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On the Dirichlet Problem for the Heat Equation in Non-Symmetric Conical Domains of $\mathbb{R}^{N+1}$

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Abstract. In this paper we give new results of existence, uniqueness and maximal regularity of the solution to the N-dimensional heat equation $\partial_t u - \Delta u = f$, with Cauchy-Dirichlet boundary conditions in a time-dependent domain.

1 Introduction

Let Q be an open set of $\mathbb{R}^{N+1}$ defined by

$$Q = \left\{(t, x_1, x_2, ..., x_N) \in \mathbb{R}^{N+1} : (x_1, x_2, ..., x_N) \in \Omega_t, 0 < t < T\right\}$$

where T is a finite positive number and for a fixed t in the interval $]0, T[$, Ω_t is a bounded domain of $\mathbb{R}^N$ defined by

$$\Omega_t = \left\{(x_1, x_2, ..., x_N) \in \mathbb{R}^N : 0 \leq \frac{x_1^2}{h^2(t)\,\varphi^2(t)} + \frac{x_2^2}{\varphi^2(t)} + ... + \frac{x_N^2}{\varphi^2(t)} < 1\right\}.$$

Here φ is a continuous real-valued function defined on $[0, T]$, Lipschitz continuous on $]0, T]$ and such that

$$\varphi(0) = 0 \text{ and } \varphi(t) > 0$$

for every $t \in]0, T]$. h is a Lipschitz continuous real-valued function defined on $[0, T]$, such that

$$0 < \delta \leq h(t) \leq \beta \tag{1.1}$$

for every $t \in [0, T]$, where δ and β are positive constants. In Q, consider the boundary value problem

$$\begin{cases} \partial_t u - \Delta u = f \in L^2(Q), \\ u|_{\partial Q \setminus \Gamma_T} = 0, \end{cases} \tag{1.2}$$

where $\Delta u = \sum_{k=1}^N \partial_{x_k}^2 u$, ∂Q is the boundary of Q defined by

$$\partial Q = \Gamma_T \cup \left(\underset{t \in]0,T[}{\cup} \{t\} \times \partial\Omega_t\right)$$

with $\partial\Omega_t, t \in]0, T[$, is the boundary of Ω_t and Γ_T is the part of the boundary of Q where $t = T$, given by

$$\Gamma_T = \Omega_T \cup \partial\Omega_T.$$

Problem (1.2) is of interest in combustion theory, where the non-cylindrical space-time part of the boundary

$$\partial Q_{\varphi,h} = \left\{(t, (h\varphi)(t), \varphi(t), ..., \varphi(t)) \in \mathbb{R}^{N+1} : 0 < t < T\right\}$$

can be considered as an approximation of a flame front, see [13] and [15]. On the other hand, Problem (1.2) modelizes, for instance in the case $N = 2$, the diffusion of a polluant in a flow of a river with variable width and depth, see [11] and [12].

Observe that the domain Q considered here is nonstandard since it shrinks at $t=0$, $(\varphi(0)=0)$. This prevents the non-symmetric cone Q to be transformed into a usual cylindrical domain by means of a smooth transformation. On the other hand, the semi group generating the solution cannot be defined since the initial condition is defined on a set measure zero, $\{(0,0,...0)\}$. It is well known that there are two main approaches for the study of boundary value problems in such non-smooth domains. We can work directly in the non-regular domains and we obtain singular solutions, or we impose conditions on the non-regular domains to obtain regular solutions (see, for example [18] and [8]). It is the second approach that we follow in this work. So, we impose a sufficient condition on the function φ near 0, that is,

$$\varphi'(t)\,\varphi(t) \quad \to \quad 0 \qquad \text{as } t\to 0, \tag{1.3}$$

and we obtain existence and regularity results for Problem (1.2) by using the domain decomposition method. More precisely, we will prove that Problem (1.2) has a solution with optimal regularity, that is a solution u belonging to the anisotropic Sobolev space

$$H_0^{1,2}(Q) := \left\{ u \in H^{1,2}(Q) : \, u|_{\partial Q\setminus \Gamma_T} = 0 \right\},$$

with

$$H^{1,2}(Q) = \left\{ u \in L^2(Q) : \partial_t u, \partial_{x_1}^{i_1}\partial_{x_2}^{i_2}...\partial_{x_N}^{i_N} u \in L^2(Q), 1 \le i_1+i_2+...+i_N \le 2 \right\}.$$

The space $H^{1,2}(Q)$ is equipped with the natural norm, that is

$$\|u\|_{H^{1,2}(Q)} = \left(\|u\|_{L^2(Q)}^2 + \|\partial_t u\|_{L^2(Q)}^2 + \sum_{\substack{i_1,i_2,\ldots,i_N=0\\ 1\le i_1+i_2+\ldots+i_N\le 2}}^{2} \left\|\partial_{x_1}^{i_1}\partial_{x_2}^{i_2}...\partial_{x_N}^{i_N} u\right\|_{L^2(Q)}^2 \right)^{1/2}.$$

Examples of sections Ω_t of Q satisfying condition (1.3) are $\varphi(t) = t^{\frac{1}{2}+\epsilon}$, for each $\epsilon > 0$. However the condition (1.3) holds false in the case $\epsilon = 0$. Our main result is

Theorem 1.1. *Problem (1.2) admits a unique solution $u \in H^{1,2}(Q)$ in one of these two cases:*
1) h and φ verify the conditions (1.1) and (1.3).
2) h verifies the condition (1.1) and $(h\varphi)$ and φ are increasing functions in a neighborhood of 0.

In [10] the same problem has been studied in the case of a symmetric conical domain, i.e., in the case where $h(t)=1$. The case $N=2$ is studied in [19] and [9] both in symmetric and non-symmetric conical domains. Alkhutov [2] has studied the case of the heat equation in a ball in some weighted L^p-spaces, but his class of domains is much smaller, as he only considers domains which corresponds here to $(\varphi(t), h(t)) = (\sqrt{t(2R-t)}, 1)$ where R is the radius of the ball. It is clear that these functions satisfy conditions (1.1) and (1.3). Further references on the analysis of parabolic problems in non-cylindrical domains are: Paronetto [17], Alkhutov [1], [2], Degtyarev [4], Sadallah [18]. There are many other works concerning boundary value problems in non-smooth domains (see, for example, Grisvard [6] and the references therein).

The organization of this article is as follows. In Section 2, first we prove an uniqueness result for Problem (1.2), then we derive some technical lemmas which will allow us to prove a uniform estimate (in a sense to be defined later). Section 3 is devoted to the proof of Theorem 1.1. We divide the proof into three main steps. First, we prove that Problem (1.2) admits a (unique) solution in the case of a domain which can be transformed into a cylinder. Second, for T small enough, we prove that the result holds true in the case of a conical domain under the above mentioned assumptions on functions φ and h. Here, we restart the solution at later time $t=\frac{1}{n}$ and show its local existence over $\left[\frac{1}{n}, T\right]$. Then using a passage to the limit and energy estimates, we show that we can get solution on Q by letting $n\to\infty$. Finally, by using a trace result, we complete the proof of Theorem 1.1.

2 Preliminaries

Proposition 2.1. *The solution of Problem (1.2) is unique.*

Proof. Let us consider $u \in H_0^{1,2}(Q)$ a solution of Problem (1.2) with a null right-hand side term. So, the calculations show that the inner product $\langle \partial_t u - \Delta u, u\rangle$ in $L^2(Q)$ gives

$$0 = \int_{\Gamma_T} \tfrac{1}{2} |u|^2 \, dx_1 dx_2 ... dx_N + \int_Q |\nabla u|^2 dt \, dx_1 dx_2 ... dx_N.$$

This implies that $|\nabla u|^2 = 0$ and consequently $\Delta u = 0$. Then, the equation of Problem (1.2) gives $\partial_t u = 0$. Thus, u is constant. The boundary conditions imply that $u = 0$ in Q. This proves the uniqueness of the solution of Problem (1.2). □

Remark 2.2. In the sequel, we will be interested only by the question of the existence of the solution of Problem (1.2).

The following result is well known (see, for example, [16])

Lemma 2.3. *Let $B(0,1)$ be the unit ball of $\mathbb{R}^N$. Then, the Laplace operator $\Delta : H^2(B(0,1)) \cap H_0^1(B(0,1)) \longrightarrow L^2(B(0,1))$ is an isomorphism. Moreover, there exists a constant $C > 0$ such that*

$$\|v\|_{H^2(B(0,1))} \leq C \|\Delta v\|_{L^2(B(0,1))}, \; \forall v \in H^2(B(0,1)).$$

In the previous lemma, H^2 and H_0^1 are the usual Sobolev spaces defined, for instance, in Lions-Magenes [16]. In section 3, we will need the following result.

Lemma 2.4. *For a fixed $t \in \left]0, T\right[$, there exists a constant $C > 0$ such that for each $u \in H^2(\Omega_t) \cap H_0^1(\Omega_t)$, we have*

$$\left\|\partial_{x_k}^j u\right\|_{L^2(\Omega_t)}^2 \leq C\varphi^{2(2-j)}(t) \|\Delta u\|_{L^2(\Omega_t)}^2, \; j = 0, 1; \; k = 1, 2, ..., N,$$

where $\partial_{x_k}^0 u = u$, $k = 1, 2, ..., N$.

Proof. It is a direct consequence of Lemma 2.3. Indeed, let $t \in \left]0, T\right[$ and define the following change of variables

$$B(0,1) \to \Omega_t; \; (x_1, x_2, ..., x_N) \longmapsto (h(t)\varphi(t)x_1, \varphi(t)x_2, ..., \varphi(t)x_N) = (x_1', x_2', ..., x_N').$$

Set $v(x_1, x_2, ..., x_N) = u(h(t)\varphi(t)x_1, \varphi(t)x_2, ..., \varphi(t)x_N)$, then if $v \in H^2(B(0,1))$, u belongs to $H^2(\Omega_t)$.

i) For $j = 0, 1$, we have

$$\begin{aligned}
\left\|\partial_{x_1}^j v\right\|_{L^2(B(0,1))}^2 &= \int_{B(0,1)} \left(\partial_{x_1}^j v\right)^2 (x_1, x_2, ..., x_N) \, dx_1 dx_2 ... dx_N \\
&= \int_{\Omega_t} \left(\partial_{x_1'}^j u\right)^2 (x_1', x_2', ..., x_N') \, h^{2j}(t)\varphi^{2j}(t) \tfrac{1}{\varphi^N(t)h(t)} dx_1' dx_2' ... dx_N' \\
&= h^{2j-1}(t)\varphi^{2j-N}(t) \int_{\Omega_t} \left(\partial_{x_1'}^j u\right)^2 (x_1', x_2', ..., x_N') \, dx_1' dx_2' ... dx_N' \\
&= h^{2j-1}(t)\varphi^{2j-N}(t) \left\|\partial_{x_1'}^j u\right\|_{L^2(\Omega_t)}^2.
\end{aligned}$$

On the other hand, we have

$$\begin{aligned}
\|\Delta v\|_{L^2(B(0,1))}^2 &= \int_{B(0,1)} \left[\left(\partial_{x_1}^2 v + \partial_{x_2}^2 v + ... + \partial_{x_N}^2 v\right)(x_1, x_2, ..., x_N)\right]^2 \, dx_1 dx_2 ... dx_N \\
&= \int_{\Omega_t} \left(h^2(t)\varphi^2(t)\partial_{x_1'}^2 u + \textstyle\sum_{k=2}^N \varphi^2(t)\partial_{x_k'}^2 u\right)^2 \tfrac{1}{\varphi^N(t)h(t)} dx_1' dx_2' ... dx_N' \\
&= \tfrac{\varphi^{4-N}(t)}{h(t)} \int_{\Omega_t} \left(h^2(t)\partial_{x_1'}^2 u + \textstyle\sum_{k=2}^N \partial_{x_k'}^2 u\right)^2 (x_1', x_2', ..., x_N') \, dx_1' dx_2' ... dx_N' \\
&\leq \tfrac{1}{\delta}\varphi^{4-N}(t) \|\Delta u\|_{L^2(\Omega_t)}^2,
\end{aligned}$$

where δ is the constant which appears in (1.1). Using the inequality

$$\left\|\partial_{x_1}^j v\right\|_{L^2(B(0,1))}^2 \leq C \|\Delta v\|_{L^2(B(0,1))}^2, \; j = 0, 1,$$

of Lemma 2.3 and the condition (1.1), we obtain the desired inequality

$$\left\|\partial_{x_1}^j u\right\|_{L^2(\Omega_t)}^2 \leq C\varphi^{2(2-j)}(t)\,\|\Delta u\|_{L^2(\Omega_t)}^2, j=0,1.$$

ii) For $k=2,...,N$ and $j=0,1$, we have

$$\begin{aligned}
\left\|\partial_{x_k}^j v\right\|_{L^2(B(0,1))}^2 &= \int_{B(0,1)} \left(\partial_{x_k}^j v\right)^2 (x_1,x_2,...,x_k,...,x_N)\; dx_1dx_2...dx_k...dx_N \\
&= \int_{\Omega_t} \left(\partial_{x_k'}^j u\right)^2 (x_1',x_2',...,x_k',...,x_N')\,\varphi^{2j}(t)\; \tfrac{1}{\varphi^N(t)h(t)}dx_1'dx_2'...dx_k'...dx_N' \\
&= \tfrac{\varphi^{2j-N}(t)}{h(t)} \int_{\Omega_t} \left(\partial_{x_k'}^j u\right)^2 (x_1',x_2',...,x_k',...,x_N')\,dx_1'dx_2'...dx_k'...dx_N' \\
&= \tfrac{\varphi^{2j-N}(t)}{h(t)} \left\|\partial_{x_k'}^j u\right\|_{L^2(\Omega_t)}^2.
\end{aligned}$$

On the other hand, we have

$$\|\Delta v\|_{L^2(B(0,1))}^2 \leq \frac{1}{\delta}\varphi^{4-N}(t)\,\|\Delta u\|_{L^2(\Omega_t)}^2.$$

Using Lemma 2.3 and the condition (1.1), we obtain the desired inequality. □

3 Proof of Theorem 1.1

We divide the proof of Theorem 1.1 into three steps.

3.1 Step 1: Case of a truncated domain Q_α which can be transformed into a cylinder

In this subsection, we replace Q by $Q_\alpha,\ \alpha>0$:

$$Q_\alpha = \left\{(t,x_1,x_2,...,x_N)\in\mathbb{R}^{N+1} : \frac{1}{\alpha}<t<T, 0\leq \frac{x_1^2}{h^2(t)\varphi^2(t)} + \frac{x_2^2}{\varphi^2(t)} + ... + \frac{x_N^2}{\varphi^2(t)} < 1\right\}.$$

Theorem 3.1. *The following problem admits a unique solution $u\in H^{1,2}(Q_\alpha)$*

$$\begin{cases} \partial_t u - \Delta u = f \in L^2(Q_\alpha), \\ u|_{\partial Q_\alpha \setminus \Gamma_T} = 0. \end{cases} \tag{3.1}$$

Proof. The change of variables

$$(t,x_1,x_2,...,x_N) \;\mapsto\; (t,y_1,y_2,...,y_N) = \left(t, \tfrac{x_1}{h(t)\varphi(t)}, \tfrac{x_2}{\varphi(t)}, ..., \tfrac{x_N}{\varphi(t)}\right)$$

transforms Q_α into the cylinder $P_\alpha = \left]\frac{1}{\alpha}, T\right[\times B(0,1)$, where $B(0,1)$ is the unit ball of $\mathbb{R}^N$. Putting $u(t,x_1,x_2,...,x_N) = v(t,y_1,y_2,...,y_N)$ and $f(t,x_1,x_2,...,x_N) = g(t,y_1,y_2,...,y_N)$, then Problem (3.1) is transformed, in P_α into the following variable-coefficient parabolic problem

$$\begin{cases} \partial_t v - \frac{1}{h^2(t)\varphi^2(t)}\partial_{y_1}^2 v - \frac{1}{\varphi^2(t)}\sum_{k=2}^N \partial_{y_k}^2 v - \frac{(h\varphi)'(t)y_1}{h(t)\varphi(t)}\partial_{y_1} v - \frac{\varphi'(t)}{\varphi(t)}\sum_{k=2}^N y_k\partial_{y_k} v = g \\ v_{/\partial P_\alpha - \Gamma_T} = 0. \end{cases}$$

This change of variables conserves the spaces L^2 and $H^{1,2}$ since $\frac{1}{h^2\varphi^2}$, $\frac{1}{\varphi^2}$, $\frac{(h\varphi)'}{h\varphi}$ and $\frac{\varphi'}{\varphi}$ are bounded functions on each Q_α. In other words

$$f\in L^2(Q_\alpha) \Longleftrightarrow g\in L^2(P_\alpha) \;\text{ and }\; u\in H^{1,2}(Q_\alpha) \Longleftrightarrow v\in H^{1,2}(P_\alpha).$$

□

Proposition 3.2. *The following operator is compact*

$$\mathcal{B}_1 := -\left[\frac{(h\varphi)'(t)\,y_1}{h(t)\,\varphi(t)}\partial_{y_1} + \frac{\varphi'(t)}{\varphi(t)}\sum_{k=2}^N y_k\partial_{y_k}\right] : H_0^{1,2}(P_\alpha) \longrightarrow L^2(P_\alpha).$$

Proof. P_α has the horn property of Besov (see [3]). So, for $j = 1, 2, ..., N$

$$\partial_{y_j} : H_0^{1,2}(P_\alpha) \longrightarrow H^{\frac{1}{2},1}(P_\alpha)\,;\; v \longmapsto \partial_{y_j} v,$$

is continuous. Since P_α is bounded, the canonical injection is compact from $H^{\frac{1}{2},1}(P_\alpha)$ into $L^2(P_\alpha)$ (see for instance [3]), where

$$H^{\frac{1}{2},1}(P_\alpha) = L^2\left(\frac{1}{\alpha}, T; H^1(B(0,1))\right) \cap H^{\frac{1}{2}}\left(\frac{1}{\alpha}, T; L^2(B(0,1))\right).$$

For the complete definitions of the Hilbertian Sobolev spaces $H^{r,s}$ see for instance [16]. Consider the composition

$$\partial_{y_j} : H_0^{1,2}(P_\alpha) \to H^{\frac{1}{2},1}(P_\alpha) \to L^2(P_\alpha)\,;\; v \mapsto \partial_{y_j} v \mapsto \partial_{y_j} v,$$

then ∂_{y_j} is a compact operator from $H_0^{1,2}(P_\alpha)$ into $L^2(P_\alpha)$. Since $-\frac{\varphi'(t)}{\varphi(t)}$, $-\frac{(h\varphi)'(t)}{h(t)\varphi(t)}$ are bounded functions on each P_α, the operators $-\frac{(h\varphi)'(t)y_1}{h(t)\varphi(t)}\partial_{y_1}$, $-\frac{\varphi'(t)y_k}{\varphi(t)}\partial_{y_k}$, $k = 2, 3, ..., N$ are also compact from $H_0^{1,2}(P_\alpha)$ into $L^2(P_\alpha)$. Consequently, $\mathcal{B}_1$ is compact from $H_0^{1,2}(P_\alpha)$ into $L^2(P_\alpha)$. □

So, in order to complete the proof of Theorem 3.1, it is sufficient to show that the operator

$$\mathcal{B}_2 := \partial_t - \frac{1}{h^2(t)\varphi^2(t)}\partial_{y_1}^2 - \frac{1}{\varphi^2(t)}\sum_{k=2}^{N}\partial_{y_k}^2$$

is an isomorphism from $H_0^{1,2}(P_\alpha)$ into $L^2(P_\alpha)$.

Lemma 3.3. *The operator $\mathcal{B}_2$ is an isomorphism from $H_0^{1,2}(P_\alpha)$ into $L^2(P_\alpha)$.*

Proof. Since the coefficients $\frac{1}{h^2(t)\varphi^2(t)}$ and $\frac{1}{\varphi^2(t)}$ are bounded in $\overline{P_\alpha}$, the optimal regularity is given by Ladyzhenskaya-Solonnikov-Ural'tseva [14, Theorem 9.1, pp. 341-342]. □

We shall need the following result in order to justify the calculus of this section.

Lemma 3.4. *The space*

$$\left\{u \in H^4(P_\alpha) : u|_{\partial_p P_\alpha} = 0\right\}$$

is dense in the space

$$\left\{u \in H^{1,2}(P_\alpha) : u|_{\partial_p P_\alpha} = 0\right\}.$$

Here, $\partial_p P_\alpha$ is the parabolic boundary of P_α and H^4 stands for the usual Sobolev space defined, for instance, in Lions-Magenes [16].

The proof of the above lemma may be found in [7].

Remark 3.5. In Lemma 3.4, we can replace P_α by Q_α with the help of the change of variables defined above.

3.2 Step 2: Case of a conical type domain

A uniform estimate

In this case, we define Q by

$$Q = \left\{(t, x_1, x_2, ..., x_N) \in \mathbb{R}^{N+1} : 0 < t < T, 0 \le \frac{x_1^2}{h^2(t)\varphi^2(t)} + \frac{x_2^2}{\varphi^2(t)} + ... + \frac{x_N^2}{\varphi^2(t)} < 1\right\}$$

with

$$\varphi(0) = 0,\; \varphi(t) > 0,\; t \in \left]0, T\right].$$

We assume that the functions h and φ satisfy conditions (1.1) and (1.3). For each $n \in \mathbb{N}^*$, we define Q_n by

$$Q_n = \left\{(t, x_1, x_2, ..., x_N) \in \mathbb{R}^{N+1} : \frac{1}{n} < t < T, 0 \leq \frac{x_1^2}{h^2(t)\varphi^2(t)} + \frac{x_2^2}{\varphi^2(t)} + ... + \frac{x_N^2}{\varphi^2(t)} < 1\right\}$$

and we denote $f_n = f|_{Q_n}$ and $u_n \in H^{1,2}(Q_n)$ the solution of Problem (1.2) in Q_n. Such a solution exists by Theorem 3.1.

Proposition 3.6. *For T small enough, there exists a constant K_1 independent of n such that*

$$\|u_n\|_{H^{1,2}(Q_n)} \leq K_1 \|f_n\|_{L^2(Q_n)} \leq K_1 \|f\|_{L^2(Q)}.$$

In order to prove Proposition 3.6, we need the following result which is a consequence of Lemma 2.4 and Grisvard-Looss [5, Theorem 2.2].

Lemma 3.7. *There exists a constant $C > 0$ independent of n such that*

$$\sum_{\substack{i_1, i_2, ..., i_N = 0 \\ i_1 + i_2 + ... + i_N = 2}}^{2} \left\|\partial_{x_1}^{i_1} \partial_{x_2}^{i_2} ... \partial_{x_N}^{i_N} u_n\right\|_{L^2(Q_n)}^2 \leq C \|\Delta u_n\|_{L^2(Q_n)}^2.$$

Proof of Proposition 3.6: Let us denote the inner product in $L^2(Q_n)$ by $\langle ., . \rangle$, then we have

$$\begin{aligned} \|f_n\|_{L^2(Q_n)}^2 &= \langle \partial_t u_n - \Delta u_n, \partial_t u_n - \Delta u_n \rangle \\ &= \|\partial_t u_n\|_{L^2(Q_n)}^2 + \|\Delta u_n\|_{L^2(Q_n)}^2 - 2\langle \partial_t u_n, \Delta u_n \rangle \end{aligned}$$

Estimation of $-2\langle \partial_t u_n, \Delta u_n \rangle$: We have

$$\partial_t u_n . \Delta u_n = \sum_{k=1}^{N} \partial_{x_k} \left(\partial_t u_n \partial_{x_k} u_n\right) - \frac{1}{2} \sum_{k=1}^{N} \partial_t \left(\partial_{x_k} u_n\right)^2.$$

Then

$$\begin{aligned} -2\langle \partial_t u_n, \Delta u_n \rangle &= -2 \int_{Q_n} \partial_t u_n . \Delta u_n dt dx_1 dx_2 ... dx_N \\ &= -2 \int_{Q_n} \sum_{k=1}^{N} \partial_{x_k} \left(\partial_t u_n \partial_{x_k} u_n\right) dt dx_1 dx_2 ... dx_N \\ &\quad + \int_{Q_n} \sum_{k=1}^{N} \partial_t \left(\partial_{x_k} u_n\right)^2 dt dx_1 dx_2 ... dx_N \\ &= \int_{\partial Q_n} \left[|\nabla u_n|^2 \nu_t - 2\partial_t u_n \sum_{k=1}^{N} \partial_{x_k} u_n \nu_{x_k}\right] d\sigma \end{aligned}$$

where ν_t, ν_{x_1}, ..., ν_{x_N} are the components of the unit outward normal vector at ∂Q_n. We shall rewrite the boundary integral making use of the boundary conditions. On the part of the boundary of Q_n where $t = \frac{1}{n}$, we have $u_n = 0$ and consequently $\partial_{x_k} u_n = 0$, $k = 1, ..., N$. The corresponding boundary integral vanishes. On the part of the boundary where $t = T$, we have $\nu_{x_k} = 0$, $k = 1, ..., N$ and $\nu_t = 1$. Accordingly the corresponding boundary integral

$$\int_{\Gamma_T} |\nabla u_n|^2 (T, x_1, x_2, ..., x_N) \, dx_1 dx_2 ... dx_N$$

is nonnegative. On the part of the boundary where $\frac{x_1^2}{h^2(t)\varphi^2(t)} + \frac{x_2^2}{\varphi^2(t)} + ... + \frac{x_N^2}{\varphi^2(t)} = 1$, we have

$$\nu_{x_1} = \frac{\cos\theta_1}{\sqrt{\left(\varphi'(t)\, h(t) \sin^2\theta_1 + (h\varphi)'(t) \cos^2\theta_1\right)^2 + (h(t)\sin\theta_1)^2 + \cos^2\theta_1}},$$

$$\nu_{x_k} = \frac{h(t)\sin\theta_1 ... \sin\theta_{k-1}\cos\theta_k}{\sqrt{\left(\varphi'(t)\, h(t) \sin^2\theta_1 + (h\varphi)'(t) \cos^2\theta_1\right)^2 + (h(t)\sin\theta_1)^2 + \cos^2\theta_1}}, \; k = 2, 3, ..., N-1,$$

$$\nu_{x_N} = \frac{h(t)\sin\theta_1 ... \sin\theta_{N-2}\sin\theta_{N-1}}{\sqrt{\left(\varphi'(t)\,h(t)\sin^2\theta_1 + (h\varphi)'(t)\cos^2\theta_1\right)^2 + (h(t)\sin\theta_1)^2 + \cos^2\theta_1}},$$

$$\nu_t = \frac{-\left(\varphi'(t)\,h(t)\sin^2\theta_1 + (h\varphi)'(t)\cos^2\theta_1\right)}{\sqrt{\left(\varphi'(t)\,h(t)\sin^2\theta_1 + (h\varphi)'(t)\cos^2\theta_1\right)^2 + (h(t)\sin\theta_1)^2 + \cos^2\theta_1}}$$

and

$$u_n(t, h(t)\,\varphi(t)\cos\theta_1, \varphi(t)\sin\theta_1\cos\theta_2, ..., \varphi(t)\sin\theta_1\sin\theta_2 ... \sin\theta_{N-2}\cos\theta_{N-1},$$
$$\varphi(t)\sin\theta_1\sin\theta_2 ... \sin\theta_{N-2}\sin\theta_{N-1}) = 0.$$

Differentiating with respect to $t, \theta_1, ..., \theta_{N-2}$ and θ_{N-1} we obtain

$$\begin{aligned}\partial_t u_n &= -\varphi'(t)\left[\textstyle\sum_{k=2}^{N-1}\sin\theta_1 ... \sin\theta_{k-1}\cos\theta_k.\partial_{x_k}u_n + \sin\theta_1 ... \sin\theta_{N-1}.\partial_{x_N}u_n\right] \\ &\quad - (h\varphi)'(t)\cos\theta_1.\partial_{x_1}u_n,\end{aligned}$$

$$h\sin\theta_1\partial_{x_1}u_n = \cos\theta_1\left[\sum_{k=2}^{N-1}\sin\theta_2 ... \sin\theta_{k-1}\cos\theta_k.\partial_{x_k}u_n + \sin\theta_2 ... \sin\theta_{N-1}.\partial_{x_N}u_n\right],$$

$$\sin\theta_j\partial_{x_j}u_n = \cos\theta_j\left[\sum_{k=j+1}^{N-1}\sin\theta_{j+1} ... \sin\theta_{k-1}\cos\theta_k.\partial_{x_k}u_n + \sin\theta_{j+1} ... \sin\theta_{N-1}.\partial_{x_N}u_n\right]$$

for $j = 2, 3, ..., N-2$ and

$$\sin\theta_{N-1}.\partial_{x_{N-1}}u_n = \cos\theta_{N-1}.\partial_{x_N}u_n.$$

Consequently the corresponding boundary integral is

$$\begin{aligned}&J_n \\ &= -2\int_0^{2\pi} ... \int_0^{\pi}\int_0^{\pi}\int_{\frac{1}{n}}^{T}\partial_t u_n \times \{\cos\theta_1.\partial_{x_1}u_n + h\textstyle\sum_{k=2}^{N-1}\sin\theta_1 ... \sin\theta_{k-1}\cos\theta_k.\partial_{x_k}u_n \\ &+h\sin\theta_1 ... \sin\theta_{N-2}\sin\theta_{N-1}.\partial_{x_N}u_n\} \times \varphi(t)\,dtd\theta_1 d\theta_2 ... d\theta_{N-1} \\ &- \int_0^{2\pi} ... \int_0^{\pi}\int_0^{\pi}\int_{\frac{1}{n}}^{T}|\nabla u_n|^2\left(\left(\varphi'(t)\,h(t)\sin^2\theta_1 + (h\varphi)'(t)\cos^2\theta_1\right)\right) \times \varphi(t)\,dtd\theta_1 d\theta_2 ... d\theta_{N-1} \\ &= 2\int_0^{2\pi} ... \int_0^{\pi}\int_0^{\pi}\int_{\frac{1}{n}}^{T}\{\varphi'(t)\left[\textstyle\sum_{k=2}^{N-1}\sin\theta_1 ... \sin\theta_{k-1}\cos\theta_k.\partial_{x_k}u_n + \sin\theta_1 ... \sin\theta_{N-1}.\partial_{x_N}u_n\right] \\ &\times\left(\cos\theta_1.\partial_{x_1}u_n + h\textstyle\sum_{k=2}^{N-1}\sin\theta_1 ... \sin\theta_{k-1}\cos\theta_k.\partial_{x_k}u_n + h\sin\theta_1 ... \sin\theta_{N-2}\sin\theta_{N-1}.\partial_{x_N}u_n\right) \\ &+ (h\varphi)'(t)\cos\theta_1.\partial_{x_1}u_n\} \times \varphi(t)\,dtd\theta_1 d\theta_2 ... d\theta_{N-1} \\ &- \int_0^{2\pi} ... \int_0^{\pi}\int_0^{\pi}\int_{\frac{1}{n}}^{T}|\nabla u_n|^2\left(\left(\varphi'(t)\,h(t)\sin^2\theta_1 + (h\varphi)'(t)\cos^2\theta_1\right)\right) \times \varphi(t)\,dtd\theta_1 d\theta_2 ... d\theta_{N-1} \\ &= \int_0^{2\pi} ... \int_0^{\pi}\int_0^{\pi}\int_{\frac{1}{n}}^{T}|\nabla u_n|^2\left(\left(\varphi'(t)\,h(t)\sin^2\theta_1 + (h\varphi)'(t)\cos^2\theta_1\right)\right) \times \varphi(t)\,dtd\theta_1 d\theta_2 ... d\theta_{N-1}.\end{aligned}$$

Here, we have used the equality

$$\begin{aligned}&(h\varphi)'(t)\cos\theta_1.\partial_{x_1}u_n \\ &+\varphi'(t)\left[\textstyle\sum_{k=2}^{N-1}\sin\theta_1 ... \sin\theta_{k-1}\cos\theta_k.\partial_{x_k}u_n + \sin\theta_1 ... \sin\theta_{N-2}\sin\theta_{N-1}.\partial_{x_N}u_n\right] \\ &\times\left(\cos\theta_1.\partial_{x_1}u_n + h\textstyle\sum_{k=2}^{N-1}\sin\theta_1 ... \sin\theta_{k-1}\cos\theta_k.\partial_{x_k}u_n + h\sin\theta_1 ... \sin\theta_{N-1}.\partial_{x_N}u_n\right) \\ &= |\nabla u_n|^2\left(\varphi'(t)\,h(t)\sin^2\theta_1 + (h\varphi)'(t)\cos^2\theta_1\right)\end{aligned}$$

which can be obtained from the above mentioned boundary conditions. Finally

$$\begin{aligned}&-2\langle\partial_t u_n, \Delta u_n\rangle \\ &= \int_0^{2\pi} ... \int_0^{\pi}\int_0^{\pi}\int_{\frac{1}{n}}^{T}|\nabla u_n|^2\left(\varphi'(t)\,h(t)\sin^2\theta_1 + (h\varphi)'(t)\cos^2\theta_1\right) \times \varphi(t)\,dtd\theta_1 d\theta_2 ... d\theta_{N-1} \\ &+ \int_{\Gamma_T}|\nabla u_n|^2(T, x_1, x_2, ..., x_N)\,dx_1 dx_2 ... dx_N.\end{aligned} \tag{3.2}$$

Remark 3.8. Observe that the integral $\int_{\Gamma_T} |\nabla u_n|^2 (T, x_1, x_2, ..., x_N)\, dx_1 dx_2 ... dx_N$ which appears in the last formula is nonnegative. This is a good sign for our estimate because we can deduce immediately

$$\begin{aligned}
&\|f_n\|^2_{L^2(Q_n)}\\
&\geq \|\partial_t u_n\|^2_{L^2(Q_n)} + \|\Delta u_n\|^2_{L^2(Q_n)}\\
&+ \int_0^{2\pi} ... \int_0^{\pi} \int_0^{\pi} \int_{\frac{1}{n}}^{T} |\nabla u_n|^2 \left(\varphi'(t)\, h(t) \sin^2\theta_1 + (h\varphi)'(t) \cos^2\theta_1\right) \times \varphi(t)\, dt d\theta_1 d\theta_2 ... d\theta_{N-1}.
\end{aligned}$$

So, if $(h\varphi)$ and φ are increasing functions in the interval $\left(\frac{1}{n}, T\right)$, then

$$\int_0^{2\pi} ... \int_0^{\pi} \int_0^{\pi} \int_{\frac{1}{n}}^{T} |\nabla u_n|^2 \left(\varphi'(t)\, h(t) \sin^2\theta_1 + (h\varphi)'(t) \cos^2\theta_1\right) \times \varphi(t)\, dt d\theta_1 d\theta_2 ... d\theta_{N-1} \geq 0.$$

Consequently,

$$\|f_n\|^2_{L^2(Q_n)} \geq \|\partial_t u_n\|^2_{L^2(Q_n)} + \|\Delta u_n\|^2_{L^2(Q_n)}. \tag{3.3}$$

But, thanks to Lemma 2.4 and since φ is bounded in $(0, T)$, there exists a constant $C' > 0$ such that

$$\max\left(\|u\|^2_{L^2(Q_n)}, \|\partial_{x_k} u\|^2_{L^2(Q_n)}\right) \leq C' \|\Delta u\|^2_{L^2(Q_n)}, \; k = 1, 2, ..., N.$$

Taking into account Lemma 3.7 and estimate (3.3), this proves the desired estimate of Proposition 3.6. So, it remains to prove the estimate of Proposition 3.6 under the hypothesis (1.3). For this purpose, we need the following lemma

Lemma 3.9. *One has*

$$\begin{aligned}
&-2\langle \partial_t u_n, \Delta u_n \rangle\\
&= 2\int_{Q_n} \left(\frac{(h\varphi)'}{h\varphi} x_1 \partial_{x_1} u_n + \frac{\varphi'}{\varphi} \sum_{k=2}^{N} x_k \partial_{x_k} u_n\right) \Delta u_n dt dx_1 dx_2 ... dx_N\\
&- \int_{Q_n} \left((N-2) \frac{(h\varphi)'}{h\varphi} (\partial_{x_1} u_n)^2 + \left(\frac{(h\varphi)'}{h\varphi} + (N-1)\frac{\varphi'}{\varphi}\right) \sum_{k=2}^{N} (\partial_{x_k} u_n)^2\right) dt dx_1 dx_2 ... dx_N\\
&+ \int_{\Gamma_T} |\nabla u_n|^2 (T, x_1, x_2, ..., x_N)\, dx_1 dx_2 ... dx_N.
\end{aligned}$$

Proof. The proof of this lemma can be found in the Appendix. □

Now, we continue the proof of Proposition 3.6. We have

$$\begin{aligned}
&\left|\int_{Q_n} \left(\frac{(h\varphi)'}{h\varphi} x_1 \partial_{x_1} u_n + \frac{\varphi'}{\varphi} \sum_{k=2}^{N} x_k \partial_{x_k} u_n\right) \Delta u_n dt dx_1 dx_2 ... dx_N\right|\\
&\leq \|\Delta u_n\|_{L^2(Q_n)} \left\|\frac{(h\varphi)'}{h\varphi} x_1 \partial_{x_1} u_n\right\|_{L^2(Q_n)}\\
&+ \sum_{k=2}^{N} \|\Delta u_n\|_{L^2(Q_n)} \left\|\frac{\varphi'}{\varphi} x_k \partial_{x_k} u_n\right\|_{L^2(Q_n)},
\end{aligned}$$

but Lemma 2.4 yields for $k = 2, ..., N$

$$\begin{aligned}
\left\|\frac{\varphi'}{\varphi} x_k \partial_{x_k} u_n\right\|^2_{L^2(Q_n)} &= \int_{\frac{1}{n}}^{T} \varphi'^2(t) \int_{\Omega_t} \left(\frac{x_k}{\varphi(t)}\right)^2 (\partial_{x_k} u_n)^2 dt dx_1 dx_2 ... dx_N\\
&\leq \int_{\frac{1}{n}}^{T} \varphi'^2(t) \int_{\Omega_t} (\partial_{x_k} u_n)^2 dt dx_1 dx_2 ... dx_N\\
&\leq C^2 \int_{\frac{1}{n}}^{T} (\varphi(t)\varphi'(t))^2 \int_{\Omega_t} (\Delta u_n)^2 dt dx_1 dx_2 ... dx_N\\
&\leq C^2 \epsilon^2 \|\Delta u_n\|^2_{L^2(Q_n)},
\end{aligned}$$

since $(\varphi(t)\varphi'(t)) \leq \epsilon$. Similarly, we have also

$$\left\|\frac{(h\varphi)'}{h\varphi} x_1 \partial_{x_1} u_n\right\|^2_{L^2(Q_n)} \leq C^2 \epsilon^2 \|\Delta u_n\|^2_{L^2(Q_n)}.$$

Then

$$\left|\int_{Q_n} \left(\frac{(h\varphi)'}{h\varphi} x_1 \partial_{x_1} u_n + \frac{\varphi'}{\varphi} \sum_{k=2}^{N} x_k \partial_{x_k} u_n\right) \Delta u_n dt dx_1 dx_2 ... dx_N\right| \leq NC\epsilon \|\Delta u_n\|^2_{L^2(Q_n)}.$$

By using a similar argument, we show that

$$\left|\int_{Q_n}\left((N-2)\frac{(h\varphi)'}{h\varphi}(\partial_{x_1}u_n)^2+\left(\frac{(h\varphi)'}{h\varphi}+(N-1)\frac{\varphi'}{\varphi}\right)\sum_{k=2}^{N}(\partial_{x_k}u_n)^2\right)dtdx_1dx_2...dx_N\right|$$
$$\leq K\epsilon\left\|\Delta u_n\right\|^2_{L^2(Q_n)}.$$

Therefore, Lemma 3.9 shows that

$$\begin{aligned}
&|2\langle\partial_t u_n,\Delta u_n\rangle|\\
&\geq -2\left|\int_{Q_n}\left(\frac{(h\varphi)'}{h\varphi}x_1\partial_{x_1}u_n+\frac{\varphi'}{\varphi}\sum_{k=2}^{N}x_k\partial_{x_k}u_n\right)\Delta u_n dtdx_1dx_2...dx_N\right|\\
&-\left|\int_{Q_n}\left((N-2)\frac{(h\varphi)'}{h\varphi}(\partial_{x_1}u_n)^2+\left(\frac{(h\varphi)'}{h\varphi}+(N-1)\frac{\varphi'}{\varphi}\right)\sum_{k=2}^{N}(\partial_{x_k}u_n)^2\right)dtdx_1dx_2...dx_N\right|\\
&+\int_{\Gamma_T}\left|\nabla u_n\right|^2(T,x_1,x_2,...,x_N)\,dx_1dx_2...dx_N\\
&\geq-(K+2NC)\,\epsilon\left\|\Delta u_n\right\|^2_{L^2(Q_n)}.
\end{aligned}$$

Hence

$$\begin{aligned}
\left\|f_n\right\|^2_{L^2(Q_n)} &= \left\|\partial_t u_n\right\|^2_{L^2(Q_n)}+\left\|\Delta u_n\right\|^2_{L^2(Q_n)}-2\langle\partial_t u_n,\Delta u_n\rangle\\
&\geq \left\|\partial_t u_n\right\|^2_{L^2(Q_n)}+(1-(K+2NC)\,\epsilon)\left\|\Delta u_n\right\|^2_{L^2(Q_n)}.
\end{aligned}$$

Then, it is sufficient to choose ϵ such that$(1-(K+2NC)\,\epsilon)>0$ to get a constant $K_0>0$ independent of n such that

$$\left\|f_n\right\|_{L^2(Q_n)}\geq K_0\left\|u_n\right\|_{H^{1,2}(Q_n)},$$

and since

$$\left\|f_n\right\|_{L^2(Q_n)}\leq\left\|f\right\|_{L^2(Q_n)},$$

there exists a constant $K_1>0$, independent of n satisfying

$$\left\|u_n\right\|_{H^{1,2}(Q_n)}\leq K_1\left\|f_n\right\|_{L^2(Q_n)}\leq K_1\left\|f\right\|_{L^2(Q)}.$$

This completes the proof of Proposition 3.6.

Passage to the limit

We are now in position to prove the first main result of this work.

Theorem 3.10. *Assume that the functions h and φ are as in Theorem 1.1. Then, for T small enough, Problem (1.2) admits a unique solution $u\in H^{1,2}(Q)$.*

Proof. Choose a sequence Q_n $n=1,2,...$ of truncated conical type domains (see subsection 3.2) such that $Q_n\subseteq Q$. Then we have $Q_n\to Q$, as $n\to\infty$. Consider the solution $u_n\in H^{1,2}(Q_n)$ of the Cauchy-Dirichlet problem

$$\begin{cases}\partial_t u_n-\Delta u_n=f & \text{in } Q_n\\ u_n|_{\partial Q_n\setminus\Gamma_T}=0,\end{cases}$$

with Γ_T is the part of the boundary of Q_n where $t=T$. Such a solution u_n exists by Theorem 3.1. Let $\widetilde{u_n}$ the 0-extension of u_n to Q. In virtue of Proposition 3.6, we know that there exists a constant C such that

$$\left\|\widetilde{u_n}\right\|_{L^2(Q)}+\left\|\widetilde{\partial_t u_n}\right\|_{L^2(Q)}+\sum_{\substack{i_1,i_2,...,i_N=0\\1\leq i_1+i_2+...+i_N\leq 2}}^{2}\left\|\widetilde{\partial_{x_1}^{i_1}\partial_{x_2}^{i_2}...\partial_{x_N}^{i_N}u_n}\right\|_{L^2(Q)}\leq C\left\|f\right\|_{L^2(Q)}.$$

This means that $\widetilde{u_n}$, $\widetilde{\partial_t u_n}$, $\widetilde{\partial_{x_1}^{i_1}\partial_{x_2}^{i_2}...\partial_{x_N}^{i_N}u_n}$ for $1\leq i_1+i_2+...+i_N\leq 2$ are bounded functions in $L^2(Q)$. So for a suitable increasing sequence of integers n_k, $k=1,2,...$, there exist functions

$$u,\ v \text{ and } v_{i_1,i_2,...,i_N}\ \ 1\leq i_1+i_2+...+i_N\leq 2$$

in $L^2(Q)$ such that

$$\widetilde{u_{n_k}} \rightharpoonup u,\ \widetilde{\partial_t u_{n_k}} \rightharpoonup v \text{ and } \partial_{x_1}^{i_1}\widetilde{\partial_{x_2}^{i_2}...\partial_{x_N}^{i_N}} u_{n_k} \rightharpoonup v_{i_1,i_2,...,i_N}$$

$1 \le i_1 + i_2 + ... + i_N \le 2$, weakly in $L^2(Q)$, as $k \to \infty$. Clearly,

$$v = \partial_t u,\ v_{i_1,i_2,...,i_N} = \partial_{x_1}^{i_1}\partial_{x_2}^{i_2}...\partial_{x_N}^{i_N} u\ ,\ 1 \le i_1 + i_2 + ... + i_N \le 2$$

in the sense of distributions in Q and so in $L^2(Q)$. So, $u \in H^{1,2}(Q)$ and

$$\partial_t u - \Delta u = f \text{ in } Q.$$

On the other hand, the solution u satisfies the boundary conditions $u|_{\partial Q\setminus\Gamma_T} = 0$ since

$$\forall n \in \mathbb{N}^*,\ u|_{Q_n} = u_n.$$

This proves the existence of a solution to Problem (1.2). □

3.3 Step 3: Case of an arbitrary T

Let T be any positive real number and $T_1 < T$ small enough. Set $Q = D_1 \cup D_2 \cup \Gamma_{T_1}$ where

$$D_1 = \underset{t\in]0,T_1[}{\cup} \{t\} \times \Omega_t,\ D_2 = \underset{t\in]T_1,T[}{\cup} \{t\} \times \Omega_t \text{ and } \Gamma_{T_1} = \Omega_{T_1} \cup \partial\Omega_{T_1}.$$

In the sequel, f stands for an arbitrary fixed element of $L^2(Q)$. We have to solve Problem (1.2) in Q. We know (see Theorem 3.10) that the Cauchy-Dirichlet problem

$$\begin{cases} \partial_t v_1 - \Delta v_1 = f|_{D_1} \in L^2(D_1) \\ v_1|_{\partial D_1\setminus\Gamma_{T_1}} = 0 \end{cases} \tag{3.4}$$

has a unique solution $v_1 \in H^{1,2}(D_1)$. Hereafter, we denote the trace $v_{1/\Gamma_{T_1}}$ by ψ which is in the Sobolev space $H^1(\Gamma_{T_1})$ because $v_1 \in H^{1,2}(D_1)$ (see [16]). Now, consider the following problem in D_2

$$\begin{cases} \partial_t v_2 - \Delta v_2 = f|_{D_2} \in L^2(D_2) \\ v_2|_{\Gamma_{T_1}} = \psi \\ v_2|_{\partial D_2\setminus(\Gamma_{T_1}\cup\Gamma_T)} = 0. \end{cases} \tag{3.5}$$

We use the following result, which is a consequence of [16, Theorem 4.3, Vol. 2], to solve Problem (3.5).

Proposition 3.11. *Let Q be the cylinder $]0,T[\times B(0,1)$, where $B(0,1)$ is the unit ball of $\mathbb{R}^N$, $f \in L^2(Q)$ and $\psi \in H^1(\gamma_0)$. Then, the problem*

$$\begin{cases} \partial_t u - \Delta u = f \text{ in } Q, \\ u|_{\gamma_0} = \psi, \\ u|_{\gamma_0\cup\gamma_1} = 0, \end{cases}$$

where $\gamma_0 = \{0\} \times B(0,1)$, $\gamma_1 =]0,T[\times \partial B(0,1)$, admits a (unique) solution $u \in H^{1,2}(Q)$.

Remark 3.12. We have ψ lies in $H^1(\Sigma_{T_1})$, then $\partial_{x_1}\psi$ is (only) in $L^2(\Sigma_{T_1})$ and its pointwise values should not make sense. So, in the application of [16, Theorem 4.3, Vol.2], there are no compatibility conditions to satisfy.

Thanks to the transformation

$$(t, x_1, x_2, ..., x_N) \longmapsto (t, y_1, y_2, ..., y_N) = (t, h(t)\varphi(t)x_1, \varphi(t)x_2, ..., \varphi(t)x_N),$$

we deduce the following result

Proposition 3.13. *Problem (3.5) admits a (unique) solution* $v_2 \in H^{1,2}(D_2)$.

Now, define the function u in Q by

$$u = \begin{cases} v_1 \text{ in } D_1, \\ v_2 \text{ in } D_2, \end{cases}$$

where v_1 and v_2 are the solutions of Problem (3.4) and Problem (3.5) respectively. Then, u is the (unique) solution of Problem (1.2) for an arbitrary T. This completes the proof of Theorem 1.1.

Appendix: Proof of Lemma 3.9

For $\frac{1}{n} < t < T$, consider the following parametrization of the domain Ω_t

$$(0,\pi)\times(0,\pi)\times...\times(0,\pi)\times(0,2\pi) \longrightarrow \Omega_t;\ (\theta_1,\theta_2,...,\theta_{N-2},\theta_{N-1}) \longmapsto (x_1,x_2,...,x_{N-1},x_N),$$

where

$$(x_1,x_k,x_N) = \varphi(t)\,(h(t)\cos\theta_1, \sin\theta_1 ... \sin\theta_{k-1}\cos\theta_k, \sin\theta_1\sin\theta_2 ... \sin\theta_{N-2}\sin\theta_{N-1}),$$

$k = 2,3,4,...,N-1$. Let us denote the inner product in $L^2(\Omega_t)$ by $\langle .,. \rangle$, and set

$$I_n = \left\langle \Delta u_n, \frac{(h\varphi)'}{h\varphi} x_1 \partial_{x_1} u_n + \frac{\varphi'}{\varphi} \sum_{k=2}^{N} x_k \partial_{x_k} u_n \right\rangle,$$

then we have

$$\begin{aligned} I_n &= \int_{\Omega_t} \left(\sum_{k=1}^N \partial_{x_k}^2 u_n\right)\left(\tfrac{(h\varphi)'}{h\varphi} x_1\partial_{x_1}u_n + \tfrac{\varphi'}{\varphi}\sum_{k=2}^N x_k\partial_{x_k}u_n\right) dx_1 dx_2 ... dx_N \\ &= \int_{\Omega_t}\left(\tfrac{(h\varphi)'}{h\varphi} x_1\partial_{x_1}u_n \sum_{k=1}^N \partial_{x_k}^2 u_n + \tfrac{\varphi'}{\varphi}\sum_{k=2}^N x_k\partial_{x_k}u_n\partial_{x_k}^2 u_n\right) dx_1 dx_2 ... dx_N \\ &\quad + \int_{\Omega_t}\left(\tfrac{\varphi'}{\varphi}\sum_{k=2}^N x_k\partial_{x_k}u_n\partial_{x_1}^2 u_n + \tfrac{\varphi'}{\varphi}\sum_{j=2}^N x_j\partial_{x_j}u_n \sum_{k=2,k\neq j}^N \partial_{x_k}^2 u_n\right) dx_1 ... dx_N. \end{aligned}$$

Using Green formula, we obtain

$$\begin{aligned} I_n &= \tfrac{1}{2}\int_{\Omega_t}\left(\tfrac{(h\varphi)'}{h\varphi} x_1\partial_{x_1}(\partial_{x_1}u_n)^2 + \tfrac{\varphi'}{\varphi}\sum_{k=2}^N x_k\partial_{x_k}(\partial_{x_k}u_n)^2\right) dx_1 dx_2 ... dx_N \\ &\quad + \int_{\Omega_t}\left(\tfrac{\varphi'}{\varphi}\sum_{k=2}^N x_k\partial_{x_k}u_n\partial_{x_1}^2 u_n + \tfrac{\varphi'}{\varphi}\sum_{j=2}^N x_j\partial_{x_j}u_n\left(\sum_{k=2,k\neq j}^N \partial_{x_k}^2 u_n\right)\right) dx_1 ... dx_N \\ &\quad + \int_{\Omega_t}\tfrac{(h\varphi)'}{h\varphi}\sum_{k=2}^N x_1\partial_{x_1}u_n\partial_{x_k}^2 u_n dx_1 dx_2 ... dx_N \\ &= \tfrac{1}{2}\int_{\partial\Omega_t}\left(\tfrac{(h\varphi)'}{h\varphi} x_1\nu_{x_1}(\partial_{x_1}u_n)^2 + \tfrac{\varphi'}{\varphi}\sum_{k=2}^N x_k\nu_{x_k}(\partial_{x_k}u_n)^2\right) d\sigma \\ &\quad -\tfrac{1}{2}\int_{\Omega_t}\left(\tfrac{(h\varphi)'}{h\varphi}(\partial_{x_1}u_n)^2 + \tfrac{\varphi'}{\varphi}\sum_{k=2}^N(\partial_{x_k}u_n)^2\right) dx_1 dx_2 ... dx_N \\ &\quad + \int_{\partial\Omega_t}\left(\tfrac{\varphi'}{\varphi}\sum_{j=2}^{N-1}\sum_{k=j+1}^N (x_j\nu_{x_k} + x_j\nu_{x_k})\partial_{x_j}u_n\partial_{x_k}u_n\right) d\sigma \\ &\quad - \int_{\Omega_t}\left(\tfrac{\varphi'}{\varphi}\sum_{j=2}^{N-1}\sum_{k=j+1}^N x_j\partial_{x_k}u_n\partial_{x_j x_k}u_n\right) dx_1 dx_2 ... dx_N \\ &\quad + \int_{\partial\Omega_t}\left(\tfrac{(h\varphi)'}{h\varphi}\sum_{k=2}^N (x_1\nu_{x_k} + x_k\nu_{x_1})\partial_{x_1}u_n\partial_{x_k}u_n\right) d\sigma \\ &\quad - \int_{\Omega_t}\left(\tfrac{(h\varphi)'}{h\varphi}\sum_{k=2}^N (x_1\partial_{x_k}u_n\partial_{x_1x_k}u_n + x_k\partial_{x_1}u_n\partial_{x_1x_k}u_n)\right) dx_1 dx_2 ... dx_N \\ &\quad + \int_{\partial\Omega_t}\left(\tfrac{\varphi'}{\varphi}\sum_{k=2}^N (x_1\nu_{x_k} + x_k\nu_{x_1})\partial_{x_1}u_n\partial_{x_k}u_n\right) d\sigma \\ &\quad - \int_{\Omega_t}\left(\tfrac{\varphi'}{\varphi}\sum_{k=2}^N (x_1\partial_{x_k}u_n\partial_{x_1x_k}u_n + x_k\partial_{x_1}u_n\partial_{x_1x_k}u_n)\right) dx_1 dx_2 ... dx_N \end{aligned}$$

where $\nu_{x_1}, \nu_{x_2}, ..., \nu_{x_N}$ are the components of the unit outward normal vector at $\partial\Omega_t$. Then

$$\begin{array}{rcl} I_n & = & \frac{1}{2}\int_{\partial\Omega_t}\left(\frac{(h\varphi)'}{h\varphi}x_1\nu_{x_1}\left(\partial_{x_1}u_n\right)^2+\frac{\varphi'}{\varphi}\sum_{k=2}^{N}x_k\nu_{x_k}\left(\partial_{x_k}u_n\right)^2\right)d\sigma \\ & & -\frac{1}{2}\int_{\Omega_t}\left(\frac{(h\varphi)'}{h\varphi}\left(\partial_{x_1}u_n\right)^2+\frac{\varphi'}{\varphi}\sum_{k=2}^{N}\left(\partial_{x_k}u_n\right)^2\right)dx_1dx_2...dx_N \\ & & +\int_{\partial\Omega_t}\left(\frac{\varphi'}{\varphi}\sum_{j=2}^{N-1}\sum_{k=j+1}^{N}\left(x_j\nu_{x_k}+x_j\nu_{x_k}\right)\partial_{x_j}u_n\partial_{x_k}u_n\right)d\sigma \\ & & +\int_{\partial\Omega_t}\left(\frac{(h\varphi)'}{h\varphi}\sum_{k=2}^{N}\left(x_1\nu_{x_k}+x_k\nu_{x_1}\right)\partial_{x_1}u_n\partial_{x_k}u_n\right)d\sigma \\ & & +\int_{\partial\Omega_t}\left(\frac{\varphi'}{\varphi}\sum_{k=2}^{N}\left(x_1\nu_{x_k}+x_k\nu_{x_1}\right)\partial_{x_1}u_n\partial_{x_k}u_n\right)d\sigma \\ & & -\frac{1}{2}\int_{\Omega_t}\left(\frac{\varphi'}{\varphi}\sum_{j=2}^{N-1}\sum_{k=j+1}^{N}x_j\partial_{x_j}\left(\partial_{x_k}u_n\right)^2\right)dx_1dx_2...dx_N \\ & & -\frac{1}{2}\int_{\Omega_t}\left(\frac{(h\varphi)'}{h\varphi}\sum_{k=2}^{N}\left(x_1\partial_{x_1}\left(\partial_{x_k}u_n\right)^2+x_k\partial_{x_k}\left(\partial_{x_1}u_n\right)^2\right)\right)dx_1dx_2...dx_N \\ & & -\frac{1}{2}\int_{\Omega_t}\left(\frac{\varphi'}{\varphi}\sum_{k=2}^{N}\left(x_1\partial_{x_1}\left(\partial_{x_k}u_n\right)^2+x_k\partial_{x_k}\left(\partial_{x_1}u_n\right)^2\right)\right)dx_1dx_2...dx_N. \end{array}$$

Thus,

$$\begin{array}{rcl} I_n & = & \frac{1}{2}\int_{\partial\Omega_t}\left(\frac{(h\varphi)'}{h\varphi}x_1\nu_{x_1}\left(\partial_{x_1}u_n\right)^2+\frac{\varphi'}{\varphi}\sum_{k=2}^{N}x_k\nu_{x_k}\left(\partial_{x_k}u_n\right)^2\right)d\sigma \\ & & -\frac{1}{2}\int_{\Omega_t}\left(\frac{(h\varphi)'}{h\varphi}\left(\partial_{x_1}u_n\right)^2+\frac{\varphi'}{\varphi}\sum_{k=2}^{N}\left(\partial_{x_k}u_n\right)^2\right)dx_1dx_2...dx_N \\ & & +\int_{\partial\Omega_t}\left(\frac{\varphi'}{\varphi}\sum_{j=2}^{N-1}\sum_{k=j+1}^{N}\left(x_j\nu_{x_k}+x_j\nu_{x_k}\right)\partial_{x_j}u_n\partial_{x_k}u_n\right)d\sigma \\ & & +\int_{\partial\Omega_t}\left(\frac{(h\varphi)'}{h\varphi}\sum_{k=2}^{N}\left(x_1\nu_{x_k}+x_k\nu_{x_1}\right)\partial_{x_1}u_n\partial_{x_k}u_n\right)d\sigma \\ & & +\int_{\partial\Omega_t}\left(\frac{\varphi'}{\varphi}\sum_{k=2}^{N}\left(x_1\nu_{x_k}+x_k\nu_{x_1}\right)\partial_{x_1}u_n\partial_{x_k}u_n\right)d\sigma \\ & & -\frac{1}{2}\int_{\partial\Omega_t}\left(\frac{\varphi'}{\varphi}\sum_{j=2}^{N-1}x_j\nu_{x_j}\sum_{k=2,k\neq j}^{N}\left(\partial_{x_k}u_n\right)^2\right)d\sigma \\ & & -\frac{1}{2}\int_{\partial\Omega_t}\frac{(h\varphi)'}{h\varphi}\sum_{k=2}^{N}\left(x_1\nu_{x_1}\left(\partial_{x_k}u_n\right)^2+x_k\nu_{x_k}\left(\partial_{x_1}u_n\right)^2\right)d\sigma \\ & & -\frac{1}{2}\int_{\partial\Omega_t}\frac{\varphi'}{\varphi}\sum_{k=2}^{N}\left(x_1\nu_{x_1}\left(\partial_{x_k}u_n\right)^2+x_k\nu_{x_k}\left(\partial_{x_1}u_n\right)^2\right)d\sigma \\ & & +\frac{1}{2}\int_{\Omega_t}\frac{\varphi'}{\varphi}\sum_{j=2}^{N-1}\left(\sum_{k=2,k\neq j}^{N}\left(\partial_{x_k}u_n\right)^2\right)dx_1dx_2...dx_N \\ & & +\frac{1}{2}\int_{\Omega_t}\frac{(h\varphi)'}{h\varphi}\left(\sum_{k=2}^{N}\left(\left(\partial_{x_k}u_n\right)^2+\left(\partial_{x_1}u_n\right)^2\right)\right)dx_1dx_2...dx_N \\ & & +\frac{1}{2}\int_{\Omega_t}\frac{\varphi'}{\varphi}\left(\sum_{k=2}^{N}\left(\left(\partial_{x_k}u_n\right)^2+\left(\partial_{x_1}u_n\right)^2\right)\right)dx_1dx_2...dx_N \end{array}$$

and then

$$\begin{array}{rcl} I_n & = & \frac{1}{2}\int_{\partial\Omega_t}\left(\frac{(h\varphi)'}{h\varphi}x_1\nu_{x_1}\left(\partial_{x_1}u_n\right)^2+\frac{\varphi'}{\varphi}\sum_{k=2}^{N}x_k\nu_{x_k}\left(\partial_{x_k}u_n\right)^2\right)d\sigma \\ & & +\int_{\partial\Omega_t}\left(\frac{\varphi'}{\varphi}\sum_{j=2}^{N-1}\sum_{k=j+1}^{N}\left(x_j\nu_{x_k}+x_j\nu_{x_k}\right)\partial_{x_j}u_n\partial_{x_k}u_n\right)d\sigma \\ & & +\int_{\partial\Omega_t}\left(\frac{(h\varphi)'}{h\varphi}\sum_{k=2}^{N}\left(x_1\nu_{x_k}+x_k\nu_{x_1}\right)\partial_{x_1}u_n\partial_{x_k}u_n\right)d\sigma \\ & & +\int_{\partial\Omega_t}\left(\frac{\varphi'}{\varphi}\sum_{k=2}^{N}\left(x_1\nu_{x_k}+x_k\nu_{x_1}\right)\partial_{x_1}u_n\partial_{x_k}u_n\right)d\sigma \\ & & -\frac{1}{2}\int_{\partial\Omega_t}\left(\frac{\varphi'}{\varphi}\sum_{j=2}^{N-1}x_j\nu_{x_j}\sum_{k=2,k\neq j}^{N}\left(\partial_{x_k}u_n\right)^2\right)d\sigma \\ & & -\frac{1}{2}\int_{\partial\Omega_t}\frac{(h\varphi)'}{h\varphi}\sum_{k=2}^{N}\left(x_1\nu_{x_1}\left(\partial_{x_k}u_n\right)^2+x_k\nu_{x_k}\left(\partial_{x_1}u_n\right)^2\right)d\sigma \\ & & -\frac{1}{2}\int_{\partial\Omega_t}\frac{\varphi'}{\varphi}\sum_{k=2}^{N}\left(x_1\nu_{x_1}\left(\partial_{x_k}u_n\right)^2+x_k\nu_{x_k}\left(\partial_{x_1}u_n\right)^2\right)d\sigma \\ & & +\frac{1}{2}\int_{\Omega_t}\left((N-2)\frac{(h\varphi)'}{h\varphi}\left(\partial_{x_1}u_n\right)^2+\left(\frac{(h\varphi)'}{h\varphi}+(N-1)\frac{\varphi'}{\varphi}\right)\sum_{k=2}^{N}\left(\partial_{x_k}u_n\right)^2\right)dx_1...dx_N. \end{array}$$

Consequently,

$$
\begin{aligned}
&I_n \\
&= \tfrac{\varphi h\varphi}{2}\int_0^{2\pi}...\int_0^{\pi}\int_0^{\pi}\{\tfrac{(h\varphi)'}{h\varphi}\left(\cos\theta_1.\partial_{x_1}u_n\right)^2 \\
&+\tfrac{\varphi'}{\varphi}\left(\left(\sin\theta_1...\sin\theta_{N-1}.\partial_{x_N}u_n\right)^2+\textstyle\sum_{k=2}^{N-1}\left(\sin\theta_1...\sin\theta_{k-1}\cos\theta_k.\partial_{x_k}u_n\right)^2\right)\}d\theta_1 d\theta_2...d\theta_{N-1} \\
&+2\varphi h\varphi\int_0^{2\pi}...\int_0^{\pi}\int_0^{\pi}\tfrac{\varphi'}{\varphi}\textstyle\sum_{j=1}^{N-1}\{\sum_{k=j+1}^{N}\sin\theta_1...\sin\theta_{j-1}\cos\theta_j.\sin\theta_1...\sin\theta_{k-1}\cos\theta_k\partial_{x_k}u_n \\
&+\sin\theta_1...\sin\theta_{j-1}\cos\theta_j.\sin\theta_1...\sin\theta_{N-2}\sin\theta_{N-1}.\partial_{x_N}u_n\}\partial_{x_j}u_n d\theta_1 d\theta_2...d\theta_{N-1} \\
&+2\varphi h\varphi\int_0^{2\pi}...\int_0^{\pi}\int_0^{\pi}\left(\tfrac{(h\varphi)'}{h\varphi}+\tfrac{\varphi'}{\varphi}\right)\textstyle\sum_{k=2}^{N}\cos\theta_1.\sin\theta_1...\sin\theta_{k-1}\cos\theta_k\partial_{x_k}u_n\partial_{x_1}u_n d\theta_1...d\theta_{N-1} \\
&-\tfrac{\varphi h\varphi}{2}\int_0^{2\pi}...\int_0^{\pi}\int_0^{\pi}\tfrac{\varphi'}{\varphi}\textstyle\sum_{j=2}^{N-1}\left(\sin\theta_1...\sin\theta_{j-1}\cos\theta_j\right)^2\left(\sum_{k=2,k\neq j}^{N}\left(\partial_{x_k}u_n\right)^2\right)d\theta_1 d\theta_2...d\theta_{N-1} \\
&-\tfrac{\varphi h\varphi}{2}\int_0^{2\pi}...\int_0^{\pi}\int_0^{\pi}\left(\tfrac{(h\varphi)'}{h\varphi}+\tfrac{\varphi'}{\varphi}\right)\{\left(\cos\theta_1\partial_{x_N}u_n\right)^2+\left(\sin\theta_1...\sin\theta_{N-2}\sin\theta_{N-1}\partial_{x_1}u_n\right)^2 \\
&+\textstyle\sum_{k=2}^{N-1}\left(\left(\cos\theta_1\partial_{x_k}u_n\right)^2+\left(\sin\theta_1...\sin\theta_{k-1}\cos\theta_k\partial_{x_1}u_n\right)^2\right)\}d\theta_1 d\theta_2...d\theta_{N-1} \\
&+\tfrac{1}{2}\int_{\Omega_t}\left((N-2)\tfrac{(h\varphi)'}{h\varphi}\left(\partial_{x_1}u_n\right)^2+\left(\tfrac{(h\varphi)'}{h\varphi}+(N-2)\tfrac{\varphi'}{\varphi}\right)\textstyle\sum_{k=2}^{N}\left(\partial_{x_k}u_n\right)^2\right)dx_1 dx_2...dx_N.
\end{aligned}
$$

By using the boundary condition

$$
\begin{gathered}
u_n(t,h(t)\varphi(t)\cos\theta_1,\varphi(t)\sin\theta_1\cos\theta_2,...,\varphi(t)\sin\theta_1\sin\theta_2...\sin\theta_{N-2}\cos\theta_{N-1}, \\
\varphi(t)\sin\theta_1\sin\theta_2...\sin\theta_{N-2}\sin\theta_{N-1})=0
\end{gathered}
$$

and particulary the relations

$$
\sin\theta_{N-1}\cos\theta_{N-1}.\partial_{x_{N-1}}u_n\partial_{x_N}u_n=\left(\cos\theta_{N-1}.\partial_{x_N}u_n\right)^2=\left(\sin\theta_{N-1}.\partial_{x_{N-1}}u_n\right)^2,
$$

$$
h\left(\sin\theta_1\partial_{x_1}u_n\right)^2=\sin\theta_1\cos\theta_1\left[\sum_{k=2}^{N-1}\sin\theta_2...\sin\theta_{k-1}\cos\theta_k.\partial_{x_k}u_n+\sin\theta_2...\sin\theta_{N-1}.\partial_{x_N}u_n\right],
$$

$$
\begin{aligned}
\left(\sin\theta_j\partial_{x_j}u_n\right)^2 \;=\; &\left[\textstyle\sum_{k=j+1}^{N-1}\sin\theta_{j+1}...\sin\theta_{k-1}\cos\theta_k.\partial_{x_k}u_n+\sin\theta_{j+1}...\sin\theta_{N-1}.\partial_{x_N}u_n\right] \\
&\times\sin\theta_j\cos\theta_j
\end{aligned}
$$

for $j=2,3,...,N-2$, we obtain

$$
\begin{aligned}
I_n \;=\; &\int_0^{2\pi}...\int_0^{\pi}\int_0^{\pi}|\nabla u_n|^2\left(\left(\varphi'(t)\,h(t)\sin^2\theta_1+(h\varphi)'(t)\cos^2\theta_1\right)\right)\times\varphi(t)\,d\theta_1...d\theta_{N-1} \\
&+\tfrac{1}{2}\int_{\Omega_t}\left((N-2)\tfrac{(h\varphi)'}{h\varphi}\left(\partial_{x_1}u_n\right)^2+\left(\tfrac{(h\varphi)'}{h\varphi}+(N-1)\tfrac{\varphi'}{\varphi}\right)\textstyle\sum_{k=2}^{N}\left(\partial_{x_k}u_n\right)^2\right)dx_1...dx_N
\end{aligned}
$$

and

$$
\begin{aligned}
&\int_0^{2\pi}...\int_0^{\pi}\int_0^{\pi}\int_{\frac{1}{n}}^{T}|\nabla u_n|^2\left(\left(\varphi'(t)\,h(t)\sin^2\theta_1+(h\varphi)'(t)\cos^2\theta_1\right)\right)\times\varphi(t)\,dt d\theta_1 d\theta_2...d\theta_{N-1} \\
&+\int_{Q_n}\left((N-2)\tfrac{(h\varphi)'}{h\varphi}\left(\partial_{x_1}u_n\right)^2+\left(\tfrac{(h\varphi)'}{h\varphi}+(N-1)\tfrac{\varphi'}{\varphi}\right)\textstyle\sum_{k=2}^{N}\left(\partial_{x_k}u_n\right)^2\right)dt dx_1 dx_2...dx_N \\
&\qquad=2\int_{Q_n}\left(\tfrac{(h\varphi)'}{h\varphi}x_1\partial_{x_1}u_n+\tfrac{\varphi'}{\varphi}\textstyle\sum_{k=2}^{N}x_k\partial_{x_k}u_n\right)\Delta u_n dt dx_1 dx_2...dx_N.
\end{aligned}
$$

Finally, in virtue of relationship (3.2), it follows

$$
\begin{aligned}
&-2\langle\partial_t u_n,\Delta u_n\rangle \\
&=2\int_{Q_n}\left(\tfrac{(h\varphi)'}{h\varphi}x_1\partial_{x_1}u_n+\tfrac{\varphi'}{\varphi}\textstyle\sum_{k=2}^{N}x_k\partial_{x_k}u_n\right)\Delta u_n dt dx_1 dx_2...dx_N \\
&-\int_{Q_n}\left((N-2)\tfrac{(h\varphi)'}{h\varphi}\left(\partial_{x_1}u_n\right)^2+\left(\tfrac{(h\varphi)'}{h\varphi}+(N-1)\tfrac{\varphi'}{\varphi}\right)\textstyle\sum_{k=2}^{N}\left(\partial_{x_k}u_n\right)^2\right)dt dx_1 dx_2...dx_N \\
&+\int_{\Gamma_T}|\nabla u_n|^2(T,x_1,x_2,...,x_N)\,dx_1 dx_2...dx_N.
\end{aligned}
$$

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