma(R)$ are investigated. S. P. Redmond generalized this by introducing the ideal-based zero-divisor graph [14]. For a commutative ring R and an ideal I of R, *the ideal-based zero-divisor graph* is an undirected graph $\Gamma_I(R)$ with vertices $\{x \in R - I : xy \in I \text{ for some } y \in R - I\}$, where distinct vertices x and y are adjacent if and only if $xy \in I$. He investigated the relationship between these graphs and zero-divisor graphs $\frac{R}{I}$. H. R. Maimani and David F. Anderson have also studied this concept [5, 10].

Throughout this paper, the rings are commutative and I is an ideal of R. For any subset X of a ring R, $|X|$ denote the number of elements in X and $X^* = X - \{0\}$. For any element $x \in R$, $ann(x)$ denote the annihilator of x in R and is defined as $ann(x) = \{y \in R : xy = 0\}$. Let $a \in R$. If $a^n = 0$, for some positive integer n, then a is said to be *nilpotent element of nilpotency* n. A ring R is said to be *Noetherian* if it satisfies the following three equivalent conditions: (1) Every non-empty set of ideals in R has a maximal element. (2) Every ascending chain of ideals in R is stationary. (3) Every ideal in R is finitely generated. A ring R is said to be *decomposable* if R can be written as $R_1 \times R_2$, where R_1 and R_2 are rings; otherwise R is said to be *reduced*.

The graphs G considered in this paper are simple. The vertex set of G will be denoted by $V(G)$. For a graph G, the *degree* of a vertex v in G is the number of edges incident with v. Denote the degree of the vertex v in $\Gamma_I(R)$ by $\deg(v)$ and that of $\Gamma(R)$ by $\deg_\Gamma(v)$. We denote the *complete graph* with n vertices and *complete bipartite graph* with two parts of sizes m and n, by K_n and $K_{m,n}$, respectively. The graph $K_{1,m}$ is called a *star graph*. The *distance* between any two vertices x and y, denoted $d(x, y)$, is the length of the shortest $x - y$ path. The *diameter* of a connected graph G is the maximum distance between two distinct vertices of G. Let G be a graph and H be subset of G. The *induced subgraph* H in G, $\langle H\rangle$ is a graph with vertex set H and two vertices of H are adjacent if they are adjacent in G.

The main aim of this article is to find the central sets of $\Gamma_I(R)$. First we prove that for a commutative Noetherian ring R, radius of $\Gamma_I(R)$ is either 1 or 2. In Sec. 2, we give the definitions and theorems which are needed for subsequent sections. In Sec. 3 we determine the radius of $\Gamma_I(R)$. We also find necessary and sufficient condition for $diam(\Gamma_I(R)) = 1$ and also for $diam(\Gamma_I(R)) = 2$. In Sec. 4 ,we find median of $\Gamma_I(R)$ and relation between center and median of $\Gamma_I(R)$. In Sec.5, we determine domination number of $\Gamma_I(R)$.

2 Preliminaries

Definition 2.1. Let G be a connected graph and $x \in V(G)$. Then $e(x) = \max_{y \in V(G)} d(x,y)$. The radius of G, $rad(G) = \min_{x \in V(G)} e(x)$ and the center of G, $C(G) = \{x \in V(G) : e(x) = rad(G)\}$. The diameter of G, $diam(G) = \max_{x \in V(G)} e(x)$.

Definition 2.2. A graph G is *self centered* if $V(G)$ is the center of G.

If a connected graph G has radius r and diameter d, then $r \leq d \leq 2r$. We denote the center, eccentricity of $\Gamma(\frac{R}{I})$ by $C(\Gamma(\frac{R}{I})), e_\Gamma(x)$ respectively and that of $\Gamma_I(R)$ by $C(\Gamma_I(R))$, $e(x)$ respectively.

Definition 2.3. The *status* $s(x)$ of a vertex x of a connected graph G is the sum of the distances from x to the other vertices of G, i.e, $s(x) = \sum_{y \in V(G)} d(x,y)$. The set of vertices with minimum status is called the *median* of the graph.

If G has no edges, then median of G is $V(G)$. We denote the status of every vertex x of $\Gamma_I(R)$ by $s(x)$ and that of every vertex $x+I$ of $\Gamma(\frac{R}{I})$ by $s_\Gamma(x+I)$.

Definition 2.4. A *dominating set* of a graph G is a subset S of $V(G)$ such that each vertex of G is either in S or adjacent to an element of S. The *domination number* of a graph G is the size of the smallest possible dominating set and is denoted by $\gamma(G)$.

Definition 2.5. A dominating set S of G is called *connected* if the subgraph induced by S is connected. The *connected domination number* of G is the size of the smallest connected dominating set and is denoted by $\gamma_c(G)$.

S. P. Redmond had introduced the concept of an Ideal-Based zero-divisor graph as follows.

Definition 2.6. [14] Let R be a commutative ring and let I be an ideal of R. The *ideal-based zero-divisor graph* is an undirected graph $\Gamma_I(R)$ with vertices $\{x \in R - I : xy \in I$ for some $y \in R - I\}$, where distinct vertices x and y are adjacent if and only if $xy \in I$.

Example 2.7. For $R \cong \mathbb{Z}_{24}$ and $I \cong (8)$, $\Gamma_I(R)$ is shown in Figure 1.

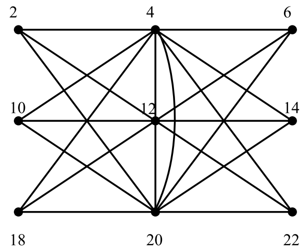

Figure 1

Theorem 2.8. [14] *Let I be an ideal of a ring R, and let $x, y \in R - I$. Then*
(a) if $x+I$ is adjacent to $y+I$ in $\Gamma(\frac{R}{I})$, then x is adjacent to y in $\Gamma_I(R)$.
(b) if x is adjacent to y in $\Gamma_I(R)$ and $x+I \neq y+I$, then $x+I$ is adjacent to $y+I$ in $\Gamma(\frac{R}{I})$.
(c) if x is adjacent to y in $\Gamma_I(R)$ and $x+I = y+I$, then $x^2, y^2 \in I$.

Corollary 2.9. [14] *If x and y are (distinct) adjacent vertices in $\Gamma_I(R)$, then all (distinct) elements of $x+I$ and $y+I$ are adjacent in $\Gamma_I(R)$. If $x^2 \in I$, then all the distinct elements of $x+I$ are adjacent in $\Gamma_I(R)$*

Remark 2.10. [14] Clearly there is a strong relationship between $\Gamma(\frac{R}{I})$ and $\Gamma_I(R)$. Let I be an ideal of a ring R. One can verify that the following method can be used to construct the graph $\Gamma_I(R)$. Let $\{a_\lambda\}_{\lambda\in\Lambda} \subseteq R$ be a set of coset representatives of the vertices of $\Gamma(\frac{R}{I})$. For each $i \in I$, define a graph G_i with vertices $\{a_\lambda + i : \lambda \in \Lambda\}$, where edges are defined by the relationship $a_\lambda + i$ is adjacent to $a_\beta + i$ in G_i if and only if $a_\lambda + I$ is adjacent to $a_\beta + I$ in $\Gamma(\frac{R}{I})$ (i.e., $a_\lambda a_\beta \in I$).

Define the graph G to have as its vertex set $V = \bigcup\limits_{i\in I} G_i$. We define the edge set of G to be:

(1) all edges contained in G_i for each $i \in I$.

(2) for distinct $\lambda, \beta \in \Lambda$ and for any $i, j \in I$, $a_\lambda + i$ is adjacent to $a_\beta + j$ if and only if $a_\lambda + I$ is adjacent to $a_\beta + I$ in $\Gamma(\frac{R}{I})$ (i.e.,$a_\lambda a_\beta \in I$).

(3) for $\lambda \in \Lambda$ and distinct $i, j \in I, a_\lambda + i$ is adjacent to $a_\lambda + j$ if and only if $a_\lambda^2 \in I$.

Definition 2.11. [14] Using the notation as in the above construction, we call the subset $a_\lambda + I$ a column of $\Gamma_I(R)$. If $a_\lambda^2 \in I$, then we call $a_\lambda + I$ a connected column of $\Gamma_I(R)$.

Remark 2.12. [14] Denote the vertices of $\Gamma(\frac{R}{I})$ by $V(\Gamma(\frac{R}{I})) = \{a_i + I : i \in \Lambda\}$. By remark 2.5, we can denote the vertex set of $\Gamma_I(R)$ as $V(\Gamma_I(R)) = \{a_i + h : i \in \Lambda, h \in I\}$ and so $|V(\Gamma_I(R))| = |I|\,\big|V(\Gamma(\frac{R}{I}))\big|$.

Lemma 2.13. [11] *Let I be an ideal of a ring R. Then in $\Gamma_I(R)$,*

$$\deg(a) = \begin{cases} |I| \deg_\Gamma(a+I) & \text{if } \; a^2 \notin I. \\ |I| \deg_\Gamma(a+I) + |I| - 1 & \text{if } \; a^2 \in I. \end{cases}$$

Theorem 2.14. [14, Theorem 2.4] *Let I be an ideal of a ring R. Then $\Gamma_I(R)$ is connected with $diam(\Gamma_I(R)) \leq 3$. Furthermore, if $\Gamma_I(R)$ contains a cycle, then $gr(\Gamma_I(R)) \leq 7$.*

Theorem 2.15. [14, Theorem 5.7] *Let I be a nonzero ideal of a ring R. Then $\Gamma_I(R)$ is bipartite if and only if either (a) $gr(\Gamma_I(R)) = \infty$ or (b) $gr(\Gamma_I(R)) = 4$ and $\Gamma(\frac{R}{I})$ is bipartite.*

Theorem 2.16. [15, Corollary 2.2] *Let R be a commutative Noetherian ring with identity. The radius of $\Gamma(R)$ is 0 if and only if either $R \cong \mathbb{Z}_4$ or $R \cong \frac{\mathbb{Z}_2[X]}{\langle x^2\rangle}$. The radius of $\Gamma(R)$ is 1 if and only if either $R \cong \mathbb{Z}_2 \times A$, where A is an integral domain, or $Z(R)$ is an ideal of R. If, in addition, R is finite, then the radius of $\Gamma(R)$ is 1 if and only if either $R \cong \mathbb{Z}_2 \times \mathbb{F}$, where $\mathbb{F}$ is a finite field, or R is local.*

Theorem 2.17. [15, Theorem 2.3] *Let R be a commutative Noetherian ring with identity that is not an integral domain. Then the radius of $\Gamma(R)$ is at most 2.*

Theorem 2.18. [15, Theorem 4.1] *Let R be a finite commutative ring with identity that is not an integral domain. Then the median and center of $\Gamma(R)$ are equal if the radius of $\Gamma(R)$ is at most 1 and the median is a subset of the center if the radius is 2.*

Theorem 2.19. [15, Corollary 4.2] *Let R be a finite commutative ring with identity that is not an integral domain. If the radius of $\Gamma(R)$ is 2, then the center equals the median if and only if R is isomorphic to a direct product of a finite number of copies of a single finite field (i.e., $R \cong \mathbb{F}^d$ for some finite field $\mathbb{F}$ and some integer $d \geq 2$).*

Theorem 2.20. [15, Theorem 5.1] *Let R be a commutative Artinian ring with identity that is not a domain. If the radius of $\Gamma(R)$ is at most 1, then the domination number of $\Gamma(R)$ is 1. If the radius is 2, then the domination number is equal to the number of factors in the Artinian decomposition of R. (In particular, the domination number is finite and at least two).*

3 Radius of $\Gamma_I(R)$

Theorem 3.1. Let $I \neq (0)$ be an ideal of R. Then radius of $\Gamma_I(R)$ can never be zero.

Proof. Clearly $\Gamma_I(R)$ has at least two vertices and by Theorem 2.14, it is connected. So $rad(\Gamma_I(R)) > 0$. □

Theorem 3.2. Let $I \neq (0)$ be an ideal of a ring R. Then the following are equivalent.
(i) There is a vertex $a+I$ of $\Gamma(\frac{R}{I})$ of nilpotency 2 that is adjacent to every other vertex.
(ii) There are at least $|I|$ vertices of $\Gamma_I(R)$ with degree $|V(\Gamma_I(R))| - 1$.

Proof. Assume (i) is true. Then $a^2 \in I$ and $\langle a+I \rangle$ is a complete subgraph of $\Gamma_I(R)$. By Theorem 2.8, $d(a+h, b) = 1$, for all b$\in V(\Gamma_I(R)), h \in I$. So $a+h$ is a vertex which is adjacent to every other vertex, for all $h \in I$. Thus (ii) holds. Conversely assume that (ii) is true. Then choose $b \in V(\Gamma_I(R))$ such that $d(b, v) = 1$, for all $v \in V(\Gamma_I(R))$. In particular $d(b, b+h) = 1$, for all $h \in I$. So $b^2 \in I$. Also for $v \neq b+h$, $d(b, v) = 1$. This implies that $d_\Gamma(b+I, v+I) = 1$ and $b+I \neq v+I$. Hence $b+I$ is a vertex of $\Gamma(\frac{R}{I})$ adjacent to every other vertex and is of nilpotency 2. □

Corollary 3.3. Let $I \neq (0)$ be an ideal of a ring R such that $\Gamma(\frac{R}{I})$ is a graph with at least two vertices. Assume R is a commutative ring satisfying any one of the conditions (i) or (ii) of Theorem 3.2. Then $rad(\Gamma_I(R)) = 1$ if and only if $rad(\Gamma(\frac{R}{I})) = 1$.

Corollary 3.4. Let $I \neq (0)$ be an ideal of R such that $\Gamma(\frac{R}{I})$ is a graph with at least two vertices. Assume $\frac{R}{I}$ is a finite local ring but not a field. Then $rad(\Gamma_I(R)) = 1$ if and only if $rad(\Gamma(\frac{R}{I})) = 1$.

Proof. Since $\frac{R}{I}$ is a finite local ring, $Ann(Z(\frac{R}{I})) \neq 0$ and so (i) holds. Hence by Corollary 3.3, the result follows. □

Theorem 3.5. Let $I \neq (0)$ be an ideal of a Noetherian ring R. Assume that $\Gamma_I(R)$ has no connected columns. Then $rad(\Gamma_I(R)) = 2$.

Proof. By Theorem 2.17, $rad(\Gamma(\frac{R}{I})) \leq 2$. Then there exist $a+I$ such that $rad(\Gamma_I(R)) = e_\Gamma(a+I)$ and so $d_\Gamma(a+I, b+I) \leq 2$, for all $b+I \in V(\Gamma(\frac{R}{I}))$. This implies that $d(a,b) \leq 2$, for all $b \in V(\Gamma_I(R))$. Since $\Gamma_I(R)$ has no connected columns, $2 \leq e(x) \leq 3$, for all $x \in V(\Gamma_I(R))$ and $d(a, a+h) = 2$. Thus $e(a) = 2$, in $\Gamma_I(R)$. Hence $rad(\Gamma_I(R)) = 2$. □

Theorem 3.6. Let $I \neq (0)$ be an ideal of R such that $\Gamma(\frac{R}{I})$ is a graph on single vertex. Then $\Gamma_I(R))$ is self centered and $rad(\Gamma_I(R)) = 1$.

Theorem 3.7. Let $I \neq (0)$ be an ideal of a Noetherian ring R. Assume that $\Gamma_I(R)$ has no connected columns. Then $diam(\Gamma(\frac{R}{I})) \leq 2$ if and only if $\Gamma_I(R)$ is self centered.

Proof. Since $\Gamma_I(R)$ has no connected columns, $2 \leq e(x) \leq 3$, for all $x \in V(\Gamma_I(R))$. Let $x \in V(\Gamma_I(R))$. Since $\Gamma_I(R)$ is connected, there exist y such that x is adjacent to y. Also $d(x, x+h) = 2$, for all $h \in I$. Suppose $d(x,z) = 3$, for some z. Clearly $x+I \neq z+I$. Let $x-y-u-z$ be a shortest path of length 3. Since $\Gamma_I(R)$ has no connected column, $y \neq x+h$ and $u \neq y+h$, for all h. If $u = x+h$ or $y = z+h$ for some h, then x is adjacent to z, a contradiction. So $u+I \neq x+I$ and $y+I \neq z+I$. So $x+I, y+I, u+I, z+I$ are distinct element of $\Gamma(\frac{R}{I})$. Hence $x+I-y+I-u+I-z+I$ is a shortest path of length 3 and $d_\Gamma(x+I, z+I) = 3$, which is a contradiction, since $diam(\Gamma(\frac{R}{I}) \leq 2$. Therefore $d(x,z) \leq 2$, for all z. Hence $e(x) = 2$, for all $x \in V(\Gamma_I(R))$. Thus the result follows. Conversely, assume that $\Gamma_I(R)$ is self centered. Then $rad(\Gamma_I(R)) = e(x)$, for all $x \in V(\Gamma_I(R))$. Clearly $rad(\Gamma_I(R)) \neq 1$. By Theorem 3.5, $rad(\Gamma_I(R)) = 2$. So $e(x) = 2$, for all $x \in V$. We have $d(x,y) \leq e(x) = 2$, for all $x \in V$. This implies that $d(x,y) \leq 2$, for all $x, y \in V$. Therefore $d_\Gamma(x+I, y+I) \leq 2$, for all $x+I, y+I \in V$. So the result follows. □

Remark 3.8. Theorem 3.7 is not true if $\Gamma_I(R)$ has a connected column. For example, if $R = \mathbb{Z}_{24}$ and $I = (8)$, $\Gamma_I(R)$ has a connected column and $diam(\Gamma(\frac{R}{I})) \leq 2$. But $\Gamma_I(R)$ is not self centered (see Figure 1).

Lemma 3.9. Let $I \neq (0)$ be an ideal of R such that $\Gamma(\frac{R}{I})$ is a graph with at least two vertices. Then the following is the relationship between $\Gamma_I(R)$ and $\Gamma(\frac{R}{I})$.

(i) If $x^2 \in I$, then $e_\Gamma(x+I) = e(x)$.

(ii) If $e_\Gamma(x+I) \neq 1$, then $e_\Gamma(x+I) \geq e(x)$.

(iii) If $x^2 \notin I$ and $e_\Gamma(x+I) = 1$, then $e_\Gamma(x+I) < e(x)$.

Theorem 3.10. Let R be a commutative Noetherian ring and I be a non zero ideal of R. Then $rad(\Gamma_I(R))$ is at most 2.

Proof. Assume $\Gamma_I(R)$ has a connected column. If $rad(\Gamma(\frac{R}{I})) = 1$, then there exist $a+I$ such that $e_\Gamma(a+I) = 1$. If $a^2 \in I$, then by Theorem 3.2, $rad(\Gamma_I(R)) = 1$. If $a^2 \notin I$, then $a - b - a + h$ is a shortest path of length 2 where $b \neq a+h$ and so $d(a, a+h) = 2, h \in I$. Thus $e(a) = 2,$ in $\Gamma_I(R)$ and so $rad(\Gamma_I(R)) \leq 2$. If $rad(\Gamma(\frac{R}{I})) = 2$, then there exist $a+I$ such that $e_\Gamma(a+I) = 2$. By Lemma 3.9, $e(a) \leq 2$ and so $rad(\Gamma_I(R)) \leq 2$. If $\Gamma_I(R)$ has no connected columns, then by Theorem 3.5, $rad(\Gamma_I(R)) \leq 2$. $\square$

Corollary 3.11. Let R be a commutative Noetherian ring with identity. If $\frac{R}{I}$ is not an integral domain, then there is a nonzero $x \in R$ such that either $xy \in I$ or $ann(x+I) \cap ann(y+I) \neq \{0\}$, for all $y \in V$.

Example 3.12. (1) Let $R \cong \mathbb{Z}_{24}$ and $I = (8)$. Then $\Gamma_I(R)$ has a connected column and $rad(\Gamma_I(R)) = 1$ (see Figure 1).

(2) Let $R \cong \mathbb{Z}_2 \times \mathbb{Z}_4 \times \mathbb{Z}_2$ and $I = \{0\} \times \{0\} \times \mathbb{Z}_2$. Since (0,2,0) is an element of nilpotency 2, $\Gamma_I(R)$ has a connected column and $rad(\Gamma_I(R)) = 2$ (see Figure 2).

(3) Let $R \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3$ and $I = \{0\} \times \{0\} \times \mathbb{Z}_3$. So $\frac{R}{I} = \mathbb{Z}_2 \times \mathbb{Z}_2 \times \{0\}$ and $\Gamma(\frac{R}{I}) \cong K_2$. Since $|I| \geq 2$, $\Gamma(\frac{R}{I})$ is a complete bipartite graph and $gr(\Gamma_I(R)) = 4$. By Theorem 2.15, $\Gamma_I(R)$ is complete bipartite graph and $rad(\Gamma_I(R)) = 2$.

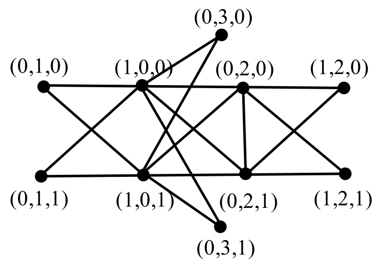

Figure 2

Theorem 3.13. Let R be a commutative Artinian ring with identity 1 and $I \neq (0)$ be an ideal of R such that $\frac{R}{I}$ is a finite ring with identity 1 and $\Gamma(\frac{R}{I})$ is a graph with at least two vertices. Then

(i) $diam(\Gamma_I(R)) > 0$.

(ii) If $rad(\Gamma_I(R))$ is 1, then $diam(\Gamma_I(R)) = 1$ if and only if $\Gamma_I(R)$ is a complete graph. Otherwise , the diameter is 2.

(iii) If $rad(\Gamma_I(R))$ is 2, then $diam(\Gamma_I(R)) = 2$ if and only if $\frac{R}{I} \cong F_1 \times F_2$, where F_1 and F_2 are both fields. Otherwise the diameter is 3.

Proof. (i) is obvious.

(ii) If $rad(\Gamma_I(R))$ is 1, then the diameter is at most 2, since for any x in the center of $\Gamma_I(R)$ and for any two vertices a and b, $a - x - b$ is a path of length 2. The diameter is 1 if and only if all the vertices of $\Gamma_I(R)$ are adjacent. Otherwise the diameter is 2.

(iii) Suppose $rad(\Gamma_I(R))$ is 2. Then $rad(\Gamma(\frac{R}{I})) = 1$ or 2. Consider the case where $rad(\Gamma(\frac{R}{I})) = 1$. Since $\frac{R}{I}$ is not local and by Theorem 2.16, $\frac{R}{I} \cong \mathbb{Z}_2 \times \mathbb{F}$, where $\mathbb{F}$ is a finite field and diameter is 2. Assume $rad(\Gamma_I(R)) = 2$. If $\frac{R}{I} \cong F_1 \times F_2$ where F_1 and F_2 are both are fields and both not isomorphic to $\mathbb{Z}_2$, then $\Gamma(\frac{R}{I})$ is a complete bipartite graph and so $gr(\Gamma_I(R)) = 4$. By Theorem 2.15, $\Gamma_I(R)$ is a complete bipartite graph. Thus $diam(\Gamma_I(R)) = 2$. Now assume $\frac{R}{I} \ncong F_1 \times F_2$, where F_1 and F_2 are both fields . Consider the Artinian decomposition

$\frac{R}{I} = R_1 \times \ldots R_n \times F_1 \times \ldots F_m$, where (R_i, M_i) is a local ring, F_j is a field, $1 \leq i \leq n$,$1 \leq j \leq m$ and $n + m \geq 2$. By choice of $\frac{R}{I}$, the case $n = 0$ and $m = 2$ is impossible. Since $\frac{R}{I}$ is not local, the cases $n = 0, m = 1$ and $n = 1, m = 0$ are impossible. Hence it is enough to consider the following cases. In all cases, an element not in the center of $\Gamma_I(R)$ will be identified.
Case 1: $n \geq 1$ and $m \geq 1$.
Let $0 \neq x \in M_1$. Let $y + I = (x, 0, \ldots 1, 0 \ldots, 0)$, where the entry in position $n + 1$ is the identity of F_1. Then $y + I$ is a zero-divisor but is not in the center of $\Gamma(\frac{R}{I})$. This implies that $d_\Gamma(y + I, z + I) = 3$, for some $z + I$, $z + I \neq y + I$ and so $d(y, z) = 3$, for some z. So $y \notin C(\Gamma_I(R))$.
Case 2: $n = 0$ and $m \geq 3$.
Then $\frac{R}{I} \cong F_1 \times \ldots F_m$. Then $y + I = (0, 1, \ldots 1)$ is a zero-divisor but is not in the center of $\Gamma(\frac{R}{I})$. This implies that $d_\Gamma(y + I, z + I) = 3$, for some $z + I$, $z + I \neq y + I$ and so $d(y, z) = 3$, for some z. So y does not lie in the center of $\Gamma_I(R)$.
Case 3: $n \geq 2$ and $m = 0$.
For each $i = 2, \ldots n$, choose $x_i \neq 0$ in M_i. Let $z + I = (1, x_2, \ldots, x_n)$. Then $z + I$ is a zero-divisor but is not in the center of $\Gamma(\frac{R}{I})$. So z does not lie in the center of $\Gamma_I(R)$. Thus, in all these cases, the center is not the entire vertex set of $\Gamma_I(R)$. Therefore, the diameter is strictly larger than the radius and $diam(\Gamma_I(R)) = 3$. □

Theorem 3.14. Let I be an ideal of a commutative Noetherian ring R such that $\frac{R}{I}$ is a finite ring and $\frac{R}{I} \ncong \mathbb{Z}_2 \times \mathbb{F}$, where $\mathbb{F}$ is a finite field. Then $C(\Gamma_I(R)) = \{x + h : x + I \in C(\Gamma(\frac{R}{I}))$ and $h \in I\}$ and $|C(\Gamma_I(R))| = |I| \left|C(\Gamma(\frac{R}{I}))\right|$.

Proof. Let $x + I \in C(\Gamma(\frac{R}{I}))$ and $h \in I$. Then $e_\Gamma(x + I) = rad(\Gamma(\frac{R}{I}))$.
Case 1: $x^2 \in I$
By Lemma 3.9, $e(x + h) = rad(\Gamma(\frac{R}{I}))$. If $rad(\Gamma(\frac{R}{I})) = 1$, then $e(x + h) = 1$ and $rad(\Gamma_I(R)) = 1$. Assume $rad(\Gamma(\frac{R}{I})) = 2$. So $\frac{R}{I}$ cannot be local and so $rad(\Gamma_I(R)) = 2$. In both cases $rad(\Gamma(\frac{R}{I})) = rad(\Gamma_I(R))$. Clearly $e(x) = rad(\Gamma_I(R))$. So $x \in C(\Gamma_I(R))$.
Case 2: $x^2 \notin I$.
If $e_\Gamma(x+I) = 1$. Then by Lemma 3.9 , $e(x) > 1$. That is $2 \leq e(x) \leq 3$. Suppose $e(x) = 3$. Then there exist y such that $d(x, y) = 3$ and $y \neq x+h, h \in I$. So $x+I \neq y+I$ and $d_\Gamma(x+I, y+I) = 3$, which is a contradiction to $e_\Gamma(x + I) = 1$. Therefore $e(x) = 2$. Since $rad(\Gamma_I(R)) \leq 2$, $rad(\Gamma_I(R)) = 2 = e(x)$. Hence $x \in C(\Gamma_I(R))$. If $e_\Gamma(x + I) \neq 1$, then by Lemma 3.9 $e(x) \leq e_\Gamma(x + I)$ and $e_\Gamma(x + I) = 2$. So $e(x) \leq 2$. Since $x^2 \notin I$, $e(x) = 2 = rad(\Gamma_I(R))$. Thus $x \in C(\Gamma_I(R))$. Conversely let $x \in C(\Gamma_I(R))$. If $rad(\Gamma_I(R)) = 1$, then $e(x) = 1$ and $x^2 \in I$. By Lemma 3.9, $e_\Gamma(x + I) = 1 = rad(\Gamma(\frac{R}{I}))$. Therefore $x + I \in C(\Gamma(\frac{R}{I}))$. If $rad(\Gamma_I(R)) = 2$ and we have $\frac{R}{I} \ncong \mathbb{Z}_2 \times \mathbb{F}$, then by Corollary 3.4 $\frac{R}{I}$ is not local. So $rad(\Gamma(\frac{R}{I})) = 2$. We have $e(x) = 2$. This implies that $e_\Gamma(x + I) = 2$. So $x + I \in C(\Gamma(\frac{R}{I}))$. So the result follows. □

4 Median of $\Gamma_I(R)$

Theorem 4.1. Let R be a finite commutative ring with identity that is not an integral domain and I be an ideal of R. Then the median and center of $\Gamma_I(R)$ are equal if the radius of $\Gamma_I(R)$ is 1, and the median is a subset of the center if the radius is 2.

Proof. Assume $rad(\Gamma_I(R))$ is 1. Let $x \in C(\Gamma_I(R))$. Clearly $s(x) = |V(\Gamma_I(R))| - 1$ for all $x \in C(\Gamma_I(R))$. Let $y \in V(\Gamma_I(R))$. If $y \in C(\Gamma_I(R))$, then $s(y) = s(x)$. If not, $e(y) = 2$ or 3. This implies that $s(y) \geq |V(\Gamma_I(R))|$ and $s(x) \leq s(y)$, for all $y \in V(\Gamma_I(R))$. So $x \in M(\Gamma_I(R))$. Conversely let $z \in M(\Gamma_I(R))$. Then $s(z) \leq s(x)$, for all x. In particular $s(z) \leq s(x)$, for all $x \in C(\Gamma_I(R))$. So $s(z) = |V(\Gamma_I(R))| - 1$. Hence $e(z) = 1$ and $z \in M(\Gamma_I(R))$. So center and median coincide.

Assume that radius of $\Gamma_I(R)$ is 2. So $\frac{R}{I}$ is not local. Let $\frac{R}{I} \cong R_1 \times \ldots \times R_n \times F_1 \times \ldots F_m$ be the Artinian decomposition of $\frac{R}{I}$, where (R_i, M_i) is a local ring, F_j is a field, $1 \leq i \leq n$ and $1 \leq j \leq m$. Let z be a vertex of $\Gamma_I(R)$ that is not in the center. Then take $z+I = (a_1, \ldots, a_n, b_1, \ldots, b_m)$.

In all possible cases, a vertex x in the center is found such that $s(x) < s(z)$. If x is in the center of $\Gamma_I(R)$, then the eccentricity of x is 2. Hence,

$$s(x) = \deg(x) + 2(|V| - 1 - \deg(x)) = 2\,|V| - \deg(x) - 2 \tag{4.1}$$

From (4.1), all the vertices of the median must have the same degree. Since z is not in the center, there is some vertex w such that $d(z, w) = 3$. Thus

$$s(z) > 2\,|V| - \deg(z) - 2 \tag{4.2}$$

Case 1: $b_i \neq 0$ and $b_j \neq 0$ for some $1 \leq i < j \leq m$. Let $x + I = (0, \dots, 0, 1, 0, \dots, 0)$, where the nonzero coordinate is the identity of F_i. Then $x + I$ is in the center of $\Gamma(\frac{R}{I})$ and $ann(z+I) \subseteq ann(x+I)$. Since neither $x+I$ nor $z+I$ is nilpotent, this implies $\deg(z) < \deg(x)$. By (1) and (2), $s(z) > s(x)$.
Case 2: $b_j \neq 0$ for some $1 \leq j \leq m$ and each $a_i \in M_i$ with some $a_k \neq 0$ for some $1 \leq k \leq n$,where M_i is a maximal ideal of R_i. Let $x + I = (0, \dots, 0, a_k, 0, \dots, 0)$. Then $x + I$ is in the center of $\Gamma(\frac{R}{I})$ and $ann(z + I) \subseteq ann(x + I)$. Therefore $\deg_\Gamma(z + I) = |ann(z + I)| - 1 < |ann(x + I)| - 1 = \deg(x + I)$. Hence $\deg_\Gamma(z + I) \leq \deg_\Gamma(x + I)$. Since $b_j \neq 0$, $z^2 \notin I$. So $\deg_\Gamma(z) \leq \deg_\Gamma(x)$. By (1) and (2), $s(z) > s(x)$.
Case 3: a_i is a unit in R_i for some $1 \leq i \leq n$. Let c be a nonzero element of the maximal ideal of R_i, and let $x + I = (0, \dots, 0, c, 0, \dots 0)$. Then $x + I$ is in the center of $\Gamma(\frac{R}{I})$ and $ann(z + I) \subseteq ann(x + I)$. Therefore, $\deg(z + I) = |ann(z + I)| - 1 < |ann(x + I)| - 1$. So $\deg_\Gamma(z + I) \leq \deg_\Gamma(x + I)$. Since a_i is a unit, $z^2 \notin I$. Hence $\deg(z) \leq \deg(x)$. By (1) and (2), $s(z) > s(x)$. Hence in each of the three cases, there is a vertex x of the center with $s(x) < s(z)$. Hence z cannot be in the median. Thus the median is a subset of the center. □

Corollary 4.2. Let R be a finite commutative ring with identity that is not an integral domain and I be an ideal of a ring R. If the radius of $\Gamma_I(R)$ is 2, then the center equals the median if and only if $\frac{R}{I}$ is isomorphic to a direct product of a finite number of copies of a single finite field or $\mathbb{Z}_2 \times \mathbb{F}$ (i.e., $\frac{R}{I} \cong \mathbb{F}^d$ for some finite field $\mathbb{F}$ and some integer $d \geq 2$).

Proof. Assume $rad(\Gamma_I(R)) = 2$ and center and median coincide. Then we have two cases.
Case 1: $rad(\Gamma(\frac{R}{I})) = 1$.
Then by Theorem 2.16 , $\frac{R}{I}$ is either local or $\frac{R}{I} \cong \mathbb{Z}_2 \times \mathbb{F}$. Since $\frac{R}{I}$ cannot be local, $\frac{R}{I} \cong \mathbb{Z}_2 \times \mathbb{F}$ and $\Gamma_I(R)$ is self centered.
Case 2: $rad(\Gamma(\frac{R}{I})) = 2$.
By Theorem 2.18 , $M(\Gamma(\frac{R}{I})) \subseteq C(\Gamma(\frac{R}{I}))$. Now let $x + I \in C(\Gamma(\frac{R}{I}))$. Then by Theorem 3.14, $x \in C(\Gamma_I(R))$. By hypothesis $x \in M(\Gamma_I(R))$. Clearly

$$s(x) = 2\,|V| - \deg(x) - 2$$

and $s(x) < s(y)$, for all y. From Lemma 2.13, $s(x + I) < s(y + I)$, for all $y + I$. So $x + I \in M(\Gamma(\frac{R}{I}))$ and $C(\Gamma(\frac{R}{I})) = M(\Gamma(\frac{R}{I}))$. Also $rad(\Gamma(\frac{R}{I})) = 2$. By Theorem 2.19, $\frac{R}{I}$ is isomorphic to a direct product of a finite number of copies of a single finite field. Converse is obvious. □

Example 4.3. (1) Let $R \cong \mathbb{Z}_{24}$ and $I = (24)$. Then $C(\Gamma_I(R)) = \{4, 12, 20\} = M(\Gamma_I(R))$ and $rad(\Gamma_I(R)) = 1$ (see Figure 1).

(2) Let $R \cong \mathbb{Z}_2 \times \mathbb{Z}_4 \times \mathbb{Z}_2$ and $I = \{0\} \times \{0\} \times \mathbb{Z}_2$. Then $rad(\Gamma_I(R)) = 2$, $C(\Gamma_I(R)) = \{(1, 0, 0), (1, 0, 1), (0, 2, 0), (0, 2, 1)\}$ and $M(\Gamma_I(R)) = \{(1, 0, 0), (1, 0, 1)\}$. In this case $C(\Gamma_I(R)) \subseteq M(\Gamma_I(R))$ (see Figure 2).

(3) Let $R \cong \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_2$ and $I = \{0\} \times \{0\} \times \mathbb{Z}_2$. Then $\frac{R}{I} = \mathbb{Z}_3 \times \mathbb{Z}_3$. In this case $\Gamma(\frac{R}{I})$ is complete bipartite graph and $gr(\Gamma_I(R)) = 4$. By Theorem 2.15, $\Gamma_I(R)$ is complete bipartite graph. So $rad(\Gamma_I(R)) = 2$ and $C(\Gamma_I(R)) = M(\Gamma_I(R))$.

5 Domination Number of $\Gamma_I(R)$

Theorem 5.1. Let R be a commutative ring and $I \neq (0)$ be an ideal of R such that $\frac{R}{I}$ is an Artinian ring with identity.

(i) If $rad(\Gamma_I(R))$ is 1, then $\gamma(\Gamma_I(R))$ is 1.
(ii) If $rad(\Gamma_I(R))$ is 2 and $\frac{R}{I} \ncong \mathbb{Z}_2 \times \mathbb{F}$, then $\gamma(\Gamma_I(R))$ is number of factors in the Artinian decomposition of $\frac{R}{I}$, where $\mathbb{F}$ is a finite field.
(iii) If $\frac{R}{I} \cong \mathbb{Z}_2 \times \mathbb{F}$, then $\gamma(\Gamma_I(R))$ is 2, where $\mathbb{F}$ is a finite field.

Proof. (i) Assume that $rad(\Gamma_I(R)) = 1$. Then any element in the center forms a dominating set and so $\gamma(\Gamma_I(R))$ is 1.

(ii) Assume that $rad(\Gamma_I(R)) = 2$ and $\frac{R}{I} \ncong \mathbb{Z}_2 \times \mathbb{F}$. Then $\frac{R}{I}$ is not local. So $rad(\Gamma(\frac{R}{I})) = 2$ and by Theorem 2.20, domination number of $\Gamma(\frac{R}{I})$ is number of factors in the Artinian decomposition of $\frac{R}{I}$, say m. In [15, Corollary 5.3] it was observed that the connected domination number of $\Gamma(\frac{R}{I})$ equals the domination number of $\Gamma(\frac{R}{I})$. Let $S = \{x_i + I : 1 \leq i \leq m\}$ be a dominating set of $\Gamma(\frac{R}{I})$. Then the subgraph induced by the set S is connected. Consider the set $\{x_1, \ldots, x_m\}$. Let $y \in V(\Gamma_I(R))$. Suppose $y = x_i + h$, where $h \in I$ and $1 \leq i \leq m$. Since S is a connected dominating set, y is dominated by $x_j, j \neq i$. If not, the vertex $y + I$ is dominated by $x_i + I$, for some i. Then y is dominated by x_i. So $\gamma(\Gamma_I(R)) \leq m$. Now suppose that $D = \{z_1, \ldots, z_{m-1}\}$ is a dominating set of $\Gamma_I(R)$. Then every vertex x of $V \setminus D$ is dominated by z_i, for some i. Then $xz_i \in I$, for some i. If $x + I = z_i + I$, then $x + I$ lies in the dominating set. If not, by Theorem 2.8, every vertex $x + I$ of $\Gamma_I(R)$ is dominated by $z_i + I$, for some i. This implies that $\{z_1 + I, \ldots, z_{m-1} + I\}$ is a dominating set of $\Gamma(\frac{R}{I})$, which is a contradiction. So the result follows.

(iii) Assume that $\frac{R}{I} \cong \mathbb{Z}_2 \times \mathbb{F}$. Since $\frac{R}{I}$ is reduced, $\Gamma_I(R)$ has no connected columns and not complete. Let $x_i = (0, \ldots, 0, 1, 0, \ldots, 0)$, where the non-zero coordinate is the identity of $\frac{R_i}{I_i} \cong \mathbb{Z}_2$. Let $x_j = (0, \ldots, 0, 1, 0, \ldots, 0)$, where the non-zero coordinate is the identity of $\frac{R_j}{I_j} \cong \mathbb{F}$. Then $\{x_i, x_j\}$ is a minimal dominating set and so $\gamma(\Gamma_I(R))$ is 2. □

Corollary 5.2. Let R be a finite commutative ring with identity that is not a domain and $I \neq (0)$ be an ideal of R. Then the domination number of $\Gamma_I(R)$ equals the number of distinct maximal ideals of $\frac{R}{I}$.

Corollary 5.3. Let R be a finite commutative ring with identity that is not a domain and I be a non zero ideal of R. Then the connected domination number of $\Gamma_I(R)$ equals the number of distinct maximal ideals of $\frac{R}{I}$.

Proof. In [15, Corollary 5.3] it was observed that the connected domination number of $\Gamma(\frac{R}{I})$ equals the domination number of $\Gamma(\frac{R}{I})$. By Theorem 2.8(a), connected domination number of $\Gamma_I(R)$ equals the domination number of $\Gamma_I(R)$. Hence the result follows from Corollary 5.2 □

Example 5.4. (1) Let $R \cong \mathbb{Z}_8$ and $I = (24)$. Then $\gamma(\Gamma_I(R)) = 1 = \gamma_c(\Gamma_I(R))$ (see Figure 1).
(2) Let $R \cong \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_2$ and $I = \{0\} \times \{0\} \times \mathbb{Z}_2$. So $\frac{R}{I} = \mathbb{Z}_3 \times \mathbb{Z}_3 \times \{0\}$. Since $|I| \geq 2$, $\Gamma(\frac{R}{I})$ is a complete bipartite graph and $gr(\Gamma_I(R)) = 4$. By Theorem 2.15, $\Gamma_I(R)$ is complete bipartite graph and $\gamma(\Gamma_I(R)) = 2 = \gamma_c(\Gamma_I(R))$.
(3) Let $R \cong \mathbb{Z}_2 \times \mathbb{F}_4 \times \mathbb{Z}_2$ and $I = \{0\} \times \{0\} \times \mathbb{Z}_2$. Let $\mathbb{F}_4 = \{0, 1, a, b\}$. Then $\frac{R}{I} \cong \mathbb{Z}_2 \times \mathbb{F}_4$. The set $D = \{(1,0,0), (0,1,0)\}$ is a dominating set. So $\gamma(\Gamma_I(R)) = 2$. Also D is a minimal connected dominating set and $\gamma_c(\Gamma_I(R)) = 2$ (see Figure 3).

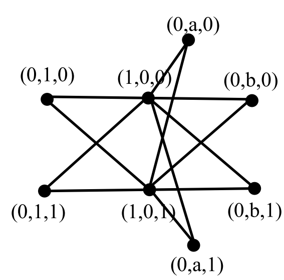

Figure 3

References

[1] S. Akbari, A. Mohammadian, On the Zero-divisor graph of a commutative Ring, *J. Algebra*, **Vol 274** (2), 847-855 (2004).

[2] D. D. Anderson, M. Naseer, Beck's coloring of a commutative ring, *J. Algebra*, **Vol 159** (2), 500-514 (1993).

[3] D. F. Anderson, A. Frazier, A. Lauve, P.S. Livingston, The zero-divisor graph of a commutative ring, II, in: *Lecture Notes in Pure and Appl. Math.*, Marcel Dekker, New York,**Vol 220**, pp. 61-72 (2001).

[4] D. F. Anderson, P. S. Livingston, The zero-divisor graph of a commutative ring, *J. Algebra* **Vol 217** (2), 434-447 (1999).

[5] David F. Anderson, Sara Shirinkam , Some Remarks on the Graph $\Gamma_I(R)$, *Comm. Alg*, **Vol 42**, 545-562 (2014).

[6] M. F. Atiyah, I. G. Macdonald, *Introduction to Commutative Algebra*, Addison-Wesley, Reading, MA, 1969.

[7] I. Beck, Coloring of commutative rings, *J. Algebra* **Vol 116**,(1) 208-226 (1988).

[8] Gary Chartrand, Ping Zhang, *Introduction to Graph Theory*, Tata McGraw-Hill (2006).

[9] I. Kaplansky, *Commutative Rings*. Washington, NJ: Polygonal Publishing House (1974).

[10] H. R. Maimani, M. R. Pournaki, S. Yassemi, Zero-divisor graph with respect to an ideal, *Comm. Alg*, **Vol 34**, 923-929 (2006).

[11] A. Mallika, R. Kala, K. Selvakumar, A note on ideal-based zero-divisor graph of a commutative ring, communicated.

[12] S. P. Redmond, "Generalizations of the Zero-Divisor Graph of a Ring", Doctoral Dissertation, The University of Tennessee, Knoxville,TN (2001).

[13] S. P. Redmond, The zero-divisor graph of a non-commutative ring, *Internat. J. Commutative Rings* **Vol 1** (4), 203-211 (2002) .

[14] S. P. Redmond, An Ideal-Based Zero-Divisor Graph of a Commutative Ring, *Comm. Alg*. **Vol 31** (9), 4425-4443 (2003) .

[15] S. P. Redmond, Central sets and radii of the zero-divisor graphs of commutative rings, *Comm. Alg*, **Vol 34** (7), 2389-2401 (2006).

[16] Teresa W. Haynes, Stephen T. Hedetniemi, Peter J. Slater, *Fundamentals of domination in graphs*, Marcel Dekker, Inc (1988).

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