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Effect of Rotation on Radial Vibrations in an Unbounded Micro-isotropic, Micro-elastic Solid having a Spherical Cavity

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Abstract. The objective of this paper is to study the effect of rotation on the radial vibrations in an unbounded micro-isotropic, micro-elastic solid with a spherical cavity. Three dispersive relations are derived, in which one is effected by the rotation of the solid. All these relations are not encountered in any classical theory of elasticity. We noticed that the result of classical case is obtained as a particular case. The dispersion curves and phase speeds are shown graphically for against non-dimensional radius of the solid.

1 Introduction

Radial vibration studies are an important in consideration of both in theoretical and practical applications in several fields like geophysics, seismology. The problems of radial vibrations of isotropic elastic sphere and hollow sphere are discussed by Ghosh [1], Love [2] treatise contains an account of the forced vibrations of a sphere due to body forces derivable from a potential. Grey and Eringen [3] obtained the complete solution of sphere subject to dynamic surface tractions and computed the natural frequencies of the free oscillations. The sphere problem in connection with the problems of geodynamics considered by Love [4]. Propagation waves from a spherical cavity in an elastic solid with transverse isotropy about radius vector are discussed by Chakraborthy and Roy [5]. Wengler [6] studied the propagation of waves from a spherical cavity in an unbounded linear visco-elastic solid. Radial displacements of an infinite liquid saturated porous medium are derived by Kumar and Miglani [7]. In recent year, S.K.Tomar and Harinder Singh [8] studied the radial vibrations due to the spherical cavity in a micropolar elastic solid.

In this paper, we have investigated radial vibrations in a rotating unbounded micro-isotropic, micro-elastic solid with a spherical cavity. It is observed that the frequency equations are obtained which are not encountered in classical theory of elasticity. The effect of the rotation on radial frequency and phase speed are depicted graphically. Further, the result of classical case is obtained as a particular case of it.

2 Basic Equations

The basic governing equations of homogeneous isotropic, elastic medium are given [9] by the following: The balance of momentum equation is

$$(\lambda+\mu)u_{l,lk}+(\mu+K)u_{k,ll}+K\epsilon_{klm}\phi_{m,l}+\rho(f_k-\ddot{u}_k)=0. \tag{2.1}$$

The balance of stress momentum equation is

$$(\alpha+\beta)\phi_{l,lk}+\gamma\phi_{k,ll}+K\epsilon_{klm}u_{m,l}-2K\phi_k+\rho(l_k-J\ddot{\phi}_k)=0. \tag{2.2}$$

The balance of strain momentum equation is

$$B_1\phi_{pp,kk}\delta_{ij}+2B_2\phi_{(ij),kk}-A_4\phi_{pp}\delta_{ij}-2A_5\phi_{(ij)}-\frac{\rho J}{2}\ddot{\phi}_{(ij)}=0 \tag{2.3}$$

where $\vec{u}$ is the displacement vector, $\vec{\phi}$ is the micro-rotation vector, $\vec{f}$ is the body force, $\vec{l}$ is the body couple vector, ϕ_{pp}, $\phi_{(ij)}$ are micro-strains, ρ is the density, J is the micro-inertia, an index k following a comma indicates differentiation with respect to the coordinate (x_k), dot superposed on a symbol denotes differentiation with respect to the time t and $\lambda,\ \mu,\ K,\ B_1,\ B_2,\ A_4,\ A_5,\ \alpha,\ \beta,\ \gamma$ are material coefficients which satisfy the following inequalities,
$3\lambda + 2\mu + K \geq 0$, $2\mu + K \geq 0$, $K \geq 0$; $3\alpha + \beta + \gamma \geq 0$, $-\gamma < \beta < \gamma$, $\gamma \geq 0$;
$3A_4 + 2A_5 > 0$, $A_5 > 0$; $5B_1 + 4B_2$, $B_2 > 0$. The stress tensor t_{kl}, couple stress tensor m_{kl} and strain tensor $t_{k(mn)}$ are given by

$$t_{kl} = \lambda u_{r,r}\delta_{kl} + \mu(u_{k,l} + u_{l,k}) + K(u_{l,k} - \epsilon_{klr}\phi_r) \tag{2.4}$$

$$m_{kl} = \lambda\phi_{r,r}\delta_{kl} + \beta\phi_{k,l} + \gamma\phi_{l,k} \tag{2.5}$$

$$t_{k(mn)} = B_1\phi_{pp,k}\delta_{mn} + 2B_2\phi_{(mn),k} \tag{2.6}$$

where δ_{kl}, δ_{mn} are Kronecker's delta, ϵ_{klr} is the permutation symbol and () denotes the symmetric part.

3 Formulation and Solution of the Problem

We consider a spherical cavity of radius $r = a$ in a uniform micro-isotropic, micro-elastic medium of infinite extent. The medium is assumed to be rotating at a constant rate with constant angular velocity $\vec{\Omega} = (0, 0, \Omega)$ abut z-axis. When the medium undergoes dynamical deformation, the additional terms namely; the time dependent part of centripetal acceleration $\vec{\Omega} \times (\vec{\Omega} \times \vec{u})$ and the Coriolis acceleration $2(\vec{\Omega} \times \dot{\vec{u}})$.
We are interested only in radial vibrations (i.e., radial displacements, radial micro- rotation and radial micro-strain). So we take macro displacement vector, micro-rotation vector and micro strain as, $\vec{u} = (u, 0, 0)$, $\vec{\phi} = (\phi, 0, 0)$, ϕ_{rr} with

$$\vec{u} = u(r,t)\widehat{e_r} \tag{3.1}$$

$$\vec{\phi} = \phi(r,t)\widehat{e_r} \tag{3.2}$$

and

$$\phi_{rr} = \phi_{rr}(r,t) \tag{3.3}$$

where $\widehat{e_r}$ is the unit vector at the position vector in the direction of tangent to the r-curve. Under the absence of body forces and body couples the equations (2.1) to (2.6) reduce to

$$\frac{\partial^2 u}{\partial r^2} + \frac{2\partial u}{r\partial r} - \frac{2}{r^2}u = \frac{\rho}{(\lambda + 2\mu + K)}[\frac{\partial^2 u}{\partial t^2} - \Omega^2 u] \tag{3.4}$$

$$\frac{\partial^2 \phi}{\partial r^2} + \frac{2\partial \phi}{r\partial r} - \frac{2}{r^2}\phi - \frac{2K}{(\alpha + \beta + \gamma)}\phi = \frac{\rho J}{(\alpha + \beta + \gamma)}\frac{\partial^2 \phi}{\partial t^2} \tag{3.5}$$

$$B_1\nabla^2\phi_{rr} + 2B_2\nabla^2\phi_{rr} - A_4\phi_{rr} - 2A_5\phi_{rr} = \frac{\rho J}{2}\frac{\partial^2 \phi_{rr}}{\partial t^2} \tag{3.6}$$

$$B_1\nabla^2\phi_{rr} - A_4\phi_{rr} = 0 \tag{3.7}$$

$$t_{rr} = (\lambda + 2\mu + K)\frac{\partial u}{\partial r} + \frac{2\lambda}{r}u \tag{3.8}$$

$$m_{rr} = (\alpha + \beta + \gamma)\frac{\partial \phi}{\partial r} + \frac{2\alpha}{r}\phi \tag{3.9}$$

$$t_{r(rr)} = (B_1 + 2B_2)\frac{\partial \phi_{rr}}{\partial r} + B_1\frac{\partial}{\partial r}(\frac{\phi_{rr}}{r}) \tag{3.10}$$

In view of eq.3.7, eq.3.6 reduces to

$$2B_2\nabla^2\phi_{rr} - 2A_5\phi_{rr} = \frac{\rho J}{2}\frac{\partial^2\phi_{rr}}{\partial t^2} \tag{3.11}$$

where $\nabla^2 \equiv \frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r}$

3.1 Derivation of frequency equation for radial displacements

We seek the solution of eq. (3.4) of the form,

$$u(r,t) = R(r)e^{i\omega t} \tag{3.12}$$

where ω is angular frequency. Substituting eq. (3.12) in eq. (3.4) we get,

$$\frac{\partial^2 R}{\partial r^2} + \frac{2}{r}\frac{\partial R}{\partial r} - \frac{2}{r^2}R + \frac{\rho(\omega^2+\Omega^2)}{(\lambda+2\mu+K)}R = 0 \tag{3.13}$$

Suppose,

$$x = hr \tag{3.14}$$

$$\text{where } h^2 = \frac{\rho(\omega^2+\Omega^2)}{(\lambda+2\mu+K)}. \tag{3.15}$$

Under equation (3.14), the equation (3.13) reduces to

$$\frac{d^2R}{dx^2} + \frac{2}{x}\frac{dR}{dx} - \frac{2}{x^2}R + R = 0 \tag{3.16}$$

The general solution of eq. (3.16) is $R(x) = A\frac{d}{dx}(\frac{e^{ix}}{x})$, where x is given by eq. (3.14) and A is an arbitrary constant. Hence, by eq. (3.12) we obtain,

$$u(r,t) = A[\frac{i}{hr} - \frac{1}{(hr)^2}]e^{i(hr+\omega t)} \tag{3.17}$$

Substituting eq. (3.17) in the boundary condition $t_{rr} = 0$ at $r = a$, $t > 0$, we obtain

$$\rho(\omega^2+\Omega^2)(\lambda+2\mu+K)a^2 - (1-ai)(2\mu+K)^2 = 0, \tag{3.18}$$

which is the frequency equation corresponding to macro-displacement and it is depends on rotation of the solid. The frequency of classical result can be obtained as a particular case of it by allowing K tending to zero.

3.2 Derivation of frequency equation for radial micro-rotations

Now we seek the solution of eq. (3.5) in the form

$$\phi(r,t) = S(r)e^{i\omega t}. \tag{3.19}$$

Now eq. (3.5) reduces to

$$\frac{\partial^2 S}{\partial r^2} + \frac{2}{r}\frac{\partial S}{\partial r} + [\omega^2\rho J - \frac{2}{r^2} - \frac{2K}{(\alpha+\beta+\gamma)}]S = 0$$

this can be written as

$$\frac{\partial^2 S}{\partial r^2} + \frac{2}{r}\frac{\partial S}{\partial r} - \frac{2}{r^2}S + h_1^2 S = 0 \tag{3.20}$$

where

$$h_1^2 = \frac{\omega^2 \rho J - 4K}{2(\alpha+\beta+\gamma)} \tag{3.21}$$

$$\text{Let } \; y = h_1 r \tag{3.22}$$

Under eq. (3.22), the eq. (3.20) reduces to

$$\frac{d^2S}{dy^2} + \frac{2}{y}\frac{dS}{dy} - \frac{2}{y^2}S + S = 0 \tag{3.23}$$

The general solution of eq. (3.23) is, $S(y) = B\frac{d}{dy}(\frac{e^{iy}}{y})$
Hence by eq. (3.19), we obtain,

$$\phi(r,t) = B[\frac{ih_1 r - 1}{(h_1 r)^2}]e^{i(h_1 r + \omega t)} \tag{3.24}$$

where B is an arbitrary constant. Substituting eq. (3.24) in the boundary condition $m_{rr} = 0$ at $r = a$, $t > 0$, we obtain

$$a(\rho J\omega^2 - 2K)[(\alpha+\beta+\gamma)^{\frac{1}{2}} + 2i(\beta+\gamma)(\rho J\omega^2 - 2K)^{\frac{1}{2}}] = -2(2\alpha+\beta+\gamma)(\alpha+\beta+\gamma)^{\frac{3}{2}} \tag{3.25}$$

which is the frequency equation to radial micro-rotation. It is also involving elastic constants other than classical constants $\lambda,\ \mu$, so it is an additional wave which is not encountered in the classical elasticity.

3.3 Derivation of frequency equation for radial micro-strains

We seek the solution of eq. (3.11) in the form

$$\phi_{rr}(r,t) = T(r)e^{i\omega t}. \tag{3.26}$$

On substituting eq. (3.26) in eq. (3.11) we obtain,

$$\frac{\partial^2 T}{\partial r^2} + \frac{2}{r}\frac{\partial T}{\partial r} - h_2^2 T = 0 \tag{3.27}$$

where

$$h_2^2 = \frac{4A_5 - \rho J\omega^2}{4B_2} \tag{3.28}$$

Let

$$T(r) = r^{\frac{-1}{2}}U(r) \tag{3.29}$$

Substituting eq.(3.29) in eq. (3.27) we obtain,

$$r^2U'' + rU' - [h_2^2 r^2 + \frac{1}{4}]U = 0 \tag{3.30}$$

which can be expressed as

$$r^2U'' + rU' + [(ih_2)^2 - (\frac{1}{2})^2]U = 0 \tag{3.31}$$

It is Bessel equation, whose solution is

$$U(r) = L_1 J_{\frac{1}{2}}(ih_2 r) + L_2 Y_{\frac{1}{2}}(ih_2 r) \tag{3.32}$$

where $J_{\frac{1}{2}}()$, $Y_{\frac{1}{2}}()$ are Bessel functions with imaginary arguments and is written as,

$$U(r) = L_1 I_{\frac{1}{2}}(h_2 r) + L_2 K_{\frac{1}{2}}(h_2 r) \tag{3.33}$$

where L_1, L_2 are arbitrary constants. Substituting eq. (3.33) in eq. (3.29) we get

$$T(r) = r^{\frac{-1}{2}}[L_1 I_{\frac{1}{2}}(h_2 r) + L_2 K_{\frac{1}{2}}(h_2 r)] \tag{3.34}$$

Substituting eq. (3.34) in eq. (3.26) we obtain,

$$\phi_{rr}(r,t) = r^{\frac{1}{2}}[L_1 I_{\frac{1}{2}}(h_2 r) + L_2 K_{\frac{1}{2}}(h_2 r)]e^{i\omega t} \tag{3.35}$$

As $r \to \infty$, $\phi_{rr} \to \infty$, which is possible only, if $L_1 = 0$.
For large values of z we have $K_{\frac{1}{2}}(z) = (\frac{\pi}{2z})^{\frac{1}{2}} e^{-z}$.

So,

$$K_{\frac{1}{2}}(h_2 r) = (\frac{\pi}{2h_2 r})^{\frac{1}{2}} e^{-h_2 r} \tag{3.36}$$

Inserting eq. (3.36) in eq. (3.35) we obtain,

$$\phi_{rr}(r,t) = L_2(\frac{\pi}{2h_2})^{\frac{1}{2}} \frac{1}{r} e^{i\omega t - h_2 r} \tag{3.37}$$

Inserting eq. (3.37) in the boundary condition
$t_{r(rr)} = 0$ at $r = a$, we obtain,

$$B_1[(1 + h_2 a)(1 + a) + 1] + 2B_2(1 + h_2 a)a = 0 \tag{3.38}$$

which is a frequency equation corresponding to micro-strains and it is an additional wave not encountered in classical theory of elasticity.

4 Numerical Results and Discussion

In order to study numerically the dispersion relations and square phase speed $v^2 = \frac{\omega^2 a^2}{\pi^2}$, of micro-rotation, micro-strain and effect of angular rotation on dispersion relation and v^2 of radial displacements, we have consider the frequency equations (3.18), (3.25) and (3.38) with neglecting imaginary parts. To understand the problem in great detail numerically, we take the relevant values [10], are $\lambda = 7.59 \times 10^{10}\ dyne/cm^2$; $\mu = 1.89 \times 10^{10}\ dyne/cm^2$ $K = 0.014 \times 10^{10}\ dyne/cm^2$; $\beta = 0.0226 \times 10^{10}\ dyne/cm^2$; $\gamma = 0.0263 \times 10^{10}\ dyne/cm^2$; $J = 0.00196$; $\rho = 2.192$; $\alpha = 0.0214 \times 10^{10}\ dyne/cm^2$ (not mentioned in [10]); $B_1 = 0.0123$; $B_2 = 0.0156$; $A_5 = 0.0173$ (also not mentioned in [10]). We study the variation of frequency, square phase speed (for the effect of rotation Ω=0, 0.5,1) versus non- dimensional radius a with $10 \times 10^2 \le a \le 30 \times 10^2$. The variation of frequency, phase speed for angular rotation Ω are shown in fig.1, fig.2 respectively. We observe that natural frequency decrease and phase speed increase with increasing of rotation. The macro-displacement, micro-rotational frequency curves are shown in fig. 3, the micro-rotational, micro- isotropical frequencies are shown in fig.4. The macro-displacement, micro-rotational phase speeds are shown in fig. 5 and the micro-rotational, micro-isotropical phase speeds ares shown in fig.6 and we noticed that isotropical waves are propagate with constant speed and micro-rotational waves are slower than macro-displacement and isotropical waves.

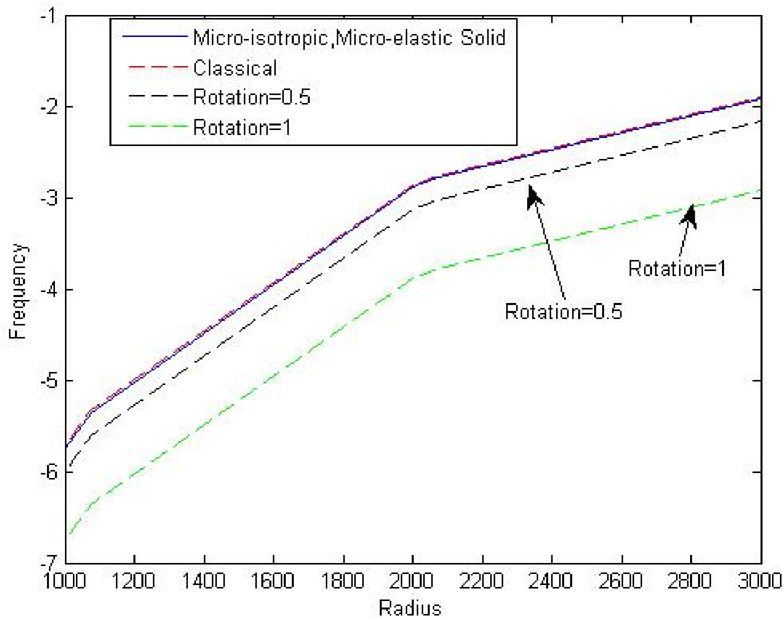

Figure 1. Variation of frequency versus radius

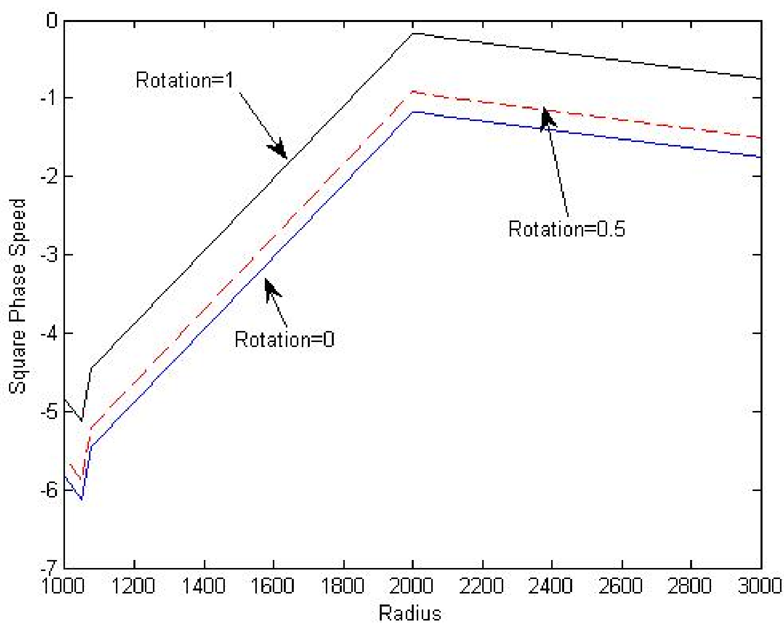

Figure 2. Variation of phase speed versus radius

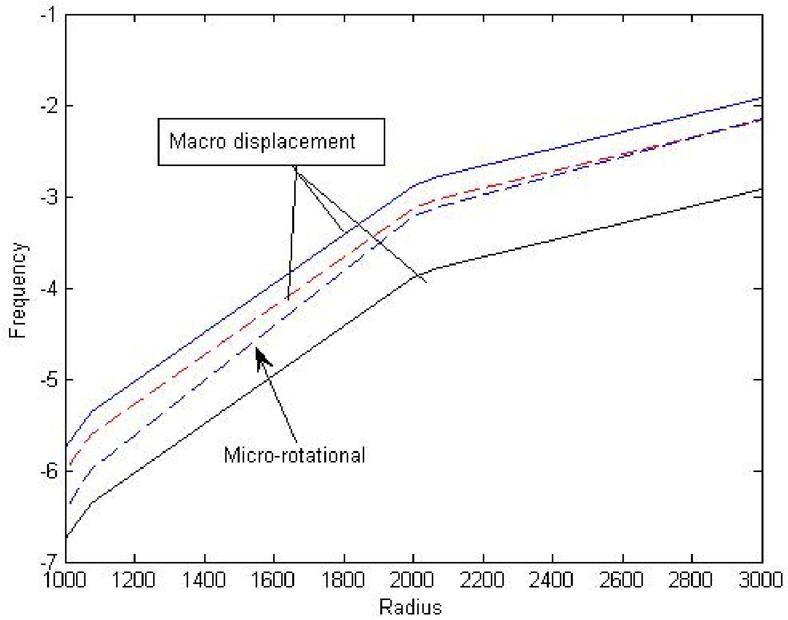

Figure 3. Variation of frequency versus radius

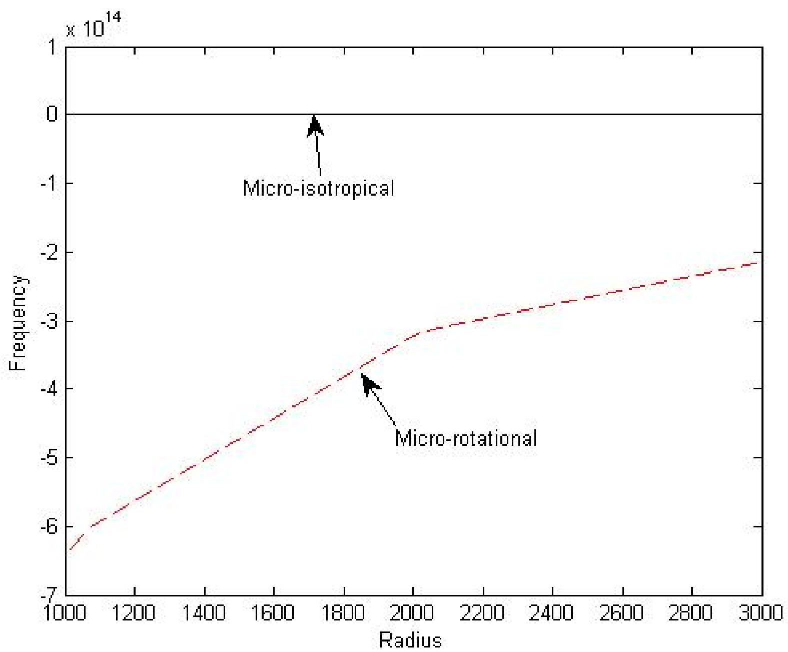

Figure 4. Variation of frequency versus radius

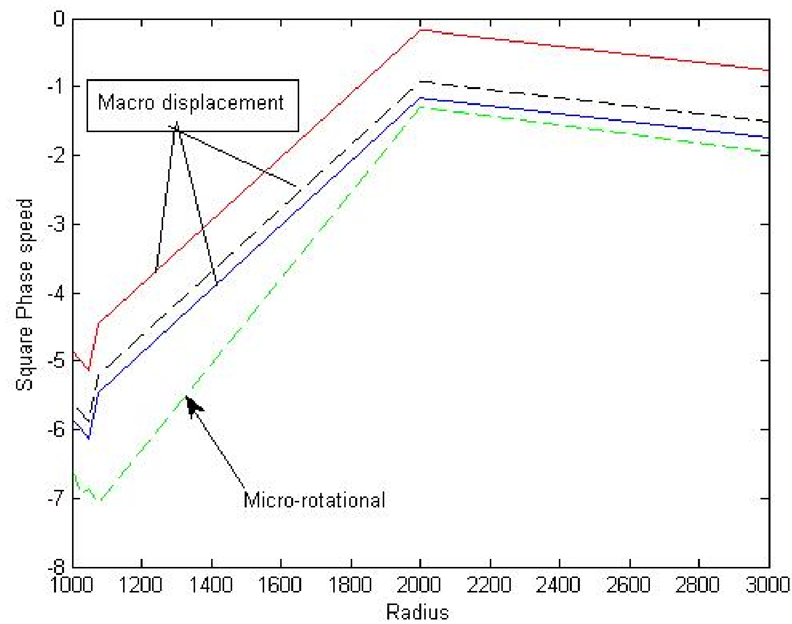

Figure 5. Variation of phase speed versus radius

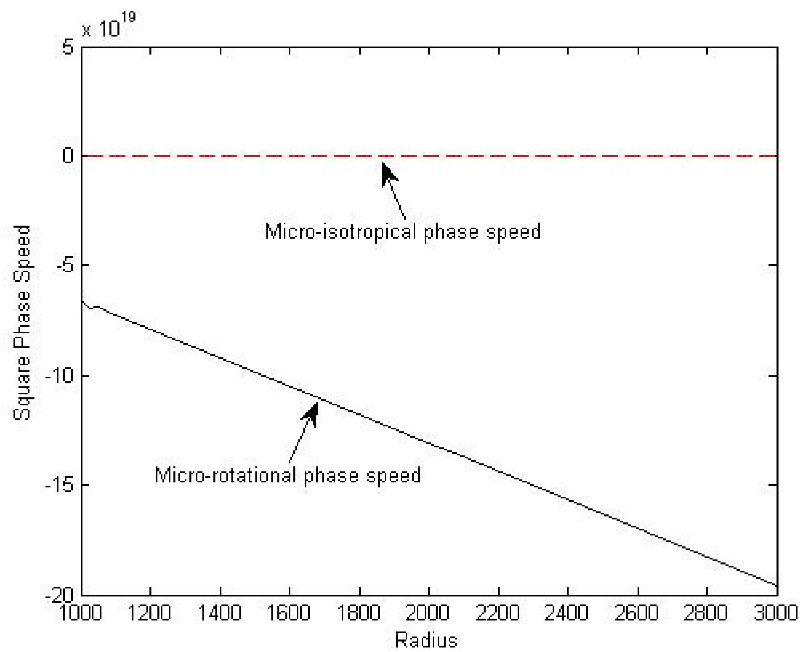

Figure 6. Variation of phase speed versus radius

5 Conclusions

This paper considers unbounded micro-isotropic, micro-elastic solid having a spherical cavity with measurable radius. In the study of radial vibrations, it is observed that: (i) Three frequency equations are derived, in which one is effected by angular rotation of the solid.
(ii) Comparative results are shown in graphically.
(iii) The natural frequency is decrease, while the phase speed is increase with the increasing rotation.
(iv) Isotropical waves are propagate with a constant speed.
(v) Micro-rotational waves are slower than macro-displacement and isotropical waves.

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