Palestine Journal of Mathematics

Vol. 6(1)(2017) , 1–5

ON r-JORDAN MAPS OF TRIANGULAR ALGEBRAS

Rachid Tribak and Driss Aiat Hadj Ahmed

Communicated by N. Mahdou

MSC 2010 Classifications: Primary 47B49, 47L35; Secondary 16W99.

Keywords and phrases: Triangular algebra, r-Jordan map, additivity.

Abstract. Let R be a commutative ring such that $\frac{1}{2} \in R$. We prove that if r is a fixed invertible element of R and Φ is a bijective map from a triangular R-algebra $\mathcal{T}$ onto an arbitrary R-algebra which satisfies

$$\Phi(r(XY + YX)) = r(\Phi(X)\Phi(Y) + \Phi(Y)\Phi(X)) \ (\forall X, Y \in \mathcal{T}),$$

then Φ is automatically additive.

1 Introduction

Throughout this paper R will denote a commutative ring with $\frac{1}{2} \in R$. Let $\mathcal{A}$ and $\mathcal{B}$ be unital algebras over the ring R. Let $\mathcal{M}$ be a unital $(\mathcal{A}, \mathcal{B})$-bimodule, which is faithful as a left $\mathcal{A}$-module as well as a right $\mathcal{B}$-module, that is, for any $a \in \mathcal{A}$ and $b \in \mathcal{B}$, $a\mathcal{M} = \mathcal{M}b = \{0\}$ imply $a = 0$ and $b = 0$. The R-algebra

$$\mathcal{T} = Tri(\mathcal{A}, \mathcal{M}, \mathcal{B}) = \left\{ \begin{pmatrix} a & m \\ & b \end{pmatrix} : a \in \mathcal{A}, b \in \mathcal{B}, m \in \mathcal{M} \right\}$$

under the usual matrix operations is called a triangular algebra (see [3] or [4]).

Let $\mathcal{C}$ and $\mathcal{C}'$ be unital R-algebras and let $r \in R$. A map $\Phi : \mathcal{C} \to \mathcal{C}'$ is called an r-Jordan map if it is a bijective map which satisfes

$$\Phi(r(XY + YX)) = r(\Phi(X)\Phi(Y) + \Phi(Y)\Phi(X)) \ \forall X, Y \in \mathcal{C}.$$

Recently, several authors have studied the additivity of r-Jordan maps. In [7], Molnár showed that every $\frac{1}{2}$-Jordan map between standard operator algebras is additive. In [6], Lu showed that if $R = \mathbb{Q}$ the field of rational numbers and r is a nonzero rational number, then every r-Jordan map from a unital prime algebra containing a nontrivial idempotent, or a standard operator algebra, or a unital algebra which has a system of matrix units, onto an arbitrary algebra is additive.

In the present paper, we study the additivity of r-Jordan maps on triangular algebras. We will prove that if r is an invertible element of R, then every r-Jordan map from $\mathcal{T}$ onto an arbitrary R-algebra is additive.

2 Main result

The following theorem is our main result.

Theorem 2.1. *Let $\mathcal{A}$ and $\mathcal{B}$ be unital algebras over the ring R. Let $\mathcal{M}$ be a unital $(\mathcal{A}, \mathcal{B})$-bimodule which is faithful as a left $\mathcal{A}$-module and also as a right $\mathcal{B}$-module. Let $\mathcal{T} = Tri(\mathcal{A}, \mathcal{M}, \mathcal{B})$ be the triangular algebra, and $\mathcal{C}$ be an algebra over R. Let r be an invertible element of R. Assume that $\Phi : \mathcal{T} \to \mathcal{C}$ is an r-Jordan map, that is, Φ is a bijective map satisfying*

$$\Phi(r(XY + YX)) = r(\Phi(X)\Phi(Y) + \Phi(Y)\Phi(X)) \ \forall X, \ Y \in \mathcal{T}.$$

Then Φ is additive.

We have divided the proof of the last theorem into a sequence of lemmas.

Let $a \in \mathcal{A}$, $b \in \mathcal{B}$ and $u \in \mathcal{M}$. Throughout this paper we shall use the following notations: $E_a = \begin{pmatrix} a & 0 \\ & 0 \end{pmatrix}$, $F_b = \begin{pmatrix} 0 & 0 \\ & b \end{pmatrix}$ and $X_u = \begin{pmatrix} 0 & u \\ & 0 \end{pmatrix}$.

We begin with the following lemma which will be used frequently in the sequel.

Lemma 2.2. *Let a, $a' \in \mathcal{A}$, b, $b' \in \mathcal{B}$ and u, $u' \in \mathcal{M}$. The following relations hold:*
(i) $E_aE_{a'} = E_{aa'}$, $E_aF_b = 0$, $E_aX_u = X_{au}$.
(ii) $F_bE_a = 0$, $F_bF_{b'} = F_{bb'}$, $F_bX_u = 0$.
(iii) $X_uE_a = 0$, $X_uF_b = X_{ub}$, $X_uX_{u'} = 0$.

Proof. The proof is straightforward. □

Throughout the remainder of this section, Φ is a map which satisfies the assumptions of Theorem 2.1.

Lemma 2.3. *We have $\Phi(0) = 0$.*

Proof. Since Φ is surjective, there exists $A \in \mathcal{T}$ such that $\Phi(A) = 0$. Thus

$$\begin{aligned} \Phi(0) &= \Phi(r(0A + A0)) \\ &= r(\Phi(0)\Phi(A) + \Phi(A)\Phi(0)) \\ &= r(\Phi(0)0 + 0\Phi(0)) = 0. \Box \end{aligned}$$

Lemma 2.4. *Let $a \in \mathcal{A}$, $b \in \mathcal{B}$ and $u \in \mathcal{M}$. Then there exist $\alpha \in \mathcal{A}$, $\beta \in \mathcal{B}$ and $v \in \mathcal{M}$ such that $\Phi(A) = \Phi(E_a) + \Phi(F_b) + \Phi(X_u)$, where $A = E_\alpha + F_\beta + X_v$. Moreover, for every $T \in \mathcal{T}$, we have $\Phi(r(AT + TA)) = \Phi(r(E_aT + TE_a)) + \Phi(r(F_bT + TF_b)) + \Phi(r(X_uT + TX_u))$.*

Proof. The first part follows easily from the surjectivity of Φ. The second part follows from the fact that $\Phi(r(XY + YX)) = r(\Phi(X)\Phi(Y) + \Phi(Y)\Phi(X))\ \forall X,\ Y \in \mathcal{T}$. □

Lemma 2.5. *Let $a \in \mathcal{A}$ and $u \in \mathcal{M}$. Then $\Phi(E_a + X_u) = \Phi(E_a) + \Phi(X_u)$.*

Proof. By Lemma 2.4, there exist $\alpha \in \mathcal{A}$, $\beta \in \mathcal{B}$ and $v \in \mathcal{M}$ such that $\Phi(A) = \Phi(E_a) + \Phi(X_u)$, where $A = E_\alpha + F_\beta + X_v$. Moreover, for any $T \in \mathcal{T}$, we have

$$\Phi(r(AT + TA)) = \Phi(r\,(E_aT + TE_a)) + \Phi(r\,(X_uT + TX_u)).$$

If we take $T = F_1$, we get $\Phi(r(F_{2\beta} + X_v)) = \Phi(0) + \Phi(rX_u)$ by Lemma 2.2. Hence $\Phi(r(F_{2\beta} + X_v)) = \Phi(rX_u)$ by Lemma 2.3. The injectivity of Φ gives $u = v$ and $\beta = 0$. Now replacing T by X_m with $m \in \mathcal{M}$, we obtain $\Phi(rX_{\alpha m}) = \Phi(0) + \Phi(rX_{am}) = \Phi(rX_{am})$ by Lemmas 2.2 and 2.3. Again by the injectivity of Φ, we get $\alpha m = am$ for every $m \in \mathcal{M}$. Since $\mathcal{M}$ is a faithful left $\mathcal{A}$-module, we have $\alpha = a$. It follows that $\Phi(E_a + X_u) = \Phi(E_a) + \Phi(X_u)$. □

Lemma 2.6. *Let $b \in \mathcal{B}$ and $u \in \mathcal{M}$. Then $\Phi(F_b + X_u) = \Phi(F_b) + \Phi(X_u)$.*

Proof. By Lemma 2.4, there exist $\alpha \in \mathcal{A}$, $\beta \in \mathcal{B}$ and $v \in \mathcal{M}$ such that $\Phi(A) = \Phi(F_b) + \Phi(X_u)$, where $A = E_\alpha + F_\beta + X_v$. Moreover, for any $T \in \mathcal{T}$, we have

$$\Phi(r(AT + TA)) = \Phi(r\,(F_bT + TF_b)) + \Phi(r\,(X_uT + TX_u)).$$

If $T = E_1$, then $\Phi(r(E_{2\alpha} + X_v)) = \Phi(0) + \Phi(rX_u) = \Phi(rX_u)$ by Lemmas 2.2 and 2.3. Now, the injectivity of Φ implies $u = v$ and $\alpha = 0$.

If $T = X_m$ with $m \in \mathcal{M}$, then $\Phi(rX_{m\beta})) = \Phi(0) + \Phi(rX_{mb})$, and the injectivity of Φ yields $m\beta = mb$ for all $m \in \mathcal{M}$. Hence $\beta = b$ since $\mathcal{M}$ is a faithful right $\mathcal{B}$-module. This completes the proof. □

Lemma 2.7. *Let $a \in \mathcal{A}$, $b \in \mathcal{B}$ and $u \in \mathcal{M}$. Then $\Phi(E_a+F_b+X_u)=\Phi(E_a)+\Phi(F_b)+\Phi(X_u)$.*

Proof. By Lemma 2.4, we can find $\alpha \in \mathcal{A}$, $\beta \in \mathcal{B}$ and $v \in \mathcal{M}$ such that $\Phi(A)=\Phi(E_a)+\Phi(F_b)+\Phi(X_u)$, where $A=E_\alpha+F_\beta+X_v$. By Lemma 2.2, we have $AE_1+E_1A=E_{2\alpha}+X_v$. Hence $\Phi(r(AE_1+E_1A))=\Phi(r(E_{2\alpha}+X_v))$. On the other hand, replacing T by E_1 in Lemma 2.4, we get

$$\Phi(r(AE_1+E_1A))=\Phi(r\,(E_aE_1+E_1E_a))+\Phi(r\,(F_bE_1+E_1F_b))+\Phi(r\,(X_uE_1+E_1X_u)).$$

So by Lemmas 2.2 and 2.3, we have $\Phi(r(E_{2\alpha}+X_v))=\Phi(rE_{2a})+\Phi(rX_u)$. From Lemma 2.5, it follows that $\Phi(r(E_{2\alpha}+X_v))=\Phi(r(E_{2a}+X_u))$. By the injectivity of Φ, we have $\alpha=a$ and $u=v$. Similarly, by using Lemmas 2.3 and 2.6, we can show that $\Phi(r(AF_1+F_1A))=\Phi(r(F_{2\beta}+X_v))=\Phi(r(F_{2b}+X_u))$ and hence $\beta=b$. Consequently, $\Phi(E_a+F_b+X_m)=\Phi(E_a)+\Phi(F_b)+\Phi(X_m)$. □

Lemma 2.8. *Let u, $v \in \mathcal{M}$. Then $\Phi(X_u+X_v)=\Phi(X_u)+\Phi(X_v)$.*

Proof. It is easy to check that Lemma 2.2 gives

$$\begin{aligned} X_u+X_v &= E_1X_u+X_vF_1 \\ &= (E_1+X_v)\,(X_u+F_1) \\ &= (E_1+X_v)\,(X_u+F_1)+(X_u+F_1)\,(E_1+X_v)\,. \end{aligned}$$

Thus we have

$$\begin{aligned} \Phi\,(X_u+X_v) &= \Phi\left(r\left(\left(\frac{1}{r}E_1+\frac{1}{r}X_v\right)(X_u+F_1)+(X_u+F_1)\left(\frac{1}{r}E_1+\frac{1}{r}X_v\right)\right)\right) \\ &= r\left(\Phi\left(\frac{1}{r}E_1+\frac{1}{r}X_v\right)\Phi\,(X_u+F_1)+\Phi\,(X_u+F_1)\,\Phi\left(\frac{1}{r}E_1+\frac{1}{r}X_v\right)\right). \end{aligned}$$

It follows from Lemma 2.7 that

$$\begin{aligned} \Phi(X_u+X_v) &= r\left(\left(\Phi\left(\frac{1}{r}E_1\right)+\Phi\left(\frac{1}{r}X_v\right)\right)(\Phi\,(X_u)+\Phi\,(F_1))\right) \\ &+ r\left((\Phi\,(X_u)+\Phi\,(F_1))\left(\Phi\left(\frac{1}{r}E_1\right)+\Phi\left(\frac{1}{r}X_v\right)\right)\right) \\ &= r\left(\Phi\left(\frac{1}{r}E_1\right)\Phi\,(X_u)+\Phi\,(X_u)\,\Phi\left(\frac{1}{r}E_1\right)\right) \\ &+ r\left(\Phi\left(\frac{1}{r}E_1\right)\Phi\,(F_1)+\Phi\,(F_1)\,\Phi\left(\frac{1}{r}E_1\right)\right) \\ &+ r\left(\Phi\left(\frac{1}{r}X_v\right)\Phi\,(X_u)+\Phi\,(X_u)\,\Phi\left(\frac{1}{r}X_v\right)\right) \\ &+ r\left(\Phi\left(\frac{1}{r}X_v\right)\Phi\,(F_1)+\Phi\,(F_1)\,\Phi\left(\frac{1}{r}X_v\right)\right). \end{aligned}$$

Therefore,

$$\begin{aligned} \Phi(X_u+X_v) &= \Phi\,(E_1X_u+X_uE_1)+\Phi\,(E_1F_1+F_1E_1) \\ &+ \Phi\,(X_vX_u+X_uX_v)+\Phi\,(X_vF_1+F_1X_v)\,. \end{aligned}$$

This implies that $\Phi(X_u+X_v)=\Phi\,(X_u)+\Phi\,(X_v)$ by Lemma 2.2. □

Lemma 2.9. *Let a, $a' \in \mathcal{A}$. Then $\Phi(E_a+E_{a'})=\Phi(E_a)+\Phi(E_{a'})$.*

Proof. By Lemma 2.4, there exist $\alpha \in \mathcal{A}$, $\beta \in \mathcal{B}$ and $v \in \mathcal{M}$ such that $\Phi(A) = \Phi(E_a) + \Phi(E_{a'})$, where $A = E_\alpha + F_\beta + X_v$. Moreover, for any $T \in \mathcal{T}$, we have

$$\begin{aligned}\Phi(r(AT+TA)) &= r(\Phi(A)\Phi(T)+\Phi(T)\Phi(A)) \\ &= r((\Phi(E_a)+\Phi(E_{a'}))\,\Phi(T)+\Phi(T)\,(\Phi(E_a)+\Phi(E_{a'}))) \\ &= r(\Phi(E_a)\Phi(T)+\Phi(T)\Phi(E_a))+r(\Phi(E_{a'})\Phi(T)+\Phi(T)\Phi(E_{a'})) \\ &= \Phi(r\,(E_aT+TE_a))+\Phi(r\,(E_{a'}T+TE_{a'})).\end{aligned}$$

By setting $T = F_1$, we get $\Phi(r(F_{2\beta} + X_v)) = \Phi(0) + \Phi(0) = 0$ by Lemmas 2.2 and 2.3. So the injectivity of Φ gives $v = 0$ and $\beta = 0$.

By taking $T = X_m$ with $m \in \mathcal{M}$, we can get $\Phi(rX_{\alpha m}) = \Phi(rE_aX_m) + \Phi(rE_{a'}X_m) = \Phi(rX_{am}) + \Phi(rX_{a'm})$ since $\beta = 0$. Thus $\Phi(rX_{\alpha m}) = \Phi(rX_{(a+a')m})$ by Lemma 2.8. The injectivity of Φ and the fact that $\mathcal{M}$ is a faithful left $\mathcal{A}$-module show that $\alpha = a + a'$. Consequently, $\Phi(E_a + E_{a'}) = \Phi(E_a) + \Phi(E_{a'})$. □

Lemma 2.10. *For every b, $b' \in \mathcal{B}$, we have $\Phi(F_b + F_{b'}) = \Phi(F_b) + \Phi(F_{b'})$.*

Proof. The proof is similar to that of Lemma 2.9. □

Proof of Theorem 2.1. Let $S = E_a + F_b + X_u$ and $S' = E_{a'} + F_{b'} + X_{u'}$, where a, $a' \in \mathcal{A}$, b, $b' \in \mathcal{B}$ and u, $u' \in \mathcal{M}$. Combining the above lemmas, we get the following equalities:

$$\begin{aligned}\Phi(S+S') &= \Phi\left((E_a+E_{a'})+(F_b+F_{b'})+(X_u+X_{u'})\right) \\ &= \Phi\left(E_a+E_{a'}\right)+\Phi\left(F_b+F_{b'}\right)+\Phi\left(X_u+X_{u'}\right) \\ &= \Phi\left(E_a\right)+\Phi\left(E_{a'}\right)+\Phi\left(F_b\right)+\Phi\left(F_{b'}\right)+\Phi\left(X_u\right)+\Phi\left(X_{u'}\right) \\ &= \Phi(E_a+F_b+X_u)+\Phi(E_{a'}+F_{b'}+X_{u'}) \\ &= \Phi\left(S\right)+\Phi\left(S'\right).\end{aligned}$$

This proves the theorem. □

3 Applications

We begin with the following application of Theorem 2.1.

Proposition 3.1. *Let $\mathcal{A}$ and $\mathcal{B}$ be unital algebras over the ring R. Let $\mathcal{M}$ be a unital $(\mathcal{A},\mathcal{B})$-bimodule that is faithful as a left $\mathcal{A}$-module and also as a right $\mathcal{B}$-module. Let $\mathcal{T} = Tri(\mathcal{A},\mathcal{M},\mathcal{B})$ be the triangular algebra. Assume that both $\mathcal{A}$ and $\mathcal{B}$ have only trivial idempotents. If $\Phi : \mathcal{T} \to \mathcal{T}$ is a $\frac{1}{2}$-Jordan map satisfying $\Phi(\alpha X) = \alpha\Phi(X)$ for all $\alpha \in R$ and $X \in \mathcal{T}$, then Φ is either an automorphism or an anti-automorphism.*

Proof. By Theorem 2.1, Φ is additive. So Φ is a Jordan endomorphism of $\mathcal{T}$. By [1, Theorem 2.1], Φ is either an automorphism or an anti-automorphism. □

We conclude this paper by applying Theorem 2.1 to the two classical examples of triangular algebras: upper triangular matrix algebras and nest algebras.

Upper triangular matrix algebras. Let $\mathcal{M}_{l\times m}(R)$ denote the set of all $l \times m$ matrices with entries in R. We denote by $\mathcal{T}_n(R)$ the algebra of all $n \times n$ upper triangular matrices over R. For $n \geq 2$ and each $1 \leq l \leq n-1$, the algebra $\mathcal{T}_n(R)$ can be represented as a triangular algebra of the form

$$\mathcal{T}_n(R) = \begin{pmatrix} \mathcal{T}_l(R) & \mathcal{M}_{l\times(n-l)}(R) \\ & \mathcal{T}_{n-l}(R)\end{pmatrix}.$$

Corollary 3.2. *Let r be an invertible element of R and let $\mathcal{C}$ be an algebra over R. Then every r-Jordan map $\Phi : \mathcal{T}_n(R) \to \mathcal{C}$ is additive.*

Proposition 3.3. *The following conditions are equivalent:*

(i) *R contains no idempotents except* 0 *and* 1*;*

(ii) *If Φ is a 1-Jordan map from the R-algebra $\mathcal{T}_n(R)$ $(n \geq 2)$ onto an arbitrary R-algebra satisfying $\Phi(\alpha X) = \alpha\Phi(X)$ for all $\alpha \in R$ and $X \in \mathcal{T}_n(R)$, then Φ is an isomorphism or an anti-isomorphism.*

Proof. This follows from Theorem 2.1 and [2, Theorem p.198]. □

Nest algebras. (see [5]) A nest $\mathcal{N}$ is a chain of closed subspaces of a complex Hilbert space $\mathcal{H}$ containing $\{0\}$ and $\mathcal{H}$ which is closed under arbitrary intersections and closed linear spans. The nest algebra associated to $\mathcal{N}$ is the algebra

$$\mathcal{T}(\mathcal{N}) = \{T \in \mathcal{B}(\mathcal{H}) : T(N) \subset N \text{ for all } N \in \mathcal{N}\}.$$

A nest algebra $\mathcal{T}(\mathcal{N})$ is called *trivial* if $\mathcal{N} = \{0, \mathcal{H}\}$. If $\mathcal{T}(\mathcal{N})$ is a nontrivial nest algebra and $N \in \mathcal{N} \setminus \{0, \mathcal{H}\}$, then $\mathcal{T}(\mathcal{N})$ can be represented as a triangular algebra of the form

$$\mathcal{T}(\mathcal{N}) = \begin{pmatrix} \mathcal{T}(\mathcal{N}_1) & E\mathcal{T}(\mathcal{N})(1-E) \\ & \mathcal{T}(\mathcal{N}_2) \end{pmatrix},$$

where E is the orthonormal projection onto N, $\mathcal{N}_1 = E(\mathcal{N})$ and $\mathcal{N}_2 = (1-E)(\mathcal{N})$. Note that $\mathcal{N}_1$ and $\mathcal{N}_2$ are nests of N and $N^{\perp}$, respectively. Moreover, $\mathcal{T}(\mathcal{N}_1) = E\mathcal{T}(\mathcal{N})E$ and $\mathcal{T}(\mathcal{N}_2) = (1-E)\mathcal{T}(\mathcal{N})(1-E)$ are nest algebras.

Corollary 3.4. *Let $\mathcal{S}$ be an algebra over the field $\mathbb{C}$ and let r be a nonzero complex number. Then every r-Jordan map $\Phi : \mathcal{T}(\mathcal{N}) \to \mathcal{S}$ is additive.*

References

[1] D. Aiat Hadj Ahmed and L. Ben Yakoub, Jordan automorphisms, Jordan derivations of generalized triangular matrix algebra, *Internat. J. Math. Math. Sci.* **13**, 2125–2132 (2005).

[2] K. I. Beidar, M. Brešar and M. A. Chebotar, Jordan isomorphisms of triangular matrix algebras over a connected commutative ring, *Linear Algebra Appl.* **312**, 197–201 (2000).

[3] D. Benkovič and D. Eremita, Commuting traces and commutativity preserving maps on triangular algebras, *J. Algebra* **280**, 797–824 (2004).

[4] W. S. Cheung, Commuting maps of triangular algebras, *J. London Math. Soc.* **63**, 117–127 (2001).

[5] K. R. Davidson, *Nest Algebras*, Pitman Research Notes in Mathematics Series **191**, Longman, Harlow (1988).

[6] F. Lu, Jordan maps on associative algebras, *Comm. Algebra* **31**(5), 2273–2286 (2003).

[7] L. Molnár, *Jordan maps on standard operator algebras*, in: Z. Daroczy and Zs. Pales (Eds.), Functional Equations-Results and Advances, Kluwer Academic Publishers (2001).

Author information

Rachid Tribak and Driss Aiat Hadj Ahmed, Centre Régional des Métiers de l'Education et de la Formation (CRMEF)-Tanger, Avenue My Abdelaziz, Souani, BP : 3117, Tangier, Morocco.
E-mail: `tribak12@yahoo.com, ait_hadj@yahoo.com`

Received: August 23, 2015

Accepted: October 13, 2015.