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DIFFERENTIAL EQUATIONS CHARACTERIZING SPACELIKE CURVES IN THE 3-DIMENSIONAL LIGHTLIKE CONE

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Abstract. In this paper, we give some new characterizations for spacelike curves in the 3-dimensional null cone of the Minkowski 4-space E_1^4 .

Furthermore, as an example, the obtained results related to the case general and circular helix are discussed.

1 Introduction

The general theory of curves in a Riemannian manifold M have been developed a long time ago and now we have a deep knowledge of its local geometry as well as its global geometry. At the beginning of the twentieth century Einstein's theory opened a door to new geometries such as Minkowski space-time, which is simultaneously the geometry of special relativity and the geometry induced on each fixed tangent space of an arbitrary Lorentzian manifold [18] .When M is a proper semi-Riemannian manifold (or Lorentzian manifold) there exist three families of curves (spacelike, timelike, and null or lightlike curves) depending on their causal characters. It is well-known [16], that the study of timelike curves has many analogues and similarities with that of spacelike curves. In the mathematical study of relativity theory, a material particle in a spacetime is understood as a future-pointing timelike curve of unit speed in a spacetime,i.e., a connected and time-oriented 4-dimensional Lorentz manifold. The unit-speed parameter is called the proper time of a material particle. Motivated by this fundamental observation, timelike curves in Lorentzian manifolds have been studied extensively by both physicists and differential geometers [2]. Author extended spherical images of curves to a four-dimensional Lorentzian space and studied such curves in the case where the base curve is a space-like or timelike curve according to the signature (+,+,+,-). In fact, spacelike curves or timelike curves can be studied by a similar approach to that in positive definite Riemannian geometry. In the study of relativity theory, a unit speed future-pointing timelike curve in a spacetime (a connected and time-oriented 4-dimensional Lorentz manifold) is thought as a locus of a material particle in the space-time. The unit-speed parameter of this curve is called the proper time of a material particle. Analogue to timelike curves, in relativity theory a futurepointing null geodesic is thought as the locus of a lightlike particle. More generally, from the differential geometric point of view, the study of null curves has its own geometric interest.The geometry of null hypersurfaces in spacetimes has played an important role in the development of general relativity, as well as in mathematics and physics of gravitation. It is necessary, for example, to understand the causal structure of space-times, black holes, assymptotically flat systems and gravitational waves [15]. The importance of the study of null curves and its presence in the physical theories is clear from the fact, see [7], that the classical relativistic string is a surface or world-sheet Minkowski space which satisfies the Lorentzian analogue of the minimal surface equations [7] The string equations simplify to the wave equation and a couple of extra simple equations, and by solving the 2-dimensional wave equation it turns out that strings are equivalent to pairs of null curves, or a single null curve in the case of an open string (see also [7], [8]]). Motivated by the growing importance of null curves in mathematical physics. There exists a vast literature on timelike, spacelike, lightlike curves in Minkowski space, for instance [1,3,4,6,9,10,11,12,17].

In this paper, we study the differential equations characterizing spacelike curves in the 3-

dimensional null cone of the Minkowski 4-space E_1^4 . Morever, we give some new characterizations of this curve in E_1^4 .

2 Preliminaries

In the following, we use the notations and concepts from [13,14] unless otherwise stated.

Minkowski space-time E_1^4 is an Euclidean space provided with the standart flat metric given by

$$\langle X , Y \rangle = x_1 y_1 + x_2 y_2 + x_3 y_3 - x_4 y_4$$

where (x_1, x_2, x_3, x_4) and $(y_1, y_2, y_3 y_4)$ are a rectangular coordinate system in E^4.

Since $<\ ,\ >$ is an indefinite metric,recall that a vector $u \in E_1^4$ can have one of the three casual characers;

İt can be spacelike, if $< u, u > \ > 0$ or $u = 0$,

timelike, if $< u, u > \ < 0$,

null or lightlike if $< u, u > \ = 0$ and $u \neq 0$.

The norm of a vector v is given by$\| v \| = \sqrt{| \langle v, v \rangle |}$ .Therefore, v is unit vector if $\langle v, v \rangle = \mp 1$. Next, vectors v and w are said to orthogonal if $\langle v, w \rangle = 0$. Next $x = x(s)$ is a unit speed curve if $\left\langle \frac{dx(s)}{ds}, \frac{dx(s)}{ds} \right\rangle = \mp 1$.

Let c be a fixed point in E_1^4 and $r > 0$ be a constant. The pseudo-Riemannian sphere is defined by

$$S_1^3(c, r) = \{x \in E_1^4 : \ < x - c, x - c > \ = r^2\};$$

the pseudo-Riemannian hyperbolic space is defined by

$$H_0^3(c, r) = \{x \in E_1^4 : \ < x - c, x - c > \ = -r^2\};$$

the pseudo-Riemannian null cone (quadric cone) is defined by

$$Q_1^3(c) = \{x \in E_1^4 : \ < x - c, x - c > \ = 0\}.$$

When $c = 0$, we simply denote $Q_1^3(0)$ by Q^3 and call it the lightlike cone (or simply the light cone).

Let $x = x(s) : I \to Q^3 \subset E_1^4$ be a spacelike curve in the three dimensional lightlike cone Q^3 of the Minkowski 4-space E_1^4 with the arc length parameter s. We put (x_1, x_2, x_3, x_4) and have

$$x_1^2 + x_2^2 + x_3^2 - x_4^2 = 0.$$

Then from

$$x_1^2 - (ix_2)^2 = - \left(x_3^2 - x_4^2\right)$$

we get

$$\frac{x_1 + ix_2}{x_3 + x_4} = - \frac{x_3 - x_4}{x_1 - ix_2} \quad \text{or} \quad \frac{x_1 + ix_2}{x_3 - x_4} = - \frac{x_3 + x_4}{x_1 - ix_2} \tag{2.1}$$

We may assume tat

$$y(s) = - \frac{d^2x(s)}{ds^2} - \frac{1}{2} < \frac{d^2x(s)}{ds^2}, \frac{d^2x(s)}{ds^2} > x(s) \tag{2.2}$$

and we have

$$< y(s), y(s) >=< x(s), x(s) >=< y(s), \frac{dx(s)}{ds} >= 0, \ < x(s), y(s) >= 1. \tag{2.3}$$

Put $\alpha(s) = \frac{dx(s)}{ds}$ and choose $\beta(s)$ such that

$$\det \ (x(s), \alpha(s), \beta(s), \ y(s) \ ) = 1.$$

Then from (2.2), we obtain

$$\frac{d\alpha(s)}{ds} = \frac{d^2x(s)}{ds^2} = -\frac{1}{2} < \frac{d^2x(s)}{ds^2}, \frac{d^2x(s)}{ds^2} > x(s) - y(s) = k(s)\, x(s) - y(s). \tag{2.4}$$

Therefore, the Frenet formulas of spacelike curve $x = x(s) : I \to Q^3 \subset E_1^4$ can be written as

$$\begin{aligned} \frac{dx(s)}{ds} &= \alpha(s) \\ \frac{d\alpha(s)}{ds} &= \kappa(s)x(s) - y(s) \\ \frac{d\beta(s)}{ds} &= \tau(s)x(s) \\ \frac{dy(s)}{ds} &= -\kappa(s)\alpha(s) - \tau(s)\beta(s). \end{aligned} \tag{2.1}$$

The frame field $\{x(s), \alpha(s), y(s), \beta(s)\}$ is called the cone Frenet frame of the curve $x(s)$. The functions $\kappa(s)$ and $\tau(s)$ are defined as

$$k(s) = -\frac{1}{2} < \frac{d^2x(s)}{ds^2}, \frac{d^2x(s)}{ds^2} > \tag{2.6}$$

and

$$\tau(s) = \sqrt{< \frac{d^3x(s)}{ds^3}, \frac{d^3x(s)}{ds^3} > - \left(< \frac{d^2x(s)}{ds^2}, \frac{d^2x(s)}{ds^2} > \right)^2}. \tag{2.7}$$

From [13], for any asymptotic orthonormal frame $\{x(s), \alpha(s), y(s), \beta(s)\}$ of the curve $x = x(s) : I \to Q^3 \subset E_1^4$ with

$$\begin{aligned} &< x(s), x(s) >=< y(s), y(s) >= \\ &< x(s), \alpha(s) >=< x(s), \beta(s) >= \\ &< y(s), \alpha(s) >=< y(s), \beta(s) >=< \alpha(s), \beta(s) >= 0 \end{aligned} \tag{2.2}$$

$$< x(s), y(s) >=< \alpha(s), \alpha(s) >=< \beta(s), \beta(s) >= 1 \tag{2.9}$$

the Frenet formulas read

$$\frac{dx(s)}{ds} = \alpha(s) \tag{2.3}$$

$$\frac{d\alpha(s)}{ds} = \kappa(s)x(s) + \lambda(s)\beta(s) - y(s) \tag{2.4}$$

$$\begin{aligned} \frac{d\beta(s)}{ds} &= \tau(s)x(s) - \lambda(s)\alpha(s) \\ \frac{dy(s)}{ds} &= -\kappa(s)\alpha(s) - \tau(s)\beta(s). \end{aligned}$$

we know $\lambda(s) \equiv 0$ if and if only $y(s)$satisfies (2.2). Therefore some authors called the frame, satisfying (2.2), Cartan frame we know (2.2) and (2.3) are true in any dimension.

Definition 2.1. A curve $x(s)$ such that the functions$\frac{\kappa(s)}{\tau(s)} = const.$is called a general helix.

If both $\kappa(s)$ and $\tau(s)$ are constants along $x(s)$, then $x(s)$ is called a circular helix.

If $\kappa(s) = \tau(s) = 0$, then $x(s)$ is called a null cubic.

Example 2.2. Let $x(s)$ be spacelike curve in the three dimensional lightlike cone Q^3 of the Minkowski 4-space E_1^4 defined by

$$x(s) = \left(\frac{6s - s^3}{12\sqrt{2}}, \frac{s^2}{2\sqrt{3}}, 0, \frac{6s + s^3}{12\sqrt{2}} \right).$$

Then $x(s)$ is a general helix with

$$\frac{\kappa(s)}{\tau(s)} = const.$$

Example 2.3. Let $x(s)$ be spacelike curve in the three dimensional lightlike cone Q^3 of the Minkowski 4-space E_1^4 defined by

$$x(s) = \left(1, \frac{1}{\sqrt{2}}s, \frac{1}{4}s^2, \frac{1}{4}s^2 + 1 \right).$$

Then $x(s)$ is anull cubic with

$$\kappa(s) = \tau(s) = 0.$$

3 The Characterization of Curves in the Lightlike Cone

Theorem 3.1. *Let $x = x(s) : I \to Q^3 \subset E_1^4$ be a spacelike curve in the three dimensional lightlike cone Q^3 of the Minkowski 4-space E_1^4 with the arc length parameter s. If the cone frenet frame $\{x(s), \alpha(s), y(s), \beta(s)\}$ of the curve $x(s)$ is satisfies the following equation*

$$\frac{df(s)}{ds} + (g(s) - u(s)h(s)) = C, \quad C \in R \tag{3.1}$$

then we have one of the following situations, where $f(s)$, $g(s)$, $u(s), h(s)$ are differentiable functions of s.

i) the curve $x(s)$ is a general helix.
ii) $k(s) = 0$ and $\tau(s) \neq 0$.
ii) $k(s) = cons\tan t$ and $\tau(s) = 0$.

Proof. By differentiatiating (3.1) with respect to arc parameter,we obtain

$$\frac{d^2 f(s)}{ds^2} + \frac{dg(s)}{ds} - \frac{du(s)}{ds}h(s) - u(s)\frac{dh(s)}{ds} = 0. \tag{3.2}$$

If we take $f(s) = \alpha(s)$, $g(s) = y(s)$, $u(s) = \frac{k(s)}{\tau(s)}$ and $h(s) = \frac{d\beta(s)}{ds}$ in (3.2), then we have

$$\frac{dy(s)}{ds} = \frac{d}{ds}\left(\frac{k(s)}{\tau(s)}\right)\frac{d\beta(s)}{ds} + \left(\frac{k(s)}{\tau(s)}\right)\frac{d^2\beta(s)}{ds^2} - \frac{d^2\alpha(s)}{ds^2}. \tag{3.3}$$

Using the cone frenet frame formulae (2.5) in (3.3), we get

$$\left[\frac{d}{ds}\left(\frac{k(s)}{\tau(s)}\right)\tau(s) + \frac{k(s)}{\tau(s)}\frac{d\tau(s)}{ds} - \frac{dk(s)}{ds}\right] = 0. \tag{3.4}$$

The solutions of (3.4) can be written as follows:
i) the curve $x(s)$ is a general helix.
ii) $k(s) = 0$ and $\tau(s) \neq 0$.
ii) $k(s) = cons\tan t$ and $\tau(s) = 0$.

□

Theorem 3.2. *Let $x = x(s) : I \to Q^3 \subset E_1^4$ be a spacelike curve in the three dimensional lightlike cone Q^3 of the Minkowski 4-space E_1^4 with the arc length parameter s. If the curve $x(s)$ is a general helix, then*

$$\frac{d^2\beta(s)}{ds^2} - C_0\left[\frac{d^2\alpha(s)}{ds^2} + \frac{dy(s)}{ds}\right] = 0, \quad C_0 \in R. \tag{3.5}$$

Proof. By the the second equation of cone frenet frame formulae (2.5), we have

$$\frac{dy(s)}{ds} = -k(s)\alpha(s) - \frac{d\alpha(s)}{ds} \tag{3.6}$$

and the second equation of cone frenet frame formulae (2.5), we get

$$x(s) = \tau(s)\frac{d\beta(s)}{ds}. \tag{3.7}$$

from the second equation of (2.5), we have

$$k(s)x(s) = \frac{d\alpha(s)}{ds} + y(s). \tag{3.8}$$

Substituting (3.8) in (3.7) and for the curve $x(s)$ is a general helix , we obtain

$$\frac{dy(s)}{ds} = C_0\frac{d^2\beta(s)}{ds^2} - \frac{d^2\alpha(s)}{ds^2}, \ \ C_0 \in R.$$

or

$$\frac{d^2\beta(s)}{ds^2} - C_0\left[\frac{d^2\alpha(s)}{ds^2} + \frac{dy(s)}{ds}\right] = 0, \ \ C_0 \in R$$

that finishes the proof.

□

The proof of following Corrollary is obtained immediately by considering the similar way used in the proof of Theorem (3.2).

Corollary 3.3. *Let $x = x(s) : I \to Q^3 \subset E_1^4$ be a spacelike curve in the three dimensional lightlike cone Q^3 of the Minkowski 4-space E_1^4 with the arc length parameter s. If the curve $x(s)$ is a circular helix, then*

$$\frac{d^2\beta(s)}{ds^2} + C_1\frac{dy(s)}{ds} + C_2\beta(s) = 0, \ \ C_1, C_2 \in R \tag{3.9}$$

and

$$\frac{d^3x(s)}{ds^3} + C_4\alpha(s) + C_5\beta(s) = 0, \ \ C_4, C_5 \in R. \tag{3.10}$$

Theorem 3.4. *Let $x = x(s) : I \to Q^3 \subset E_1^4$ be a spacelike curve in the three dimensional lightlike cone Q^3 of the Minkowski 4-space E_1^4 with the arc length parameter s.Then the curve $x(s)$ satisfies the following differential equation*

$$\frac{d^3\alpha(s)}{ds^3} + \lambda_1(s)x(s) + \lambda_2(s)\alpha(s) + \lambda_3(s)y(s) + \lambda_4(s)\beta(s) = 0 \tag{3.11}$$

where

$$\lambda_1(s) = -\frac{dk^2(s)}{ds^2} - 2k^2(s) - \tau^2(s)$$

$$\lambda_2 y(s) = -3\frac{dk(s)}{ds}$$

$$\lambda_3(s) = 2k(s)$$

$$\lambda_4(s) = -\frac{d\tau(s)}{ds}$$

Proof. Using the cone frenet frame formulae , we have

$$\frac{d\alpha^2(s)}{ds^2} = \frac{dk(s)}{ds}x(s) + k(s)\frac{dx(s)}{ds} - \frac{dy(s)}{ds}. \tag{3.12}$$

and differentiatiating (3.12) with respect to arc parameter s, we obtain

$$\frac{d\alpha^3(s)}{ds^3} = \frac{d^2k(s)}{ds^2}x(s) + \frac{dk(s)}{ds}\frac{dx(s)}{ds} + 2\frac{dk(s)}{ds}\frac{d\alpha(s)}{ds} + 2k(s)\frac{d\alpha(s)}{ds} + \frac{d\tau(s)}{ds}\beta(s) + \frac{d\beta(s)}{ds}\tau(s) \tag{3.13}$$

Substituting equation (2.5) into the equation of (3.13) and making the necessary calculations, we have

$$\frac{d\alpha^3(s)}{ds^3} + \lambda_1(s)x(s) + \lambda_2 y(s) + \lambda_3(s)y(s) + \lambda_4(s)\beta(s) = 0$$

where

$$\lambda_1(s) = -\frac{dk^2(s)}{ds^2} - 2k^2(s) - \tau^2(s)$$

$$\lambda_2 y(s) = -3\frac{dk(s)}{ds}$$

$$\lambda_3(s) = 2k(s)$$

$$\lambda_4(s) = -\frac{d\tau(s)}{ds}$$

□

Corollary 3.5. *Let $x = x(s) : I \to Q^3 \subset E_1^4$ be a spacelike curve in the three dimensional lightlike cone Q^3 of the Minkowski 4-space E_1^4 with the arc length parameter s. If the curve $x(s)$ is a circular helix, then*

$$\frac{d\alpha^3(s)}{ds^3} + \rho(s)\frac{d\beta(s)}{ds} + 2k(s)\frac{d\alpha(s)}{ds} = 0 \tag{3.14}$$

where

$$\rho(s) = -\frac{4k^2(s) + \tau^2(s)}{\tau(s)}.$$

Proof. If the curve $x(s)$ is a circular helix, from (3.11), we get

$$\frac{d^3\alpha(s)}{ds^3} - \left(2k^2(s) + \tau^2(s)\right)x(s) - 2k(s)y(s) = 0 \tag{3.15}$$

Substituting equation (2.5) into the equation of (3.15), we obtain (3.14).

Proof.

The proof of following theorems are obtained immediately by considering the similar way used in the proof of Theorem (3.4).

Theorem 3.6. *Let $x = x(s) : I \to Q^3 \subset E_1^4$ be a spacelike curve in the three dimensional lightlike cone Q^3 of the Minkowski 4-space E_1^4 with the arc length parameter s. Then the curve $x(s)$ satisfies the following differential equation*

$$\frac{d^3\beta(s)}{ds^3} + \mu_1(s)x(s) + \mu_2(s)\alpha(s) + \mu_3(s)y(s) = 0 \tag{3.16}$$

where

$$\mu_1(s) = -\frac{d\tau^2(s)}{ds^2} - k(s)\tau(s)$$

$$\mu_2(s) = -2\frac{d\tau(s)}{ds}$$

$$\mu_3(s) = \tau(s)$$

Corollary 3.7. *Let $x = x(s) : I \to Q^3 \subset E_1^4$ be a spacelike curve in the three dimensional lightlike cone Q^3 of the Minkowski 4-space E_1^4 with the arc length parameter s. If the curve $x(s)$ is a circular helix, then*

$$\frac{d^3\beta(s)}{ds^3} - \tau(s)\frac{d\alpha(s)}{ds} = 0. \tag{3.17}$$

Theorem 3.8. *Let $x = x(s) : I \to Q^3 \subset E_1^4$ be a spacelike curve in the three dimensional lightlike cone Q^3 of the Minkowski 4-space E_1^4 with the arc length parameter s. Then the curve $x(s)$ satisfies the following differential equation*

$$\frac{d^2y(s)}{ds^2} + \gamma_1(s)x(s) + \gamma_2(s)\alpha(s) + \gamma_3(s)y(s) + \gamma_4(s)\beta(s) = 0 \tag{3.18}$$

where

$$\begin{aligned}
\gamma_1(s) &= \frac{dk(s)}{ds}, \\
\gamma_2(s) &= k^2(s) + \tau^2(s), \\
\gamma_3(s) &= -k(s), \\
\gamma_4(s) &= \frac{d\tau(s)}{ds}.
\end{aligned}$$

Corollary 3.9. *Let $x = x(s) : I \to Q^3 \subset E_1^4$ be a spacelike curve in the three dimensional lightlike cone Q^3 of the Minkowski 4-space E_1^4 with the arc length parameter s. If the curve $x(s)$ is a circular helix, then*

$$\frac{d^2y(s)}{ds^2} + \left(\frac{k^2(s) + \tau^2(s)}{\tau(s)}\right)\frac{d\beta(s)}{ds} - k(s)y(s) = 0 \tag{3.19}$$

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