

# THE WEAKLY SIGN SYMMETRIC $Q$ -MATRIX COMPLETION PROBLEM

Kalyan Sinha

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**Abstract.** In this paper, some necessary and sufficient conditions for a digraph to have weakly sign symmetric  $Q$ -completion are provided. The digraphs of order at most three that have weakly sign symmetric  $Q$ -completion are singled out.

## 1 Introduction

A *partial matrix* is a rectangular array of numbers in which some entries are specified while others are free to be chosen. A *pattern* for  $n \times n$  matrices is a subset of  $\{1, \dots, n\} \times \{1, \dots, n\}$ . A partial matrix *specifies a pattern* if its specified entries lie exactly in those positions listed in the pattern. For  $\alpha \subseteq \{1, \dots, n\}$ , the principal submatrix  $B[\alpha]$  is obtained by deleting from  $B$  all rows and columns whose indices are not in  $\alpha$ . A principal minor is the determinant of a principal submatrix.

A real  $n \times n$  matrix  $B$  is a  *$P$ -matrix* ( $P_0$ -matrix) if every principal minor of  $B$  is positive (nonnegative). A real  $n \times n$  matrix  $B = [b_{ij}]$  is a  *$Q$ -matrix* if for every  $k \in \{1, 2, \dots, n\}$ ,  $S_k(B) > 0$ , where  $S_k(B)$  is the sum of all  $k \times k$  principal minors of  $B$ . The matrix  $B$  is *weakly sign symmetric* if  $b_{ij}b_{ji} \geq 0$  for each pair of  $i, j \in \{1, \dots, n\}$ .

For a given class  $\Pi$  of matrices (e.g.,  $P$ ,  $P_0$  or  $Q$ -matrices) a *partial  $\Pi$ -matrix* is a partial matrix for which the specified entries satisfy the properties of a  $\Pi$ -matrix. Thus, a *partial  $P$ -matrix* (*partial  $P_0$ -matrix*) is one in which all fully specified principal minors are positive (nonnegative). Similarly, a partial weakly sign symmetric matrix is a matrix in which fully specified principal submatrices are weakly sign symmetric matrices.

A *completion* of a partial matrix is a specific choice of values for the unspecified entries. A  $\Pi$ -*completion* of a partial  $\Pi$ -matrix  $M$  is a completion of  $M$  which is a  $\Pi$ -matrix. For a particular class  $\Pi$  of matrices, we say a pattern has  $\Pi$ -*completion* if every partial  $\Pi$ -matrix specifying the pattern can be completed to a  $\Pi$ -matrix and the  *$\Pi$ -matrix completion problem* studies the properties and classifications of patterns having  $\Pi$ -completions. Matrix completion problems for several classes of matrices including  $P$  and  $P_0$ -matrices have been studied by a number of authors (e.g., [5, 6, 7, 8, 9, 10, 11]). For a survey of matrix completion results one may see [3].

### 1.1 Digraphs

Graph theory has played an important role in the study of matrix completion problems. Most of the graph-theoretic terms can be found in any standard reference, for example, in [1] and [4]. For our purposes, a *directed graph* or *digraph*  $D = (V_D, A_D)$  of order  $n > 0$  is a finite nonempty set  $V_D$ , with  $|V_D| = n$  of objects called *vertices* together with a (possibly empty) set  $A_D$  of ordered pairs of vertices, called *arcs* or *directed edges*. Sometimes, we simply write  $v \in D$  (resp.  $(u, v) \in D$ ) to mean  $v \in V_D$  (resp.  $(u, v) \in A_D$ ). If  $x = (u, v)$  is an arc in  $D$ , we say that  $x$  is *incident* with  $u$  and  $v$ . If  $x = (u, u)$ , then  $x$  is called a *loop* at the vertex  $u$ .

A *symmetric edge* of  $D$  is a pair of arcs  $\{(u, v), (v, u)\} \subseteq A_D$ , usually written as  $\{u, v\}$ . A digraph  $H = (V_H, A_H)$  is a *subdigraph of order  $k$*  of the digraph  $D$  if  $|V_H| = k$  and  $V_H \subseteq V_D$ ,  $A_H \subseteq A_D$ . A subdigraph  $H$  of  $D$  is an *induced subdigraph* if  $A_H = (V_H \times V_H) \cap A_D$  (*induced by  $V_H$* ) and is a *spanning subdigraph* if  $V_H = V_D$ . By  $K_n$  we denote the digraph with

vertex set  $\langle n \rangle = \{1, 2, \dots, n\}$ , and arc set  $\langle n \rangle \times \langle n \rangle$ , i.e., one with all possible arcs including loops on the vertex set  $\langle n \rangle$ . The complement of a digraph  $D$  is the digraph  $\overline{D}$ , where  $V_{\overline{D}} = V_D$  and  $(v, w) \in A_{\overline{D}}$  if and only if  $(v, w) \notin A_D$ . A digraph is called asymmetric if it does not contain a symmetric edge.

A (directed)  $u$ - $v$  path  $P$  of length  $k \geq 0$  in  $D$  is an alternating sequence  $(u = v_0, x_1, v_1, \dots, x_k, v_k = v)$  of vertices and arcs, where  $v_i, 1 \leq i \leq k$ , are distinct vertices and  $x_i = (v_{i-1}, v_i)$ . Then, the vertices  $v_i$  and the arcs  $x_i$  are said to be on  $P$ . Further, if  $k \geq 2$  and  $u = v$ , then a  $u$ - $v$  path is a cycle of length  $k$ . We then write  $C_k = \langle v_1, v_2, \dots, v_k \rangle$  and call  $C_k$  a  $k$ -cycle in  $D$ . Naturally, paths and cycles in a digraph  $D$  are considered to be subdigraphs of  $D$ .

A cycle  $C$  is even (resp. odd) if its length is even (resp. odd). A digraph  $D$  is said to be connected (resp. strongly connected) if for every pair  $u, v$  of vertices,  $D$  contains a  $u$ - $v$  path (resp. both a  $u$ - $v$  path and a  $v$ - $u$  path). The maximal connected (resp. strongly connected) subdigraphs of  $D$  are called components (resp. strong components) of  $D$ .

### 1.2 Digraphs and matrices

Let  $\pi$  be a permutation of a nonempty finite set  $V$ . The digraph  $D_\pi = (V, A_\pi)$ , where  $A_\pi = \{(v, \pi(v)) : v \in V\}$ , is called a permutation digraph. Clearly, each component of a permutation digraph is a loop or a cycle. The digraph  $D_\pi$  is said to be positive (resp. negative) if  $\pi$  is an even permutation (resp. an odd permutation). It is clear that  $D_\pi$  is negative if and only if it has odd number of even cycles.

A permutation subdigraph  $H$  (of order  $k$ ) of a digraph  $D$  is a permutation digraph that is a subdigraph of  $D$  (of order  $k$ ). A digraph  $D$  is stratified if  $D$  has a permutation subdigraph of order  $k$  for every  $k = 2, 3, \dots, |D|$ .

Let  $B = [b_{ij}]$  be an  $n \times n$  matrix. We have

$$\det(B) = \sum (\text{sgn } \pi) b_{1\pi(1)} \cdots b_{n\pi(n)} \tag{1.1}$$

where the sum is taken over all permutations  $\pi$  of  $\langle n \rangle$ .

A signing of a digraph is an assignment of a sign  $+$  or  $-$  to each arc of the digraph. The result of signing of a digraph is called a signed digraph. For an arc  $e \in D$  by  $s(e)$  we mean  $e$  has sign  $s(e)$ .

For a  $k$ -cycle in  $C_k$  in  $D$ , the sign  $s(C_k)$  is defined to be,

$$s(C_k) = (-1)^{k+1} \prod_{e \in C_k} s(e)$$

For a permutation subdigraph  $k$  of  $D$ , the sign  $s(k)$  of  $k$  is

$$s(k) = \prod_{C \in k} s(C)$$

Now if is useful to associate a partial matrix with a digraph that describes the positions of the specified entries in the partial matrix. We say that an  $n \times n$  partial matrix  $B$  specifies a digraph  $D$  if  $D = (\langle n \rangle, A_D)$ , and for  $1 \leq i, j \leq n, (i, j) \in A_D$  if and only if the entry  $b_{ij}$  of  $B$  is specified. Let  $M = [m_{ij}]$  be a partial matrix specifying  $D$ . The sign of an arc  $(i, j)$  is defined as follows:

$$\text{sgn}(i, j) = \begin{cases} 1, & \text{if } m_{ij} > 0 \\ -1 & \text{if } m_{ij} < 0 \end{cases}$$

The resulting signed digraph  $D$  is the sign pattern of a partial matrix of  $M$ . In the case of a symmetrically placed pair,  $a_{ij}$  and  $x_{ji}$ , in a partial matrix, the specified entry  $a_{ij}$  shall be referred to as the specified twin. The other member of the pair,  $x_{ij}$ , shall be referred to as the unspecified twin. If any specified twin  $a_{ij}$  of a partial matrix  $M$  specifying  $D$  is zero, we can assign suitable sign ( $+$  or  $-$ ) to the arcs of  $D$  according to our choice. In that case, we can choose also suitable sign of unspecified twin  $x_{ij}$ .

The property of being a weakly sign symmetric  $Q$ -matrix is preserved under similarity and

transposition, but it is not inherited by principal submatrices, as it can easily be verified. Thus the weakly sign symmetric  $Q$ -matrix completion problem is quite different from completion problems involving  $P$ -matrix classes, where principal submatrices inherit the properties of the class under consideration.

## 2 Partial weakly sign symmetric $Q$ -matrices and their completion problem

A *partial  $Q$ -matrix*  $M$  is a partial matrix such that  $S_k(M) > 0$  for every  $k = 1, \dots, n$  for which all  $k \times k$  principal submatrices of  $M$  are fully specified. A *partial weakly sign symmetric  $Q$ -matrix*  $M = [a_{ij}]$  is a partial  $Q$ -matrix in which all fully specified principal submatrices are weakly sign symmetric.

Let  $M = [a_{ij}]$  be a partial weakly sign symmetric matrix. If all  $1 \times 1$  principal submatrices (i.e., all diagonal entries) in  $M$  are specified, then their sum  $S_1(M)$  (the trace of  $M$ ) must be positive. If all  $k \times k$  principal submatrices are fully specified for some  $k \geq 2$ , then  $M$  is fully specified and, therefore, is a weakly sign symmetric  $Q$ -matrix. Thus, a partial weakly sign symmetric  $Q$ -matrix is characterized as follows.

**Proposition 2.1.** *Suppose  $M = [a_{ij}]$  is a partial weakly sign symmetric matrix. Then  $M$  is a partial weakly sign symmetric  $Q$ -matrix if and only if exactly one of the following holds:*

- (i) *At least one diagonal entry of  $M$  is not specified.*
- (ii) *All diagonal entries are specified, at least one diagonal entry is positive so that  $\text{Tr}(M) > 0$  and  $M$  has an off diagonal unspecified entry.*
- (iii) *All entries of  $M$  are specified and  $M$  is a  $Q$ -matrix.*

A completion  $B$  of a partial weakly sign symmetric  $Q$ -matrix  $M$  is called a *weakly sign symmetric  $Q$ -completion* of  $M$ , if  $B$  is a weakly sign symmetric  $Q$ -matrix. Since any matrix which is permutation similar to a  $Q$ -matrix is a  $Q$ -matrix, it is evident that if a partial weakly sign symmetric  $Q$ -matrix has a weakly sign symmetric  $Q$ -completion, so does any partial matrix which is permutation similar to  $M$ .

It is easy to see that any partial weakly sign symmetric matrix  $M$  with all unspecified diagonal entries has weakly sign symmetric  $Q$ -completion. A completion can be obtained by choosing sufficiently large values for the unspecified diagonal entries. Let  $M$  be a partial weakly sign symmetric  $Q$ -matrix in which the diagonal entries at  $(i, i)$  positions ( $i = k + 1, \dots, n$ ) are unspecified. In case  $M[1, \dots, k]$  is fully specified,  $M$  may not have a weakly sign symmetric  $Q$ -completion. For example, the partial matrix,

$$M = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & * \end{bmatrix},$$

where  $*$  denotes an unspecified entry, does not have weakly sign symmetric  $Q$ -completion. Indeed, for any completion  $B$  of  $M$ ,  $S_3(B) = \det B = 0$ . On the other hand, if  $M[1, \dots, k]$  has an unspecified entry and has a weakly sign symmetric  $Q$ -completion, then  $M$  has a weakly sign symmetric  $Q$ -completion. A completion of  $M$  can be obtained by choosing sufficiently large values for the unspecified diagonal entries. We list these observations in the following results.

**Theorem 2.2.** *If a matrix  $M$  omits all diagonal entries, then  $M$  has weakly sign symmetric  $Q$ -completion.*

*Proof.* Let  $M = [a_{ij}]$  be a partial weakly sign symmetric  $Q$ -matrix. For any  $t > 1$  consider a completion  $B = [b_{ij}]$  of  $M$  by setting all diagonal entries equal to  $t$  and rest of the off diagonal entries to be equal to zero. Then, any  $k \times k$  principal minor will be of the form  $t^k + p(t)$ , where  $p(t)$  is a polynomial of degree  $\leq k - 1$ . Now by choosing  $t$  large enough we have,  $S_k(B) > 0$  for all  $k \times k$  principal minors of  $B$ . Since only finitely many principal minors are to be considered, thus for sufficiently large  $t$ ,  $M$  has weakly sign symmetric  $Q$ -completion. ■

**Theorem 2.3.** *Let  $M$  be a partial weakly sign symmetric  $Q$ -matrix in which the diagonal entry at  $(r + 1, r + 1)$  position is unspecified. If the principal submatrix  $M[1, \dots, r]$  of  $M$  is not fully specified and has weakly sign symmetric  $Q$ -completion, then  $M$  has weakly sign symmetric  $Q$ -completion.*

*Proof.* Let  $M = [a_{ij}]$  be a partial weakly sign symmetric  $Q$ -matrix which omits the diagonal entry at  $(r + 1, r + 1)$  position. Then,  $M$  is of the form,

$$M = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix},$$

where,  $M_{11} = M[1, \dots, r]$  and  $M_{22} = M[r + 1, r + 1]$ .

Let  $A_1$  be the weakly sign symmetric  $Q$ -matrix completion of  $M[1, \dots, r]$ . Then,

$$M' = \begin{bmatrix} A_1 & M_{12} \\ M_{21} & M_{22} \end{bmatrix},$$

is a partial weakly sign symmetric  $Q$ -matrix, since  $M_{22}$  has an unspecified diagonal entry. Now for  $t > 0$ , consider a completion  $B = [b_{ij}]$  of  $M'$  obtained by choosing  $b_{ii} = t$ ,  $i = r + 1$  and  $b_{ij} = 0$  against all other unspecified entries in  $M'$ . Then  $B$  is of the form,

$$B = \begin{bmatrix} A_1 & B_{12} \\ B_{21} & t \end{bmatrix}.$$

Since  $A_1$  is a weakly sign symmetric  $Q$ -matrix,  $S_i(A_1) > 0$  for  $1 \leq i \leq r$ . For  $2 \leq j \leq r + 1$ ,

$$S_j(B) = S_j(A_1) + tS_{j-1}(A_1) + s_j,$$

where  $s_j$  is a constant. Now  $S_j(B) > 0$  for sufficiently large values of  $t$  and clearly  $B$  is weakly sign symmetric  $Q$ -matrix. ■

**Corollary 2.4.** *Let  $M$  be a partial weakly sign symmetric  $Q$ -matrix in which the diagonal entries at  $(i, i)$  positions ( $i = r + 1, \dots, n$ ) are unspecified. If the principal submatrix  $M[1, \dots, r]$  of  $M$  is not fully specified and has weakly sign symmetric  $Q$ -completion, then  $M$  has weakly sign symmetric  $Q$ -completion.*

The converse of Corollary 2.4 is not true which can be seen from the following example.

**Example 2.5.** Consider the partial matrix,

$$M = \begin{bmatrix} * & * & a_{13} & * \\ a_{21} & d_2 & * & * \\ * & a_{32} & * & * \\ a_{41} & * & a_{43} & d_4 \end{bmatrix},$$

where  $*$  denotes the unspecified entries. We show that for any choice of values of the specified entries  $M$  has weakly sign symmetric  $Q$ -completions, though there are occasions when  $M[1, 2, 3]$  does not have (see Example 3.3). For  $t > 0$ , consider the completion  $B(t)$  of  $M$  defined as follows:

$$B(t) = \begin{bmatrix} t & 0 & a_{13} & 0 \\ a_{21} & d_2 & b_{23} & 0 \\ 0 & a_{32} & t & b_{34} \\ b_{41} & b_{42} & a_{43} & d_4 \end{bmatrix},$$

where, we put  $b_{23} = s(a_{32})t$ ,  $b_{34} = s(a_{43})t$ ,  $b_{42} = s(a_{32}a_{43})t$ . Then,

$$\begin{aligned} S_1(B(t)) &= 2t + \sum d_i, \\ S_2(B(t)) &= t^2 + f_1(t), \\ S_3(B(t)) &= t^3 + f_2(t), \\ S_4(B(t)) &= t^4 + f_3(t), \end{aligned}$$

where  $f_i(t)$  is a polynomial in  $t$  of degree at most  $i$ ,  $i = 1, 2, 3$ . Consequently,  $B(t)$  is a weakly sign symmetric  $Q$ -matrix for sufficiently large  $t$ , and therefore,  $M$  has weakly sign symmetric  $Q$ -completion. On the other hand, the partial weakly sign symmetric  $Q$ -matrix

$$M[2, 4] = \begin{bmatrix} 0 & x_{24} \\ x_{42} & 1 \end{bmatrix},$$

with unspecified entries  $x_{24}, x_{42}$ , is the principal submatrix of  $M$  induced by its diagonal  $\Delta = \{2, 4\}$ . That  $M[2, 4]$  does not have weakly sign symmetric  $Q$ -completion is evident, because  $S_2(M[2, 4]) \leq 0$  for any completion of  $M[2, 4]$ .

### 3 Digraphs and weakly sign symmetric $Q$ -completions

We have seen that an  $n \times n$  partial matrix  $M$  specifies a digraph  $D = (\langle n \rangle, A_D)$  if for  $1 \leq i, j \leq n$ ,  $(i, j) \in A_D$  if and only if the  $(i, j)$ -th entry of  $M$  is specified. For example, the partial weakly sign symmetric  $Q$ -matrix  $M$  in Example 2.5 specifies the digraph  $D$  in Figure 1.

**Theorem 3.1.** *Suppose  $M$  is a partial weakly sign symmetric matrix specifying the digraph  $D$ . If the partial submatrix of  $M$  induced by every strongly connected induced subdigraph of  $D$  has weakly sign symmetric  $Q$ -completion, then  $M$  has weakly sign symmetric  $Q$ -completion.*

*Proof.* We prove the result for the case when  $D$  has two strong components  $D_1$  and  $D_2$ . The general result will then follow by induction. By a relabeling of the vertices of  $D$ , if required, we have

$$M = \begin{bmatrix} M_{11} & M_{12} \\ X & M_{22} \end{bmatrix},$$

where  $M_{ii}$  is a partial weakly sign symmetric  $Q$ -matrix specifying  $D_i$ ,  $i = 1, 2$ , and all entries in  $X$  are unspecified. By the hypothesis,  $M_{ii}$  has a weakly sign symmetric  $Q$ -completion  $B_{ii}$ . Consider the completion

$$B = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix},$$

by choosing all entries in  $X$  as well as all unspecified entries in  $M_{12}$  as 0. Then, for  $2 \leq k \leq |D|$  we have

$$S_k(B) = S_k(B_{11}) + S_k(B_{22}) + \sum_{r=1}^{k-1} S_r(B_{11})S_{k-r}(B_{22}) > 0,$$

Here, we mean  $S_k(B_{ii}) = 0$  whenever  $k$  exceeds the size of  $B_{ii}$ . Thus  $M$  can be completed to a weakly sign symmetric  $Q$ -matrix. ■

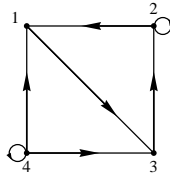
The proof of the following result is similar.

**Theorem 3.2.** *Suppose  $M$  is a partial weakly sign symmetric matrix specifying the digraph  $D$ . If the partial submatrix of  $M$  induced by each component of  $D$  has a weakly sign symmetric  $Q$ -completion, then  $M$  has a weakly sign symmetric  $Q$ -completion.*

The converse of Theorem 3.1 is not true. For example, every partial weakly sign symmetric  $Q$ -matrix specifying the digraph  $D$  in Figure 1 has weakly sign symmetric  $Q$ -completion,

although the strong component  $D_1$  induced by vertices  $\{1, 2, 3\}$  does not have weakly sign symmetric  $Q$ -completion (see Example 3.3).

**Example 3.3.** Consider the digraph  $D$  in the Figure 1. We show that  $D$  has weakly sign symmetric  $Q$ -completion, but the strong component  $D_1$  induced by vertices  $\{1, 2, 3\}$  does not have weakly sign symmetric  $Q$ -completion. Let  $M = [a_{ij}]$  be a partial weakly sign symmetric  $Q$ -



**Figure 1.** The Digraph  $D$

matrix specifying  $D$ . Then for  $t > 0$ ,  $M$  can be completed to a weakly sign symmetric  $Q$ -matrix  $B(t)$  (see Example 2.5) but the principal submatrix induced by the digraph  $D_1$  i.e.  $M[1, 2, 3]$  does not have weakly sign symmetric  $Q$ -completion. To see that  $M[1, 2, 3]$  does not have weakly sign symmetric  $Q$ -completion, consider the partial weakly sign symmetric  $Q$ -matrix

$$M[1, 2, 3] = \begin{bmatrix} x & u & -1 \\ -1 & 0 & w \\ z & -1 & y \end{bmatrix},$$

with unspecified entries  $x, u, w, y$  and  $z$ . Then for any weakly sign symmetric  $Q$ -completion  $B$  of  $M[1, 2, 3]$ , we have  $S_3(B) \leq 0$  and hence  $M[1, 2, 3]$  does not have weakly sign symmetric  $Q$ -completion.

The property of having weakly sign symmetric  $Q$ -completion is not inherited by induced subdigraphs. This can be also seen from the Example 2.5.

### 4 The weakly sign symmetric $Q$ -completion problem

We say that a digraph  $D$  has *weakly sign symmetric  $Q$ -completion*, if every partial weakly sign symmetric  $Q$ -matrix specifying  $D$  can be completed to a weakly sign symmetric  $Q$ -matrix. The *weakly sign symmetric  $Q$ -matrix completion problem* aims at studying and classifying all digraphs  $D$  which have weakly sign symmetric  $Q$ -completion.

It is clear that if a digraph  $D$  has weakly sign symmetric  $Q$ -completion, then any digraph which is isomorphic to  $D$  has weakly sign symmetric  $Q$ -completion.

#### 4.1 Necessary conditions for weakly sign symmetric $Q$ -matrix completion

In this section we provide some necessary conditions for a digraph to have weakly sign symmetric  $Q$ -completion.

**Theorem 4.1.** *Let  $D$  be a digraph with at least two vertices. If  $D$  has weakly sign symmetric  $Q$ -completion, then  $D$  omits at least two loops.*

*Proof.* Let  $D$  be a digraph with  $n$  vertices having loops at  $2, 3, \dots, n$ . Suppose  $D$  omits only one loops. Let  $M = [a_{ij}]$  be a partial weakly sign symmetric  $Q$ -matrix specifying the digraph  $D$  which is defined as follows:

$$a_{ij} = \begin{cases} 1, & \text{if } (i, j) = (1, 1), (i, j) \in D \\ 0, & \text{for all } (i, j) \neq (1, 1), (i, j) \in D. \end{cases}$$

Then for any weakly sign symmetric completion  $B$  of  $M$ ,  $S_2(B) \leq 0$ , where  $S_2(B)$  is the sum of all principal minors of order  $2 \times 2$  in  $B$ . Hence  $D$  omits at least two loops. ■

The next theorem shows that for a digraph  $D$  that omits at least two loops, stratification of  $\overline{D}$  is necessary condition for  $D$  to have weakly sign symmetric  $Q$ -completion.

**Theorem 4.2.** *Suppose  $D$  be a digraph of order  $n \geq 2$  such that  $D$  has weakly sign symmetric  $Q$ -completion, then  $\overline{D}$  is stratified.*

*Proof.* Suppose  $D$  has weakly sign symmetric  $Q$ -completion. Let  $k \geq 2$  and assume that  $\overline{D}$  has no order  $k$  permutation digraph. If  $M$  is a partial weakly sign symmetric matrix that specifies  $D$  with all specified entries zero and  $B$  is a completion of  $M$ , then all  $k \times k$  principal minors of  $B$  are zero, so  $B$  is not a  $Q$ -matrix. This implies that  $\overline{D}$  must be stratified. ■

But the condition of the Theorem 4.1 and Theorem 4.2 are not sufficient for weakly sign symmetric  $Q$ -completion which can be seen from the following Example 4.3.

**Example 4.3.** Let  $D_2$  be the digraph shown in the Figure 2 in which  $\overline{D_2}$  is stratified. Consider a

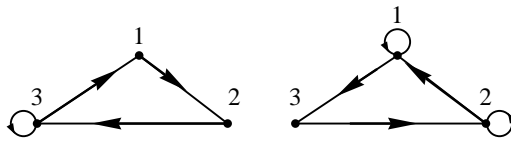


Figure 2.  $D_2, \overline{D_2}$

partial weakly sign symmetric  $Q$ -matrix,

$$M = \begin{bmatrix} 0 & 1 & x \\ y & z & 1 \\ -1 & w & u \end{bmatrix},$$

where  $x, y, z, w, u$  are unspecified entries. It is clear that  $M$  specifies  $D_2$ . Now, from sign symmetric conditions of  $M$ ,  $x \leq 0, y, w, z, u \geq 0$  and at least one of  $z$  and  $u$  is  $> 0$ . But  $M$  cannot be completed to a weakly sign symmetric  $Q$ -matrix because for any completion  $B$  of  $M$ ,  $S_3(B) \leq 0$ .

**Corollary 4.4.** *Suppose  $D$  be a digraph of order  $n$  that omits two loops and such that  $|A_D| > n(n - 1) + 2$ . Then  $D$  does not have weakly sign symmetric  $Q$ -completion.*

*Proof.* If  $D$  has more than  $n(n - 1) + 2$  arcs (including loops), then  $\overline{D}$  has fewer than  $n^2 - n(n - 1) - 2 = n - 2$  arcs. Thus  $\overline{D}$  does not contain an order  $n$  permutation digraph. Therefore by Theorem 4.2,  $D$  is not stratified and hence  $D$  does not have weakly sign symmetric  $Q$ -completion. ■

**Theorem 4.5.** *Suppose  $D$  be a digraph of order  $n > 2$  which omits only 2 loops. If  $\overline{D}$  is stratified and  $\overline{D}$  has no symmetric edge, then  $D$  does not have weakly sign symmetric  $Q$ -completion.*

*Proof.* Suppose  $M = [a_{ij}]$  be a partial weakly sign symmetric  $Q$ -matrix specifying the digraph  $D$  which is defined as follows:

$$a_{ij} = \begin{cases} 0, & \text{if } i = j, (i, j) \in D \\ -1, & \text{otherwise} \end{cases}$$

Then for completion  $B$  of  $M$ ,  $S_3(B) \leq 0$ , where  $S_3(B)$  is the sum of all principal minors of order  $3 \times 3$  in  $B$ . Hence the result follows. ■

**Example 4.6.** Consider the digraph  $D_2$  in 4.3. It is easy to see that  $D_2$  satisfies the hypothesis of the Theorem 4.5. Thus the digraph  $D_2$  does not have weakly sign symmetric  $Q$ -completion.

**4.2 Sufficient conditions for weakly sign symmetric Q-matrix completion**

**Theorem 4.7.** *If a digraph  $D \neq K_n$  of order  $n$  has weakly sign symmetric Q-completion, then any spanning subdigraph of  $D$  has weakly sign symmetric Q-completion.*

*Proof.* Suppose  $H$  be a spanning subdigraph of  $D$  and  $M_H$  be a partial weakly sign symmetric Q-matrix specifying the digraph  $H$ . Consider a partial matrix  $M_D$  obtained from  $M_H$  by specifying the entries corresponding to  $(i, j) \in A_D \setminus A_H$  as 0. Since  $D \neq K_n$ , by Proposition 2.1,  $M_D$  is a partial weakly sign symmetric Q-matrix specifying  $D$ . Let  $B$  be a weakly sign symmetric Q-completion of  $M_D$ . Clearly,  $B$  is a weakly sign symmetric Q-completion of  $M_H$ . ■

**Definition 4.8.** A digraph  $D_1$  is said to be weakly sign symmetric compatible digraph with a digraph  $D$  if the sign of the arcs of every 2-cycle  $\langle u, v \rangle$  in  $D \cup D_1$  satisfies the relation  $s(u, v)s(v, u) \geq 0$ .

**Theorem 4.9.** *Suppose  $D$  be a digraph such that  $D$  omits at least two loops and  $\overline{D}$  is stratified. If for any signing of the arcs of  $D$ ,  $\overline{D}$  is weakly sign symmetric compatible with  $D$  and every cycle of length  $\geq 3$  of  $\overline{D}$  is of positive sign, then  $D$  has weakly sign symmetric Q-completion.*

*Proof.* Let  $M = [a_{ij}]$  be a partial weakly sign symmetric Q-matrix specifying the digraph  $D$ . Now for any signing of  $D$ ,  $\overline{D}$  is weakly sign symmetric compatible with  $D$ . Then for  $t > 0$ , consider a completion  $B(t) = [b_{ij}]$  of  $M$  as follows:

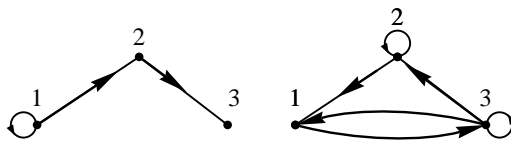
$$b_{ij} = \begin{cases} t^2, & \text{if } i = j \text{ and } (i, j) \in \overline{D} \\ \text{sgn}(i, j)t, & \text{if } (i, j) \in \overline{D} \\ a_{ij}, & \text{if } (i, j) \in D \end{cases}$$

Then for  $k = 2, \dots, n$ , we have,

$$S_2(B(t)) = \alpha t^4 + p(t) \\ S_k(B(t)) = \gamma t^k + q(t); k > 2,$$

where  $p(t), q(t)$  is a polynomial of degree at most 3 and  $k - 1$  and  $\alpha, \gamma > 0$ . Hence the result follows. ■

**Example 4.10.** Consider a digraph  $D_2$  and its complement  $\overline{D}$  in the following Figure 3. Now,



**Figure 3.**  $D, \overline{D}$

for any signing of  $D$ , it is possible to sign the arcs of  $\overline{D}$  so that  $\overline{D}$  is weakly sign symmetric compatible with  $D$ . Now, with the sign of the arcs of  $D$ , it can be possible to sign the arcs of  $\overline{D}$  so that every cycle of length  $\geq 3$  is of positive sign. Hence, by Theorem 4.9,  $D$  has weakly sign symmetric Q-completion.

**5 Classification of small digraphs as to weakly sign symmetric Q-completion**

We have examined the digraphs of order at most three to weakly sign symmetric Q-completion. Clearly, any digraph of order 1 (with or without a loop) has weakly sign symmetric Q-completion. Any digraph of order 2 and without a loop has weakly sign symmetric Q-completion.

There are only three non-isomorphic digraphs of order 3 without loops for which the digraphs obtained by attaching a loop at any of the vertices have weakly sign symmetric Q-completion (by Theorem 4.9). These digraphs are precisely are the following:





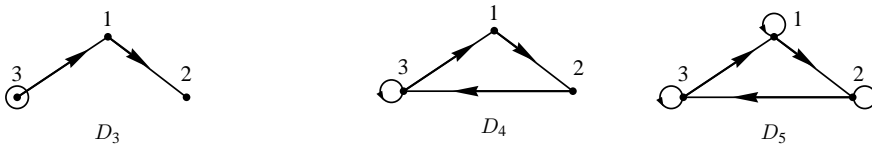
**Figure 4.** The digraphs of order 3 having weakly sign symmetric  $Q$ -completion

### 6 Comparison between $Q$ -completion and weakly sign symmetric $Q$ -completion

We know that every weakly sign symmetric  $Q$ -matrix is a  $Q$ -matrix, and every partial weakly sign symmetric  $Q$ -matrix is a partial  $Q$ -matrix, but the following examples show that the completion problems for these two classes are quite different.

**Example 6.1.** Consider the digraphs  $D_i, i = 3, 4, 5$  in Figure 5.

- (i) The digraph  $D_3$  has both  $Q$ -completion and weakly sign symmetric  $Q$ -completion.
- (ii) The digraph  $D_4$  has  $Q$ -completion, but does not have weakly sign symmetric  $Q$ -completion.
- (iii) The digraph  $D_5$  has neither  $Q$ -completion nor weakly sign symmetric  $Q$ -completion.



**Figure 5.**  $Q$ -completion vs. weakly sign symmetric  $Q$ -completion

Since the digraph  $D_3$  and  $D_4$  satisfies the Theorem 2.3 of [2], thus  $D_3$  or  $D_4$  has  $Q$ -completion. On the other hand,  $\overline{D_5}$  is not stratified, hence by Theorem 2.8 of [2],  $D_5$  does not have  $Q$ -completion. Again  $D_3$  satisfies the Theorem 4.9, therefore it has weakly sign symmetric  $Q$ -completion. But by example 3.3,  $D_4$  does not have weakly sign symmetric  $Q$ -completion. Also  $\overline{D_5}$  is not stratified, hence by Theorem 4.1,  $D_5$  does not have weakly sign symmetric  $Q$ -completion.

Suppose  $D$  is a digraph having weakly sign symmetric  $Q$ -completion. Then,  $\overline{D}$  is stratified and omits at least two loops. For all small digraphs (including all digraph of order 4) having these properties are seen to have  $Q$ -completion. Whether a stratified digraph omitting a loop necessarily have  $Q$ -completion is not known (see Question 2.9 in [2]). We do not know whether there is a digraph having weakly sign symmetric  $Q$  completion, but not  $Q$ -completion.

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### Author information

Kalyan Sinha, Department of Mathematics, PSTDS Vidyapith, Chinsurah, Hooghly, West Bengal-712305, INDIA.  
E-mail: kalyansinha90@gmail.com

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