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A Note on Nash Equilibrium in Bimatrix Games with Nonnegative Matrices

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Abstract. In this paper, we formulate a nonlinear programming problem corresponding to a bimatrix game with nonnegative matrices, and prove that a solution of this programming problem is an equilibrium for the corresponding game.

1 Introduction

The study of applied aspects of game models for different areas shows that antagonistic games are traditionally used and do not always adequately reflect the real situation. For this class of problems, bimatrix game can be used as a mathematical model. It is clear from the name that bimatrix games are described by two payoff matrices. To identify all the situations in the game with two players, choosing their pure strategies can be shown by two payoff matrices $A = [a_{ij}]$ and $B = [b_{ij}]$ of the same dimension $m \times n$ where m and n denote the number of strategies of players 1 and 2, respectively (for more details, see [7]). The principle of optimality for these games is the concept of Nash equilibrium [6]. Bimatrix games can be classified in the theory of non-cooperative games, but even they are not always solvable according to Nash or strongly solvable. To find equilibrium in bimatrix games, there are various algorithms which one of them is a method of describing the submatrices of A and B yielding all extreme points of the set of equilibrium solutions [2, 8], and some other methods reduce the problem of finding the equilibrium solutions of a bimatrix game to a problems of quadratic programming [3–5].

2 Model and Main Results

Consider the bimatrix game described by the payoff matrices $A = [a_{ij}]$ and $B = [b_{ij}]$ of dimension $m \times n$. The goal of each player is to achieve the maximum number of winnings (i.e., to maximize their own payoff). Consider the mixed extension of the bimatrix game. Let $\mathbf{x}$ and $\mathbf{y}$ denote the probability vectors such that

$$X = \left\{ \mathbf{x} = (x_1, x_2, \ldots, x_m) : x_i \geq 0 \text{ for } i = 1, 2, \ldots, m, \text{ and } \sum_{i=1}^{m} x_i = 1 \right\}, \tag{2.1}$$

$$Y = \left\{ \mathbf{y} = (y_1, y_2, \ldots, y_n) : y_j \geq 0 \text{ for } j = 1, 2, \ldots, n, \text{ and } \sum_{j=1}^{n} y_j = 1 \right\}. \tag{2.2}$$

The winnings of the players in the mixed extension are defined as follows

$$H_1(\mathbf{x}, \mathbf{y}) = \sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij} x_i y_j \quad \text{for the first player,} \tag{2.3}$$

$$H_2(\mathbf{x}, \mathbf{y}) = \sum_{i=1}^{m} \sum_{j=1}^{n} b_{ij} x_i y_j \quad \text{for the second player.} \tag{2.4}$$

Nash equilibrium in a bimatrix game is the vectors of the mixed extension of $\mathbf{x}^*$ and $\mathbf{y}^*$ for which the following inequalities hold

$$H_1(\mathbf{x}^*, \mathbf{y}^*) \geq H_1(\mathbf{x}, \mathbf{y}^*) \quad \text{for any } \mathbf{x} \in X, \tag{2.5}$$

$$H_2(\mathbf{x}^*, \mathbf{y}^*) \geq H_2(\mathbf{x}^*, \mathbf{y}) \quad \text{for any } \mathbf{y} \in Y. \tag{2.6}$$

We assume that the entries of the payoff matrices are nonnegative, otherwise, we consider the game strategically equivalent to the initial game for which the nonnegativity condition is satisfied [9]. We show that the solution of the game with nonnegative matrices is related to a nonlinear programming problem with proving the following two theorems.

Theorem 2.1. *The pair of vectors $(\mathbf{x}^*, \mathbf{y}^*)$ of the mixed extension of a bimatrix game is the Nash equilibrium if and only if there exist numbers p and q such that*

$$\sum_{j=1}^{n} a_{ij} y_j^* \leq p, \qquad i = 1, 2, \ldots, m \tag{2.7}$$

$$\sum_{i=1}^{m} b_{ij} x_i^* \leq q, \qquad j = 1, 2, \ldots, n \tag{2.8}$$

$$\sum_{i=1}^{m} \sum_{j=1}^{n} x_i^* y_j^* (a_{ij} + b_{ij}) = p + q. \tag{2.9}$$

Proof. If the pair $(\mathbf{x}^*, \mathbf{y}^*)$ is the Nash equilibrium, the expressions above are satisfied, and also p and q are the values of the price game for the first and second players, respectively, in an equilibrium situation

$$p = \sum_{i=1}^{m} \sum_{j=1}^{n} x_i^* y_j^* a_{ij} = H_1(\mathbf{x}^*, \mathbf{y}^*),$$

$$q = \sum_{i=1}^{m} \sum_{j=1}^{n} x_i^* y_j^* b_{ij} = H_2(\mathbf{x}^*, \mathbf{y}^*).$$

Conversely, let $(\mathbf{x}^*, \mathbf{y}^*), p, q$ satisfy the conditions (2.7)-(2.9). We choose a pair of vectors $(\mathbf{x}, \mathbf{y})$ of the mixed extension, and multiply the first m inequalities by the components of $\mathbf{x}$, and adding them yields

$$\sum_{i=1}^{m} x_i \sum_{j=1}^{n} a_{ij} y_j^* \leq \sum_{i=1}^{m} x_i p.$$

From (2.1) and (2.3), we have $H_1(\mathbf{x}, \mathbf{y}^*) \leq p$.

Similarly, multiply the n inequalities of (2.8) by $\mathbf{y}$, we have

$$\sum_{j=1}^{n} y_j \sum_{i=1}^{m} b_{ij} x_i^* \leq \sum_{j=1}^{n} y_j q.$$

From (2.2) and (2.4), we have $H_2(\mathbf{x}^*, \mathbf{y}) \leq q$.

If we choose $(\mathbf{x}^*, \mathbf{y}^*)$ instead of $(\mathbf{x}, \mathbf{y})$, then we have $H_1(\mathbf{x}^*, \mathbf{y}^*) \leq p$ and $H_2(\mathbf{x}^*, \mathbf{y}^*) \leq q$, and for the last inequality, (2.9), $H_1(\mathbf{x}^*, \mathbf{y}^*) = p$ and $H_2(\mathbf{x}^*, \mathbf{y}^*) = q$. The pair $(\mathbf{x}^*, \mathbf{y}^*)$ satisfies the definition of Nash equilibrium, and p and q are the same values of the price game for each player. □

Theorem 2.2. *The pair $(\mathbf{x}^*, \mathbf{y}^*)$ is an equilibrium of a bimatrix game if and only if for some nonnegative integers p and q, the collection $(\mathbf{x}^*, \mathbf{y}^*), p, q$ is a solution to the following nonlinear programming problem*

$$\begin{array}{lll} \max & \sum_{i=1}^{m} \sum_{j=1}^{n} x_i y_j (a_{ij} + b_{ij}) - p - q & \\ \text{s.t.} & \sum_{j=1}^{n} a_{ij} y_j \leq p, \quad y_j \geq 0, & i = 1, 2, \ldots, m \\ & \sum_{i=1}^{m} b_{ij} x_i \leq q, \quad x_i \geq 0, & j = 1, 2, \ldots, n \\ & \sum_{i=1}^{m} x_i = 1, \quad \sum_{j=1}^{n} y_j = 1. & \end{array} \tag{2.10}$$

Proof. For any pair of vectors $(\mathbf{x}, \mathbf{y})$ that satisfies the constraints above, the objective function will be nonpositive. Let $(\mathbf{x}^*, \mathbf{y}^*)$ be an equilibrium. We set

$$\begin{aligned} p &= \sum_{i=1}^{m} \sum_{j=1}^{n} x_i^* y_j^* a_{ij}, \\ q &= \sum_{i=1}^{m} \sum_{j=1}^{n} x_i^* y_j^* b_{ij}. \end{aligned}$$

Obviously p and q determine the equilibrium of the game. Consider a pair $(\mathbf{x}, \mathbf{y})$ satisfying the first two lines of the constraints in (2.10). We choose $\mathbf{x}$ as the m vectors of the basis of $\mathbb{R}^m$, the first line of the constraints becomes

$$\sum_{j=1}^{n} a_{ij} y_j^* \leq p, \quad i = 1, 2, \ldots, m.$$

Similarly, choose $\mathbf{y}$ as the n vectors of the basis of $\mathbb{R}^n$, the second line of the constraints becomes

$$\sum_{i=1}^{m} b_{ij} x_i^* \leq q, \quad j = 1, 2, \ldots, n.$$

Thus, the constraints of nonnegativity and normalization are satisfied for all vectors of the mixed extension.
Obviously the collection $(\mathbf{x}^*, \mathbf{y}^*), p, q$ is valid and the value of the objective function is zero, i.e., the function reaches its maximum and $(\mathbf{x}^*, \mathbf{y}^*), p, q$ is a solution.

Conversely, let $(\mathbf{x}^*, \mathbf{y}^*), p, q$ be a solution of the nonlinear programming (2.10). The objective value reaches its maximum value (zero) at these values,

$$\sum_{i=1}^{m} \sum_{j=1}^{n} x_i^* y_j^* (a_{ij} + b_{ij}) - p - q = 0.$$

Multiply the m inequalities of the first constraint in (2.10) by $\mathbf{x}^*$, we have

$$\sum_{i=1}^{m} x_i^* \sum_{j=1}^{n} y_i^* a_{ij} \leq p.$$

The pair $(\mathbf{x}^*, \mathbf{y}^*)$ is a solution, and the objective function of (2.10) is maximized, thus

$$\sum_{i=1}^{m} \sum_{j=1}^{n} x_i^* y_j^* a_{ij} = p.$$

Repeating the above process for $\mathbf{y}^*$ and the second constraint, we get

$$\sum_{i=1}^{m} \sum_{j=1}^{n} x_i^* y_j^* b_{ij} = q.$$

□

Thus, in order to determine an equilibrium in a bimatrix game, it is necessary and sufficient to solve the nonlinear programming problem in Theorem 2.2.
The fundamental difference between our result above for bimatrix games with zero-sum games is that the resulting problem in this case is nonlinear. The objective function in the resulting problem is quadratic with a system of linear constraints.

The idea of the feasible directions method is stated: it is necessary to determine the possible direction of the vector $\mathbf{d}$, such that the objective function in the direction of this vector is not getting worse, and it does not go outside the feasible region [1, Ch. 10]. Let the general form of the problem be as follows

$$\begin{aligned} \max \quad & f(\mathbf{z}) \\ \text{s.t.} \quad & \mathbf{z} = (z_1, z_2, \ldots, z_t) \\ & C\mathbf{z} \le \mathbf{s} \\ & E\mathbf{z} \le \mathbf{e}, \end{aligned}$$

where C is a matrix of dimension $r \times t$, E is an $l \times t$ matrix, and $\mathbf{s}$ and $\mathbf{e}$ are r-dimensional and l-dimensional vectors, respectively. Given a point $\mathbf{z}_k$, $C^T = (C_1^T, C_2^T)$ and $\mathbf{s}^T = (\mathbf{s}_1^T, \mathbf{s}_2^T)$, so $C_1\mathbf{z}_k = \mathbf{s}_1$ and $C_2\mathbf{z}_k < \mathbf{s}_2$; in other words, the matrix C_1 consists of the inequality constraints that are active at the point $\mathbf{z}_k$. Then, a vector $\mathbf{d}$ can be obtained from the solution of the following problem

$$\begin{aligned} [\text{P}] \quad \max \quad & \nabla f(\mathbf{z})^T \mathbf{d} \\ \text{s.t.} \quad & C_1\mathbf{d} \le 0 \\ & E\mathbf{d} = 0 \end{aligned}$$

The problem [P] is a linear programming problem, and its solution can be obtained by the simplex method. Once the feasible direction $\mathbf{d}$ is determined, we compute the following

$$\mathbf{x}^{(k+1)} = \mathbf{x}^{(k)} + \lambda_k \mathbf{d}^k,$$

where λ_k is the optimal value of the step. In general, the step is computed similar to the method of steepest descent [1]. The calculations are carried out as long as the objective function does not turn to zero at the current point.
We calculate the gradient of the objective function and the constraints matrix for the nonlinear programming to determine the situation of equilibrium in the bimatrix game.
The objective function depends on $m+n+2$ variables which are the components of $\mathbf{x}$, $\mathbf{y}$, p, and q.

We calculate the gradient

$$\nabla f = \begin{bmatrix} (a_{11}+b_{11})y_1 + (a_{12}+b_{12})y_2 + \ldots + (a_{1n}+b_{1n})y_n \\ (a_{21}+b_{21})y_1 + (a_{22}+b_{22})y_2 + \ldots + (a_{2n}+b_{2n})y_n \\ \vdots \\ (a_{m1}+b_{m1})y_1 + (a_{m2}+b_{m2})y_2 + \ldots + (a_{mn}+b_{mn})y_n \\ (a_{11}+b_{11})x_1 + (a_{21}+b_{21})x_2 + \ldots + (a_{m1}+b_{m1})x_m \\ (a_{12}+b_{12})x_1 + (a_{22}+b_{22})x_2 + \ldots + (a_{m2}+b_{m2})x_m \\ \vdots \\ (a_{1n}+b_{1n})x_1 + (a_{2n}+b_{2n})x_2 + \ldots + (a_{mn}+b_{mn})x_m \\ -1 \\ -1 \end{bmatrix} = \begin{bmatrix} (A+B)\mathbf{y} \\ (A+B)^T\mathbf{x} \\ -1 \\ -1 \end{bmatrix}.$$

The system of inequality constraints

$$\tilde{A} = \begin{bmatrix} A^T & 0 & 0 & 0 \\ 0 & B & 0 & 0 \\ 0\ldots0 & 0 & -1 \\ 0\ldots0 & -1 & 0 \end{bmatrix}$$

is an $(m+n+2) \times (m+n+2)$ matrix.
The equality constraints for vectors $\mathbf{x}$ and $\mathbf{y}$ are of the form

$$\sum_{i=1}^{m} x_i = 1, \quad \text{and} \quad \sum_{j=1}^{n} y_j = 1.$$

The matrix E for the inequality constraints consists of two rows

$$E = \begin{bmatrix} 1 & 1 & \dots & 1 & 0 & \dots & 0 & 0 & 0 \\ 0 & 0 & \dots & 0 & 1 & \dots & 1 & 0 & 0 \end{bmatrix}.$$

Recall that the game should be considered as a nonnegative payoff matrix, otherwise, we should find a number α such that

$$\begin{aligned} a_{ij} + \alpha &\geq 0, \quad \text{for } i = 1, 2, \dots, m \\ b_{ij} + \alpha &\geq 0, \quad \text{for } j = 1, 2, \dots, n \end{aligned}$$

New payment matrix entries $a'_{ij} = a_{ij} + \alpha$, $b'_{ij} = b_{ij} + \alpha$ will be strategically equivalent to the initial matrix, i.e., the equilibriums will be equal for both of them, and the equilibrium value of the game varies by the value of α.

3 Illustration

In this section, we consider an example, and explain our results in the previous section in details. We want to find the Nash equilibrium for a bimatrix game with the following payoff matrices

$$A = \begin{bmatrix} 5 & 2 & 1 & 7 \\ 3 & 8 & 6 & 1 \\ 7 & 1 & 2 & 5 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 & 6 & 7 & 0 \\ 8 & 1 & 3 & 3 \\ 3 & 7 & 2 & 5 \end{bmatrix}.$$

We formulate the nonlinear programming problem as follows

$$\begin{array}{lll} \max & \sum_{i=1}^{3} \sum_{j=1}^{4} x_i y_j \left(a_{ij} + b_{ij}\right) - p - q & \\ \text{s.t.} & \sum_{i=1}^{3} x_i b_{ij} \leq q, & j = 1, 2, 3, 4 \\ & \sum_{i=1}^{4} y_j a_{ij} \leq p, & i = 1, 2, 3 \\ & \mathbf{x} \geq 0, \ \mathbf{y} \geq 0 & \\ & \sum_{i=1}^{3} x_i = 1, \ \sum_{j=1}^{4} y_j = 1. & \end{array}$$

We define the initial approximation; consider the first column of A and find the maximum element $a_{13} = 7$, $\mathbf{y}^{(0)} = (1, 0, 0, 0)$, the initial point satisfies the constraints, $\mathbf{x}^{(0)}$ should be nonzero in third coordinate $\mathbf{x}^{(0)} = (0, 0, 1)$. We get the values $p^{(0)} = 7$ and $q^{(0)} = 3$. Now, we calculate the components of the gradient

$$\nabla f = \begin{bmatrix} 7 & 4 & 10 & 10 & 8 & 4 & 10 & -1 & -1 \end{bmatrix}^T.$$

Thus, the problem becomes

$$\begin{array}{ll} \max & 7d_1 + 4d_2 + 10d_3 + 10d_4 + 8d_5 + 4d_6 + 10d_7 - d_8 - d_9 \\ \text{s.t.} & 7d_3 - 4d_9 \leq 0 \\ & 3d_4 - 4d_8 \leq 0 \\ & d_1 + d_2 + d_3 = 0 \\ & d_4 + d_5 + d_6 + d_7 = 0. \end{array}$$

Now, implementing the dual simplex method, we find the solution

$$\mathbf{d} = \begin{bmatrix} 0 & 0.364 & -0.364 & -0.364 & 0.364 & 0 & 0 & 2.182 & 1.818 \end{bmatrix}^T,$$

and $\lambda_{\max} = 1$.

Hence, we have

$$
\begin{aligned}
\mathbf{x}^{(1)} &= \begin{bmatrix} 0 & 0.364 & 0.636 \end{bmatrix}^T, \\
\mathbf{y}^{(1)} &= \begin{bmatrix} 0.636 & 0.364 & 0 & 0 \end{bmatrix}^T, \\
p^{(1)} &= 4.818, \\
q^{(1)} &= 4.818.
\end{aligned}
$$

The objective function for these values is zero, so a solution is obtained which is an equilibrium for this game.

References

[1] M. S. Bazaraa, H. D. Sherali, C. M. Shetty, “Nonlinear Programming: Theory and Algorithms", Wiley-Interscience, 3rd ed., 2006.

[2] H. W. Kuhn, *An algorithm for equilibrium points in bimatrix games*, Proc. National Academy of Sciences of the U.S.A. **47** (1961), 1657–1662.

[3] C. E. Lemke, J. T. Howson, *Equilibrium points of bimatrix games*, Journal of the Society for Industrial and Applied Mathematics **12** (1964), 413–423.

[4] O. L. Mangasarian, *Equilibrium points in bimatrix games*, Journal of the Society for Industrial and Applied Mathematics **12** (1964), 778–780.

[5] H. Mills, *Equilibrium points in infinite games*, Journal of the Society for Industrial and Applied Mathematics **8** (1960), 397–402.

[6] J. Nash, *Non-cooperative games*, Ann. of Math. **54** (1951), 286–295.

[7] R. S. J. Savani, “Finding Nash Equilibria of Bimatrix Games", Ph.D. thesis, London School of Economics, 2006.

[8] N. N. Vorob’ev, *Equilibrium points in bimatrix games*, Theory of Probability and its Applications **3** (1958), 297–309.

[9] N. N. Vorob’ev, “Foundations of Game Theory: Noncooperative Games", Birkhäuser, 1st ed., 1994.

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