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When the Juxtaposition of Two Minimal Ring Extensions Produces No New Intermediate Rings

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Dedicated to the memory of Jack Ohm

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Abstract. Let $R \subset S$ and $S \subset T$ be minimal ring extensions of (commutative) rings. Shapiro and the author recently gave 13 mutually exclusive conditions on these minimal ring extensions and their crucial maximal ideals to characterize when $R \subset T$ satisfies FIP, that is, when $R \subset T$ has only finitely many intermediate rings. Here we show that exactly two of these 13 conditions imply that S is the only ring properly contained (via unital ring extensions) between R and T. Moreover, if one assumes, in addition, that R is quasi-local, we show that exactly two of the other 11 conditions imply that S is the only ring properly contained between R and T. In all, there are seven (of the 13) conditions which each implies that S is not the only ring properly contained between R and T. Also, for four of the 13 conditions, some examples satisfying the condition are such that S is the only ring properly contained between R and T while other examples satisfying the condition do not have this feature.

1 Introduction

This paper is a sequel to [12]. All rings considered below are commutative with identity; all subrings, inclusions of rings, ring- or algebra-homomorphisms, modules and submodules are unital. If $A \subseteq B$ is a ring extension, it is convenient to let $[A, B]$ denote the set of intermediate rings (that is, the set of rings C such that $A \subseteq C \subseteq B$). Recall from [1] that if $A \subseteq B$ is a ring extension, then $A \subseteq B$ is said to satisfy FIP if there are only finitely many rings contained between A and B (that is, if $|[A, B]| < \infty$). Whenever $A \subset B$ satisfies FIP, one has a finite (maximal) chain of rings $A = A_0 \subset \dots \subset A_i \subset A_{i+1} \subset \dots \subset A_n = B$ for some positive integer n, such that $A_i \subset A_{i+1}$ is a minimal ring extension for all $i = 0, \dots, n-1$. (As usual, $\subset$ denotes proper inclusion. Some useful background on minimal ring extensions will be given two paragraphs below.) Not all such "compositions" of minimal ring extensions produce a ring extension $A \subset B$ that satisfies FIP. In [12], Shapiro and the author focussed on the case $n = 2$. Indeed, if $R \subset S$ and $S \subset T$ are each minimal ring extensions, [12, Theorem 4.1] gave 13 mutually exclusive conditions on these minimal ring extensions and their crucial maximal ideals to characterize when $R \subset T$ satisfies FIP. As $|[R, T]| \geq 3$ in general, much of the subsequent material in [12] began to examine the relationship between each of the 13 conditions from [12, Theorem 4.1] and the possible conclusion that $|[R, T]| = 3$ (that is, the possible conclusion that S is the only ring that is properly contained between R and T).

The **main purpose of this note** is to further that examination. More precisely, we show in Theorem 2.2 that exactly two of the 13 conditions from [12, Theorem 4.1] imply that $|[R, T]| = 3$. Moreover, Theorem 2.4 shows that if one assumes, in addition, that R is quasi-local, then exactly two of the other 11 conditions from [12, Theorem 4.1] implies that $|[R, T]| = 3$. We also show in Theorem 2.9 that exactly seven of the 13 conditions from [12, Theorem 4.1] imply that $|[R, T]| > 3$. The remaining conditions are studied further in Proposition 2.10 (with complete answers for two of those conditions in Proposition 2.10 (b), (c)); see also Remark 2.11.

Recall (cf. [14]) that a ring extension $A \subset B$ is a *minimal ring extension* if there does not exist a ring properly contained between A and B. A minimal ring extension $A \subset B$ is either integrally closed (in the sense that A is integrally closed in B) or integral. If $A \subset B$

is a minimal ring extension, it follows from [14, Théorème 2.2 (i) and Lemme 1.3] that there exists a unique maximal ideal M of A (called the *crucial maximal ideal* of $A \subset B$) such that the canonical injective ring homomorphism $A_M \to B_M$ ($:= B_{A\setminus M}$) can be viewed as a minimal ring extension while the canonical ring homomorphism $A_P \to B_P$ is an isomorphism for all prime ideals P of A except M. A minimal ring extension $A \subset B$ is integrally closed if and only if $A \hookrightarrow B$ is a flat epimorphism (in the category of commutative rings). If $A \subset B$ is an integral minimal ring extension with crucial maximal ideal M, there are three possibilities: $A \subset B$ is said to be respectively *inert*, *ramified*, or *decomposed* if B/MB $(= B/M)$ is isomorphic, as an algebra over the field $K := A/M$, to a minimal field extension of K, $K[X]/(X^2)$, or $K \times K$. (As usual, X denote an indeterminate over the ambient base ring.)

The dedication of this note stems from a conversation between Jack Ohm and the author in November 1981. Having recently drafted [6] (where I had characterized the posets having a unique order-compatible topology), I asked Ohm's advice about the feasibility of characterizing the posets having only finitely many order-compatible topologies. (In view of his paper [26] with W. J. Lewis, Ohm was an expert on such matters.) His reply was that, in view of the success of Ferrand and Olivier [14] in studying minimal ring extensions, a far better question would be to ask to characterize the ring extensions $R \subset T$ having a unique proper intermediate ring. While that question remains open after more than 34 years, one can perhaps say that Theorem 2.2 gives an initial answer of sorts to Ohm's question when $R \subset T$ is viewed as having arisen from a juxtaposition of two minimal ring extensions. We would also note here the above-mentioned additional contributions in Theorem 2.4, Theorem 2.9 and Proposition 2.10. As explained in Remark 2.11 (a), a characterization of "$|[R,T]| = 3$" remains open in two of the 13 conditions listed in Theorem 2.1 (unless one settles for a trivial characterization that is recorded in Proposition 2.10 (a)). As one sees from Theorem 2.9, these are two (of the four) conditions for which there exist data satisfying this condition such that $|[R,T]| = 3$ and there exist other data satisfying this condition such that $|[R,T]| > 3$.

We assume that the reader has a copy of [12] at hand. Following [22, page 28], we let INC, LO and GU respectively denote the incomparable, lying-over and going-up properties of ring extensions. If A is a ring, then $\mathrm{Spec}(A)$ denotes the set of prime ideals of A. Any unexplained material is either taken from the Introduction of [12] or is standard, as in [16], [22].

2 Results

For ease of reference, we begin by restating the main classification result from [12]. There will be no need here to make explicit the cumbersome conditions from [12, Proposition 3.5] that are mentioned in parts (xii) and (xiii) of the following statement.

Theorem 2.1. ([12, Theorem 4.1]) Let $R \subset S$ and $S \subset T$ be minimal ring extensions, with crucial maximal ideals M and N, respectively. Then $R \subset T$ satisfies FIP if and only if (exactly) one of the following conditions holds:

(i) Both $R \subset S$ and $S \subset T$ are integrally closed.

(ii) $R \subset S$ is integral and $S \subset T$ is integrally closed.

(iii) $R \subset S$ is integrally closed, $S \subset T$ is integral, and $N \cap R \not\subseteq M$.

(iv) Both $R \subset S$ and $S \subset T$ are integral and $N \cap R \neq M$.

(v) Both $R \subset S$ and $S \subset T$ are inert, $N \cap R = M$, and either R/M is finite or there exists $\gamma \in T_M$ such that $T_M = R_M[\gamma]$.

(vi) $R \subset S$ is decomposed, $S \subset T$ is inert and $N \cap R = M$.

(vii) Both $R \subset S$ and $S \subset T$ are decomposed and $N \cap R = M$.

(viii) $R \subset S$ is inert, $S \subset T$ is decomposed, and $N \cap R = M$.

(ix) $R \subset S$ is ramified, $S \subset T$ is decomposed, and $N \cap R = M$.

(x) $R \subset S$ is decomposed, $S \subset T$ is ramified, and $N \cap R = M$.

(xi) $R \subset S$ is ramified, $S \subset T$ is inert, and $N \cap R = M$.

(xii) $R \subset S$ is inert, $S \subset T$ is ramified, $N \cap R = M$, and the two conditions stated in [12, Proposition 3.5 (a)] hold.

(xiii) Both $R \subset S$ and $S \subset T$ are ramified, $N \cap R = M$, and the two conditions stated in [12, Proposition 3.5 (b)] hold.

We next present our first main result. To follow the proof of Theorem 2.2, the reader should have at hand a copy of [12], as well as copies of several of the articles therein cited, such as [8], [9].

Theorem 2.2. Let $R \subset S$ and $S \subset T$ be minimal ring extensions, with crucial maximal ideals M and N, respectively. Then, of the 13 conditions in the statement of Theorem 2.1, the only ones of those conditions which imply that $|[R, T]| = 3$ are conditions (vi) and (xi).

Proof. (i): Let R be a one-dimensional Prüfer domain having exactly two distinct maximal ideals, say M and N. (The construction of such a domain is classical: cf. [25]; for a Noetherian, that is, a Dedekind, example of such a domain, $\mathbb{Z}_{\mathbb{Z}\setminus(2\mathbb{Z}\cup 3\mathbb{Z})}$ suffices.) Put $S := R_M$, and let T denote the quotient field of R. Since each overring of R (inside T) is an intersection of localizations of R (at prime ideals of R) by [16, Theorem 26.1 (2)], it follows that the set of overrings of R is $[R, T] = \{R, S, R_N, T\}$. Consequently, $|[R, T]| = 4$. Note that by [16, Theorem 26.1 (1)], S inherits the property of being a Prüfer domain from R. Finally, since any Prüfer domain is integrally closed (cf. [16, Theorem 24.3]), both $R \subset S$ and $S \subset T$ are integrally closed extensions.

(ii): See [12, Remark 4.2 (d)].

(iii): See [12, Remark 4.2 (e)].

(iv): See [12, Remark 4.2 (f)].

(v): Classical field theory provides numerous relevant examples. For instance, take $R := \mathbb{Q}$, $S := \mathbb{Q}(\sqrt{2})$ and $T := \mathbb{Q}(\sqrt{2}, \sqrt{3})$. Note that both $R \subset S$ and $S \subset T$ are inert since $[S : R] = 2 = [T : S]$; $N \cap R = M$ since $M = \{0\} = N$; and $T = \mathbb{Q}(\sqrt{2} + \sqrt{3}) = \mathbb{Q}[\sqrt{2} + \sqrt{3}]$, so that $\sqrt{2} + \sqrt{3}$ is a suitable γ. Finally, $|[R, T]| \neq 3$ since $\mathbb{Q}(\sqrt{3}) \in [R, T] \setminus \{R, S, T\}$.

(vi): Suppose that $R \subset S$ is decomposed, $S \subset T$ is inert and $N \cap R = M$. We will prove that $|[R, T]| = 3$. By [12, Proposition 3.1 (c), (d)], we may assume, without loss of generality, that (R, M) is quasi-local. Consider the field $K := R/M$. As $R \subset S$ is decomposed, the canonical ring homomorphism $f : K \to S/N$ is an isomorphism (cf. [9, Theorem 2.2]). It will be convenient to use f to identify S/N with K. Once again using the fact that $R \subset S$ is decomposed (and citing [9, Theorem 2.2] once again), we have that $S/M \cong R/M \times R/M$ canonically. Thus, we can identify S/M with $K \times K$. Next, note that since $S \subset T$ is inert, $L := T/N$ is a field and $K \subset L$ is a minimal field extension. Then it follows from the proof of [12, Proposition 3.3 (c)] that M is an ideal of T and $T/M \cong T/N \times R/M$ canonically (regardless of whether K is infinite). It is now harmless to identify T/M with $L \times K$. In addition, since $K \subset L$ is a minimal field extension, it is easy to verify that $K \times K \subset L \times K$ is an inert extension, with crucial maximal ideal $\{0\} \times K$. Thus, since a standard homomorphism theorem reveals that $|[R/M, T/M]| = |[R, T]|$, we can replace R with K, S with $K \times K$, and T with $L \times K$.

We will prove that if $A \in [K, L \times K] \setminus \{K\}$, then A is either $K \times K$ or $L \times K$ (that is, either S or T). If there exists $z \in (A \cap S) \setminus K$, then $A \supseteq K[z] = S$, whence the minimality of $S \subset T$ ensures that A is either S or T, as desired. Thus, without loss of generality, ($A \cap S = K$ and) there exists $w \in A \setminus S$. We can write $w = (x, y)$, with $x \in L \setminus K$ and $y \in K$. Recall that K is identified with the diagonal in $K \times K = S$. Hence, $(y, y) \in K \subset A$, and so $(x - y, 0) = (x, y) - (y, y) \in A \setminus S$. Thus, by *abus de langage*, we may take $y = 0$, so that $w = (x, 0) \in A \setminus S$ (and $x \in L \setminus K$). So,

$$A \supseteq K[w] = K[(x, 0)] = K + (xK[x] \times \{0\}).$$

Since $K[x] = L$ and x is a nonzero element of the field L, we have $xK[x] = xL = L$, and so $A \supseteq K + (L \times \{0\})$. The upshot is that if $c \in K$ and $d \in L$, then

$$(c + d, c) = (c, c) + (d, 0) \in K + (L \times \{0\}) \subseteq A.$$

Now, fix $c \in K$ and $e \in L$. Then $d := e - c \in L$, and so $(e, c) = (c + d, c) \in A$. In other words, $L \times K \subseteq A$, whence $A = T$, thus completing the proof of the assertion concerning condition (vi).

(vii): See [12, Remark 4.2 (h)].

(viii): See [12, Remark 4.2 (i)].

(ix): See [12, Remark 4.2 (j)].

(x): See [12, Remark 4.2 (k)].

(xi): Suppose that $R \subset S$ is ramified, $S \subset T$ is inert and $N \cap R = M$. We will prove that $|[R, T]| = 3$. As in the proof for condition (vi), an appeal to [12, Proposition 3.1 (c), (d)] allows us to assume, without loss of generality, that (R, M) is quasi-local. We will prove that if $A \in [R, T] \setminus \{R\}$, then A is either S or T. If there exists $z \in (A \cap S) \setminus R$, then $A \supseteq R[z] = S$, whence the minimality of $S \subset T$ ensures that A is either S or T, as desired. Thus, without loss of generality, $A \cap S = R$ and there exists $u \in A \setminus S$. Notice that $u \in T \setminus N$. Since the "inert" property of $S \subset T$ ensures that N is the unique maximal ideal of T, this means that u is a unit of T. To complete the proof concerning condition (xi), it will suffice to show that the above situation leads to a contradiction.

As $R \subset S$ is a ramified minimal ring extension, we get that $N \not\subseteq R$, whence $R + N = S$. Furthermore, since $u \notin S$, the minimality of $S \subset T$ gives that $S[u] = T$ and, moreover, that $R+N \subset R[u]+N = T$. In addition, since $R[u] \cap S = R$, it follows that $N \cap R[u] = N \cap R = M$. Consequently, we have the following ring isomorphism:

$$T/N = (R[u] + N)/N \cong R[u]/(R[u] \cap N) = R[u]/M.$$

In particular, M is an ideal of $R[u]$. Thus $R[u]M = M$, and so the (conductor) ring $(M :_T M) := \{w \in T \mid wM \subseteq M\}$ contains $R[u]$. Notice that $(M :_T M)$ also contains S, since M is an ideal of S. Hence, $(M :_T M)$ also contains $S[u]$ $(= T)$. Therefore, $(M :_T M) = T$. In other words, M is an ideal of T.

In fact, M is a common ideal of R, S and T. By [8, Lemma II.3], $R/M \subset S/M$ inherits the "minimal ring extension" property from $R \subset S$. Moreover, $R/M \subset S/M$ is ramified, with crucial maximal ideal 0. Similarly, we see that $S/M \subset T/M$ is a minimal ring extension; moreover, it is inert and has crucial maximal ideal N/M, since the extension $(S/M)/(N/M) \subset (T/M)/(N/M)$ can be identified with $S/N \subset T/N$. In addition, [8, Lemma II.3] ensures that $|[R, T]| = |[R/M, T/M]|$. Therefore, by replacing R with R/M, S with S/M, T with T/M, A with A/M, and u with the coset $u + M$, this change of notation has reduced us to the following situation: R is a field, which we conventionally denote by K; $S = K[X]/(X^2)$; $S \subset T$ is inert; $A \in [K, T] \setminus \{K\}$; and $u \in A$ is a unit of T such that $u \notin S$ and $K[u] \cap S = K$.

We claim that $L := K[u]$ is a field. Observe that $N \cap L = N \cap R[u] = N \cap R = M = \{0\}$. Hence,

$$T/N = (K[u] + N)/N = (L + N)/N \cong L/(N \cap L) = L/\{0\} \cong L.$$

The upshot is that $L \cong T/N$, which is a field, thus proving the above claim.

As usual, let x denote $X + (X^2) \in K[X]/(X^2)$ $(= S)$. Then $N = Kx$ and we can write $S = K \oplus Kx$ (additively). Also, $T = K[u] + N = L + Kx$. As $K \subset L$, we can pick $\xi \in L \setminus K$. Since T is a ring, the product $\xi x \in T$, and so there exist elements $a \in L$ and $b \in K$ such that $\xi x = a + bx$. Thus, $(\xi - b)x = a \in L$. Since $x^2 = 0$, we get that a is a nilpotent element of the field L, whence $a = 0$. It follows that $0 = (\xi - b)x \in Lx$. Therefore, since $Lx \cong L$ as L-vector spaces, $\xi - b = 0 \in L$, whence $\xi = b \in K$, the desired contradiction. This completes the proof of the assertion concerning condition (xi).

(xii): Consider the following data from [12, Remark 3.6 (a)]: $R := K$, $S := L$ and $T := L[X]/(X^2)$, where $K \subset L$ is a minimal (field) extension of finite fields. We have seen that condition (xii) applies to this set of data. It remains only to find a ring in $[R, T] \setminus \{R, S, T\}$. It is easy to verify that $(K + XL[X])/(X^2L[X])$ is such a ring.

(xiii): We will slightly modify the data from the second example constructed in [12, Remark 3.7 (b)] (which was an example that did *not* satisfy FIP). Begin with a finite field K. (This is the main difference from the second example in [12, Remark 3.7 (b)], whose construction began with an infinite field.) As before, let V be a two-dimensional K-vector space, and fix a one-dimensional K-subspace W of V. With $R := K$, consider the idealizations $S := R(+)W$ and $T := R(+)V$. Note that, contrary to the corresponding situation in [12, Remark 3.7 (b)], $R \subset T$ satisfies FIP, since the current hypotheses on K and V ensure that T is finite. Also, we can see directly that the "$N \cap R = M$" condition is satisfied, since $(0(+)W) \cap K = 0$.

On the other hand, much of the earlier argument carries over. Indeed, as in the proof of [12, Remark 3.7 (b)], it follows from [7, Remark 2.9] that $R \subset S$ is a minimal ring extension; from [27, Lemma 2.1] that both $R \subset S$ and $S \subset T$ are subintegral extensions, whence $R \subset S$ is ramified; from [27, Proposition 2.8 (3)] that $S \subset T$ is a minimal ring extension; that $S \subset T$ is an integral extension; and that $S \subset T$ is neither inert nor decomposed, and hence must be ramified. It remains only to find a ring in $[R, T] \setminus \{R, S, T\}$. To that end, pick a (necessarily one-dimensional) K-subspace U of V such that $V = W \oplus U$ as an internal direct sum of K-vector spaces. It is obvious that the ring $R(+)U$ is in $[R, T] \setminus \{R, S, T\}$. □

Despite the example given in Theorem 2.2 (i), the verdict would have been different in that context if the base ring R had been constrained to be quasi-local. (As recalled in the proof of Theorem 2.4 below, the modern theory of normal pairs essentially reduces that context to the archetypical example where R is a two-dimensional valuation domain with one-dimensional overring S and quotient field T.) This naturally leads one to ask if there are any *other* conditions from the statement of Theorem 2.1 where the verdict in Theorem 2.2 would have been different for the case of a quasi-local base ring R. Theorem 2.4 answers this question. Part (ii) of its proof will depend on Lemma 2.3, which is of some independent interest. We have benefitted from access to a preprint of [3], as [3, Corollary 2.8] established the special case of Lemma 2.3 where B is a domain. We do not believe that the domain-theoretic methodology of [3] can be adapted to yield a proof of Lemma 2.3 in the generality that is given below.

Lemma 2.3. Let $A \subset B$ be a ring extension, with A^* denoting the integral closure of A in B. Suppose that A is quasi-local, $A \subset A^*$ is a minimal ring extension, and (A^*, B) is a normal pair. Then A^* is the least element in $[A, B] \setminus \{A\}$; that is, $A^* \subseteq C$ for each $C \in [A, B] \setminus \{A\}$.

Proof. Suppose not. Then, since $A \subset A^*$ is a minimal ring extension, there exists $D \in [A, B] \setminus \{A\}$ with $A^* \cap D = A$. (The preceding observation was also made in the proof of [3, Corollary 2.8].) Pick $u \in D \setminus A$. We claim that $A \subset A[u]$ has INC. Suppose this claim fails. Then we can pick prime ideals $Q_1 \subset Q_2$ of $A[u]$ lying over the same prime ideal P of A. Since $A[u] \subseteq A^*[u]$ is an integral extension, it has LO and GU (cf. [22, Theorem 44]), and so $Q_1 \subset Q_2$ can be covered by some chain $q_1 \subset q_2$ of prime ideals of $A^*[u]$. As (A^*, B) is a normal pair, so is $(A^*, A^*[u])$. Thus, $A^* \subseteq A^*[u]$ is a P-extension, in the sense of [18]. (This follows from [23, Chapter I, Theorem 5.2]; the cited result is especially noteworthy, as it contains numerous characterizations of normal pairs of arbitrary rings.) Hence, by [5, Theorem], $A^* \subseteq A^*[u]$ has INC. So, the chain $q_1 \subset q_2$ contracts to distinct prime ideals $p_1 \subset p_2$ of A^*. This chain lies over P, contradicting the fact that $A \subseteq A^*$ (being integral) has INC. This proves the above claim.

By [5, Theorem], u is primitive over A; that is, u is the root of some polynomial in $A[X]$ having a unit coefficient. By the (u, u^{-1})-Lemma (as generalized to rings, for instance, as in [9, Lemma 3.8]), u^{-1} belongs to the maximal ideal, say m, of A. Consequently, every prime ideal of D (or, similarly, of $A[u]$) is a (prime) ideal of A; that is, $\mathrm{Spec}(A[u]) \subseteq \mathrm{Spec}(A)$ as sets. Therefore, by the claim, each prime ideal of $A[u]$ is isolated in its fiber (above itself in $\mathrm{Spec}(A)$). Since A is integrally closed in $A[u]$, it follows from Zariski's Main Theorem (as in [13]) that there exists $f \in A \setminus m$ such that $A_f = A[u]_f$ canonically (that is, the canonical A-algebra homomorphism $A_f \to A[u]_f$ is an isomorphism). As f is a unit of A, this means that $A = A[u]$ canonically, contradicting $u \notin A$. □

Theorem 2.4. Let $R \subset S$ and $S \subset T$ be minimal ring extensions, with crucial maximal ideals M and N, respectively. Assume, in addition, that R is quasi-local. Then, of the 13 conditions in the statement of Theorem 2.1, the only ones of those conditions which imply that $|[R, T]| = 3$ (given that R is quasi-local) are conditions (i), (ii), (vi) and (xi).

Proof. The arguments for the 9 conditions (iii), (iv), (v), (vii), (viii), (ix), (x), (xii) and (xiii) which were given in the proof of Theorem 2.2 either featured quasi-local R or were valid in the case of quasi-local R. Hence, each of those 9 arguments can be applied here as well. (Alternatively, instead of rereading each of those 9 arguments, one could obtain the same overall conclusion via [12, Proposition 3.1 (c), (d)].) Of course, the assertions for condition (vi) and (xi) also carry over from Theorem 2.2 (where the earlier proofs concerning (vi) and (xi) were valid regardless of whether R is quasi-local). It remains only to prove the assertions for the conditions (i) and (ii).

(i): Suppose that $R \subset S$ and $S \subset T$ are each integrally closed minimal ring extensions and that R is quasi-local. We will prove that $|[R, T]| = 3$. Much of the analysis proceeds as in [12, Remark 2.2 (a)]. Indeed, as in that earlier work, we see via [23, Theorem 5.6, Chapter I] that (R, T) is a normal pair, in the sense of [4] (that is, each $C \in [R, T]$ is integrally closed in T); via the pullback characterization of normal pairs with a quasi-local base ring [9, Theorem 6.8] that R has a divided prime ideal Q (that is, $Q \in \mathrm{Spec}(R)$ such that $QR_Q = Q$) such that $T = R_Q$ and $D := R/Q$ is a valuation domain; and that D has Krull dimension 2, its quotient field is T/Q, and the one-dimensional overring of D (inside T/Q) is S/Q.

The argument can now be finished quickly. By a standard homomorphism theorem, there is a bijection between $[R, T]$ and $[R/Q, T/Q]$. Consequently, $|[R, T]| = |[R/Q, T/Q]| = |\{R/Q, S/Q, T/Q\}| = 3$, as asserted.

(ii): Suppose that $R \subset S$ is an integral minimal ring extension, $S \subset T$ is an integrally closed minimal ring extension, and R is quasi-local. To prove that $|[R, T]| = 3$, it suffices to show that if $C \in [R, T] \setminus \{R\}$, then $S \subseteq C$ (for then the minimality of $S \subset T$ would ensure that C is either S or T). This, in turn, is an immediate consequence of Lemma 2.3 (which applies since S is the integral closure of R in T). □

Remark 2.5. (a) As was the case with [3, Corollary 2.8], we see that Lemma 2.3 generalizes the well known result [17, Theorem 2.4] of Gilmer and Heinzer that if A is a quasi-local domain with quotient field B, with A^* denoting the integral closure of A (in B), such that A^* is a Prüfer domain and $A \subset A^*$ is a minimal ring extension, then A^* is contained in each proper overring of A (which is contained in B).

(b) In [17, page 138], Gilmer and Heinzer remark that if A is a domain with quotient field B and a proper overring C (inside B) such that C is contained in each proper overring of A, then A is quasi-local. This leads us to the following question: would the assertion in Lemma 2.3 fail if one deletes the hypothesis that A is quasi-local? The answer is in the affirmative, even if one also assumes that A is Noetherian and of Krull dimension 1. To see this, it suffices to consider the data in [12, Remark 4.2 (d)], where $\mathbb{Z}[2i]$ plays the role of a suitable base ring.

(c) The fact that each of the conditions (i) and (ii) behaved differently in Theorem 2.4 than it had in Theorem 2.2 reveals one sense in which the organization of [12] cannot be made significantly more efficient. To wit: the conclusions in [12, Proposition 3.1 (c), (d)] cannot be extended to the contexts where at least one of the minimal ring extensions $R \subset S$, $S \subset T$ is not integral.

The introduction noted that Theorem 2.2 gave "an initial answer of sorts" to Ohm's question, but it seems clear that one should try to say more about the remaining ambiguities. The process of working toward a more complete answer to Ohm's question actually began in Theorem 2.4. In Theorem 2.9, we will supplement the above results with a another one of our main (classification) results. Note that we have placed the "$|[R, T]| = 3$"-related discussion of condition (ix) from Theorem 2.1 into a separate result, Theorem 2.8, because the discussion for that condition is conspicuously harder (and perhaps more interesting) than the discussions for any of the other conditions. Prior to the proof of Theorem 2.8, it will be convenient to isolate two lemmas.

Lemma 2.6. Let (R, M) and (S, N) be quasi-local rings such that $N^2 = 0$ and S is a (unital) ring extension of R. Then the following conditions are equivalent:

(1) $R \subset S$ is a(n integral) ramified (minimal) ring extension (with crucial maximal ideal M);

(2) S is R-algebra isomorphic to the idealization $R(+)R/M$.

Proof. (2) ⇒ (1): Assume (2). It is then harmless to view $S = R(+)R/M$. By [7, Corollary 2.5], $R \subset S$ is a(n integral) minimal ring extension (necessarily with crucial maximal ideal M). Lastly, to show that $R \subset S$ is ramified, one need only appeal to [27, Lemma 2.1] (which establishes the more general fact that if E is any module over a ring A, then the ring extension $A \subseteq A(+)E$ is subintegral).

(1) ⇒ (2): Assume (1). Since $R \subset S$ is ramified, [11, Proposition 2.12] provides $q \in S \setminus R$ such that $S = R[q], q^2 \in R, q^3 \in R$ and $Mq \subseteq R$. It follows easily from the proof of [11, Proposition 2.12] that we can pick such an element $q \in N$. As $N^2 = 0$, it follows that $q^2 = 0 = q^3$, whence $S = R[q] = R + Rq$.

We claim that $R \cap Rq = 0$. Suppose the claim fails. Then there exist nonzero elements $r_1, r_2 \in R$ such that $r_1 = r_2 q \in Sq \subseteq N$. As $N^2 = 0$ and $r_1 \neq 0$, we have $r_2 \in (S \setminus N) \cap R \subseteq R \setminus M$; that is, r_2 is a unit of R. Consequently, $q = (r_2)^{-1} r_1 \in R \cap N = M$, whence $S = R[q] \subseteq R[M] = R \subset S$, the desired contradiction. This completes the proof of the above claim.

Combining the facts that $S = R + Rq$, $R \cap Rq = 0$ and $q^2 = 0$, we see easily that S is R-algebra isomorphic to the idealization $R(+)Rq$. It is now harmless to view $S = R(+)Rq$. To complete the proof, it will suffice to show that $Rq \cong R/M$ as R-modules. In fact, since $R \subset R(+)Rq$ is a minimal ring extension, it follows from [7, Remark 2.9] that Rq is a simple

R-module. But, up to R-module isomorphism, R/M is the only simple R-module, whence $Rq \cong R/M$, as required. $\square$

Lemma 2.7 was motivated by the treatment in [21, Theorem 2.25] of the Chinese Remainder Theorem for "rings" possibly without identity. It is interesting to note that the converse of Lemma 2.7 can be extracted from the proof of [28, Theorem 18, page 280].

Lemma 2.7. Let A be a ring and W an A-module, with U and V being A-submodules of W. If the canonical A-module homomorphism $f : W \to W/U \oplus W/V$ is surjective, then $U + V = W$.

Proof. Suppose, on the contrary, that there exists $w \in W \setminus (U + V)$. By hypothesis, there exists $\xi \in W$ such that $f(\xi) = (w+U, V)$; that is, such that $(\xi+U, \xi+V) = (w+U, V)$. Consequently, $w - \xi \in U$ and $\xi \in V$. Hence, $w = (w - \xi) + \xi \in U + V$, the desired contradiction. $\square$

The proof of Theorem 2.8 will use the following definition from [15]. A ring extension $A \subseteq B$ is called a λ-*extension* if the poset $[A, B]$ is linearly ordered (with respect to inclusion). If $A \subseteq B$ are rings such that $|[A, B]| \leq 3$, then $A \subseteq B$ is a λ-extension; the converse is false.

Theorem 2.8. Let R, S, T, M and N satisfy condition (ix) in the statement of Theorem 2.1; that is, let $R \subset S$ be a(n integral) ramified (minimal) ring extension with crucial maximal ideal M and let $S \subset T$ be a(n integral) decomposed (minimal) ring extension with crucial maximal ideal N such that $N \cap R = M$. Then $|[R, T]| > 3$ (and $|[R, T]| < \infty$).

Proof. By [12, Proposition 3.1 (a), (c), (d)], we can assume, without loss of generality, that R is quasi-local, with unique maximal ideal M. Since $R \subset S$ is ramified, (S, N) is quasi-local, with $N^2 \subseteq M \subset N$. Also, since $S \subset T$ is decomposed, T has exactly two distinct maximal ideals, say Q_1 and Q_2; and $Q_1 \cap Q_2 = N$. Note that $Q_1 \not\subseteq N$ (for otherwise, $Q_1 \subseteq Q_2$, contradicting the fact that the extension $S \subset T$, being integral, satisfies INC); similarly, $Q_2 \not\subseteq N$.

Now, consider the conductor $C := (R : T)$. Observe that $C \subseteq M$ ($\subseteq N$). Moreover, the integral extension $R/C \subset T/C$ inherits FIP from $R \subset T$ (cf. [8, Proposition II.4]). As FIP implies the FCP condition, the extension $R/C \subset T/C$ satisfies FCP, and so [9, Theorem 4.2 (a)] ensures that R/C is an Artinian ring. Since S/C and T/C are each algebra-finite integral ring extensions, hence module-finite ring extensions, of R/C, it follows that S/C and T/C are also Artinian rings. Using a standard homomorphism theorem, it is easy to verify that $R/C \subset S/C$ (resp., $S/C \subset T/C$) is a ramified (resp., decomposed) extension having crucial maximal ideal M/C (resp., N/C). Therefore, since the assignment $A \mapsto A/C$ gives a bijection $[R, T] \to [R/C, T/C]$, one can, without loss of generality, replace the tower $R \subset S \subset T$ with the tower $R/C \subset S/C \subset T/C$. Thus, we can henceforth assume that R, S and T are Artinian rings and $(R : T) = 0$. Note that this change of notation leaves unchanged the facts that $R \subset S$ is ramified, with crucial maximal ideal M; $S \subset T$ is decomposed, with crucial maximal ideal N; and $N \cap R = M$. In addition, the above-noted facts connecting N, Q_1 and Q_2 remain unaltered.

As $M = (R : S)$, we have that $M^2 Q_i = M(MQ_i) \subseteq M(Q_1 Q_2) = M(Q_1 \cap Q_2) = MN \subseteq MS = M \subseteq R$ for $i = 1, 2$. Also, if u is a unit of T, then $M^2 u = M(uM) \subseteq M(uN) = MN \subseteq R$. Consequently, $M^2 T \subseteq R$; that is, $M^2 \subseteq (R : T) = 0$. Thus, $M^2 = 0$.

There are two cases. In the first (and much easier) case, $M = 0$; that is, R is a field. Respecting convention, we let $K := R$. As $K \subset S$ is ramified, we may also view $S := K[X]/(X^2)$. We will obtain a contradiction from the assumption that $|[K, T]| = 3$. Let us first consider the subcase where the K-algebra T is decomposable (as a direct product of at least two nonzero K-algebras). Then, since the hypothesis $|[K, T]| = 3$ ensures that $K \subset T$ is a λ-extension, [15, Theorem 3.4] gives that $T \cong K \times L$ as K-algebras, for some field L (such that $K \subset L$ is a λ-extension). It is now harmless to view $T = K \times L$. Therefore, T has no nonzero nilpotent elements, contradicting the presence of $X + (X^2) \in S \subset T$. In the remaining subcase, T is an indecomposable K-algebra. As T is an Artinian ring, it follows from the structure theorem for Artinian rings in [28, Theorem 3, page 205] that T is (quasi-)local, contradicting that Q_1 and Q_2 are (the) distinct maximal ideals of T. This completes the proof that $|[R, T]| > 3$ when $M = 0$.

In the remaining case, $M \neq 0$ $(= M^2)$. As $(R : T) = 0$, it follows that $MT \not\subseteq R$. On the other hand, the fact that $N = (S : T)$ gives that $MT \subseteq NT = N \subseteq S$, and so the minimality of the ring extension $R \subset S$ ensures that $R + MT = S$. Therefore, $R + MT = S = R + N$, with $MT \subseteq N$. We claim that, in fact, $MT = N$. To see this, consider any element $n \in N$. One has $n = r + \sum m_i t_i$ for some $r \in R$ and some finitely many elements $m_i \in M$, $t_i \in T$. Then $r = n - \sum m_i t_i \in N + MT = N$, so that $r \in N \cap R = M$ and $n \in M + MT = MT$, thus proving the claim that $N = MT$. As a consequence, we have $N^2 = M^2 T = 0T = 0$. Hence, by Lemma 2.6, S is R-algebra isomorphic to $R(+)R/M$. It is harmless to identify $S = R(+)R/M$.

We turn next to a finer description of the structure of T. As the extension $S \subset T$ is decomposed, [11, Proposition 2.12] provides $u \in T$ such that $T = S[u]$, $u^2 - u \in N$ and $uN \subseteq S$. Observe that N $(= Q_1 \cap Q_2)$ is the nilradical of T. Since idempotents can be lifted modulo any nil ideal (cf. [24, Proposition 1, page 72]), there exists $e = e^2 \in T$ such that $u - e \in N$. As $u \in e + N \subseteq S[e]$, it follows that $S[e] = T$. Note that u is neither 0 nor 1 (since $S \neq T$). As $u(1-u) = 0$, it follows that T can be expressed as the internal ring direct product $T = Tu \times T(1-u)$, where neither Tu nor $T(1-u)$ is 0. The canonical projection maps show that Tu and $T(1-u)$ are each homomorphic images of T and, hence, are each Artinian rings. In addition, Tu and $T(1-u)$ are each (quasi-)local, since T has only two maximal ideals. Let q_1 denote the unique maximal ideal of Tu and let q_2 denote the unique maximal ideal of $T(1-u)$. From now on, it will be convenient to write $A := Tu$ and $B := T(1-u)$.

Recall that the prime ideals of T are Q_1 and Q_2; they can also be described by $q_1 \times B$ and $A \times q_2$. (It will occasionally be convenient to regard the elements of T as the ordered pairs in the external direct product $Tu \times T(1-u) = A \times B$.) For the moment, we choose notation so that $Q_1 = q_1 \times B$ and $Q_2 = A \times q_2$. Regardless of whether these descriptions for Q_1 and Q_2 may eventually need to be interchanged, we have $N = Q_1 \cap Q_2 = q_1 \times q_2$.

Recall that $T = S[e]$ for some idempotent element $e \in T$. We can write $e = e_1 + e_2$, where e_1 and e_2 are idempotent elements of A and B, respectively (such that $e_1 e_2 = 0$). By applying the canonical first projection $T \to A$, we get that $A = S[e_1]$. (This notation does not necessarily mean that $S \subseteq A$; it means only that A is generated as an S-algebra by e_1.) Now, since A is quasi-local and any quasi-local ring has only 0 and 1 as its idempotent elements, it follows that A is a surjective (ring) homomorphic image of S. Therefore, A is S-algebra isomorphic to S/I for some ideal I of S. There is no harm in identifying $A = S/I$. Similarly, by using the canonical second projection $T \to B$, we can identify $B = S/J$ for some ideal J of S. Note that neither I nor J coincides with S, since T is not quasi-local (and hence cannot be isomorphic to either A or B). Consequently, I and J are each contained in N, since S is quasi-local. Furthermore, I (resp., J) is uniquely determined as the annihilator of the S-module A (respectively, B). Also, since the canonical homomorphism $h : S \to T = A \times B = S/I \times S/J$ is injective (it is essentially an inclusion map!), it follows that $I \cap J = 0$. We claim that $I + J = N$.

The above homomorphism h restricts to an injective S-module homomorphism $h^* : N \to N/I \times N/J$; we will use h^* to view $N \subseteq N/I \times N/J$. Note that N/I is the (unique) prime ideal of $S/I = A$; that is, $N/I = q_1$. Similarly, $N/J = q_2$. Therefore, $N \subseteq N/I \times N/J = q_1 \times q_2 = N$. It follows that the inclusion $N \subseteq N/I \times N/J$ is actually an equality of sets; in other words, h^* is an isomorphism and, in particular, surjective. This places our current data into the context of Lemma 2.7, an application of which gives that $I + J = N$, thus proving the above claim.

Because of the existence of the chain $R \subset T$ of minimal ring extensions, it is clear that $|[R,T]| = 3$ if and only if the poset $[R,T]$ is linearly ordered (under inclusion); that is, if and only if $R \subset T$ is a λ-extension. We now proceed by *reductio ad absurdum*. In other words, we suppose that $|[R,T]| = 3$ (equivalently, that $R \subset T$ is a λ-extension) and it will suffice to produce a contradiction. To do so, we will need to consider some (sub)cases (for each of which, we will obtain a contradiction). The analysis of these cases will depend upon a key result of Gilbert [15, Theorem 2.12]. Application of this result require the use of the following definitions from [15], as applied to the direct product $T = A \times B$. Following [15], we put $T_1 := A$ and $T_2 := B$. Let π_1 and π_2 denote the canonical projections $T \to T_1$ and $T \to T_2$, respectively. Let j denote the inclusion map $R \hookrightarrow T$. For $i = 1, 2$, consider the kernels $I_i := \ker(\pi_i \circ \mathrm{j})$ $(\subseteq R)$; and the images $R_i := \mathrm{im}(\pi_i \circ \mathrm{j})$ $(\subseteq T_i)$. Of course, $R/I_i \cong R_i$, which is a subring of T_i. Also, $J_1 := \pi_1(I_2)$, which is an ideal of R_1; and $J_2 := \pi_2(I_1)$, which is an ideal of R_2.

We can now state the application of [15, Theorem 2.12] that is appropriate to our context. The ring extension $R \subset T$ is a λ-extension if and only if (after possibly relabeling by interchanging T_1 and T_2) we have that $R_1 = T_1$ and the following set $\mathcal{T}$ is linearly ordered by inclusion: $\mathcal{T} := [R_2, T_2] \cup \{G \mid G$ is a proper, possibly zero, ideal of at least one ring in $[R_2, T_2]$ such that $G \supseteq J_2\}$. Determining the elements of $\mathcal{T}$ will require effectively using the above definitions to calculate R_2 and J_2 (or, if we needed to interchange T_1 and T_2, calculating R_1 and J_1). To do so in each of the promised (sub)cases, we will need more precise descriptions of the ideals I and J of $S(+)R/M$. We turn next to that task.

It will be convenient to let $K := R/M$. According to [19, Theorem 25.1 (1), (2)], there exist ideals $\mathcal{I}$ and $\mathcal{J}$ of R and R-submodules C and D of K such that $I = \mathcal{I}(+)C$, $J = \mathcal{J}(+)D$, $\mathcal{I}K \subseteq C$, and $\mathcal{J}K \subseteq D$. Since $I \cap J = 0$ and $I + J = N$ $(= M(+)K)$, we have that

$$\mathcal{I} \cap \mathcal{J} = 0, C \cap D = 0, \mathcal{I} + \mathcal{J} = M, C + D = K, \mathcal{I}K \subseteq C, \text{ and } \mathcal{J}K \subseteq D.$$

The final two conditions displayed above are automatically satisfied, given the other conditions, since the fact that $\mathcal{I}$ and $\mathcal{J}$ are each contained in M ensures that $\mathcal{I}K = 0 = \mathcal{J}K \subseteq C \cap D$. Next, note that since K is a simple R-module, C and D are each members of $\{0, K\}$. Thus, the restrictions on the above data are as follows: $\mathcal{I} \cap \mathcal{J} = 0$; $\mathcal{I} + \mathcal{J} = M$; and exactly one of C, D is K (while the other one of C, D is 0).

In any event, we have $I = \mathcal{I}(+)W$, for some $W \in \{0, K\}$. Then, as $S = R(+)K$ and $A = S/I$, we also have $R_1 = \text{im}(\pi_1 \circ \mathrm{j}) = (R + I)/I =$

$$((R(+)0) + (\mathcal{I}(+)W))/(\mathcal{I}(+)W) \cong (R(+)W)/(\mathcal{I}(+)W) \cong R/\mathcal{I}(+)\{0\};$$

similarly, $T_1 = S/I = (R(+)K)/(\mathcal{I}(+)W) \cong R/\mathcal{I}(+)K/W$. Therefore, $R_1 = T_1$ if and only if $K/W = 0$; that is, if and only if $I = \mathcal{I}(+)K$.

Next, it is important to mention that the proof of [15, Theorem 2.12] reveals the following. If a particular permutation/relabeling of A, B leads to $R_1 = T_1$ then *without any further permutation*, the characterization of $R \subset T$ being a λ-extension is exactly as stated above, with $\mathcal{T}$ being stated in terms of $[R_2, T_2]$ (rather than possibly in terms of $[R_1, T_1]$). Accordingly, the plentitude of (sub)cases really amounts to just one (sub)case, namely, the following: ideals $\mathcal{I}$ and $\mathcal{J}$ of R such that $I = \mathcal{I}(+)K$, $J = \mathcal{J}(+)0$, $\mathcal{I} \cap \mathcal{J} = 0$, and $\mathcal{I} + \mathcal{J} = M$. To complete the proof (by obtaining the desired contradiction), it remains only to find a pair of incomparable elements of the set $\mathcal{T}$.

We have $T_2 = B = S/J = (R(+)K)/(\mathcal{J}(+)0) \cong R/\mathcal{J}(+)K$ and, by adapting the above reasoning, $R_2 = \text{im}(\pi_2 \circ \mathrm{j}) = (R(+)0)/(\mathcal{J}(+)0) \cong R/\mathcal{J}(+)0$ (viewed canonically as a unital subring of $R/\mathcal{J}(+)K = T_2$). In addition, $I_1 = \ker(\pi_1 \circ \mathrm{j}) = \{r \in R \mid (r, 0) \in \mathcal{I}(+)K\} = \mathcal{I}$. Hence,

$$J_2 = \pi_2(I_1) = \pi_2(\mathcal{I}) = ((\mathcal{I}(+)0) + J)/J = ((\mathcal{I} + \mathcal{J})(+)0)/(\mathcal{J}(+)0),$$

which is isomorphic to $(((\mathcal{I}+\mathcal{J})/\mathcal{J})(+)0)/(\mathcal{J}(+)0) \cong (\mathcal{I}+\mathcal{J})/\mathcal{J}(+)0 = M/\mathcal{J}(+)0$, viewed inside $R/\mathcal{J}(+)0 = R_2$. Thus, according to the above application of [15, Theorem 2.12] (and the definition of the set $\mathcal{T}$), it suffices to find incomparable ideals (possibly zero, possibly unit ideals) of (possibly different) rings in $[R_2, T_2] = [R/\mathcal{J}(+)0, R/\mathcal{J}(+)K]$ which contain $M/\mathcal{J}(+)0$. As

$$\mathcal{T} = \{M/\mathcal{J}(+)0, R/\mathcal{J}(+)0, M/\mathcal{J}(+)K, R/\mathcal{J}(+)K\},$$

one (in fact, the only) pair of such incomparable objects consists of $R/\mathcal{J}(+)0$ and $M/\mathcal{J}(+)K$. □

We next provide the promised classification result.

Theorem 2.9. Consider the 13 conditions, (i)-(xiii), in the statement of Theorem 2.1. Then:

(a) If data satisfy condition (vi) or condition (xi), then $|[R, T]| = 3$.

(b) If data satisfy any of the seven conditions (iii), (iv), (vii), (viii), (ix), (x) and (xii), then $|[R, T]| > 3$ (and $|[R, T]| < \infty$).

(c) For each of the four conditions (i), (ii), (v) and (xiii), there exist data satisfying this condition for which $|[R, T]| = 3$ and there exist other data satisfying this condition for which $|[R, T]| > 3$ (and $|[R, T]| < \infty$).

Proof. For (a), see the proofs of the corresponding parts of Theorem 2.2.

For (b), all but one assertion can be handled by consulting the corresponding parts of Theorem 2.2, the therein-cited parts of [12, Remark 4.2], and Theorem 2.8. The remaining task is to verify the assertion concerning condition (xii). To that end, suppose that $R \subset S$ is an inert minimal ring extension with crucial maximal ideal M, $S \subset T$ is a ramified minimal ring extension with crucial maximal ideal N, $N \cap R = M$ and (what, in this context, becomes equivalent to the final part of the description of (xii) in Theorem 2.1) $R \subset T$ satisfies FIP. By [12, Proposition 3.1], we can assume, without loss of generality, that (R, M) is quasi-local. As $R \subset S$ is inert, it follows that $N = M$ and that (S, N) is quasi-local. Consider the fields $K := R/M$ and $S := S/M$. As $S \subset T$ is ramified, we can identify $T/M = T/N = L[X]/(X^2) = L \oplus Lx$, where $x := X + (X^2)$ is such that $x^2 = 0 \neq x$. Since M $(= N)$ is a common ideal of R and T, a standard homomorphism theorem gives a bijection $[R, T] \to [R/M, T/M]$ $(= [K, L \oplus Lx])$. Therefore, to complete a proof of the assertion concerning (xii), it suffices to prove that $[K, L \oplus Lx] \geq 4$. This, in turn, follows by considering the intermediate ring $K \oplus Lx$. This completes the proof of (b).

(c): As above, a proof of (c) can begin by consulting the corresponding parts of Theorem 2.2 and the therein-cited parts of [12, Remark 4.2]. After doing so, we are left with the task of verifying that each of the two conditions (v) and (xiii) is satisfied by some data for which $|[R, T]| = 3$. For (v), it suffices to take $R \subset S \subset T$ to be a suitable tower of finite fields, such as $\mathbb{F}_2 \subset \mathbb{F}_4 \subset \mathbb{F}_{16}$. In Remark 2.11, we will return to the principle that underlies such reasoning.

Finally, to prove the assertion concerning (xiii), we revisit a special case of the data from [12, Remark 3.6 (b)]. Specifically, take $R := K := \mathbb{F}_2 = \{0, 1\}$; $S := K[X]/(X^2) = K \oplus Kx$, where $x := X + (X^2)$ is such that $x^2 = 0 \neq x$; and $T := S[Y]/(Y^3, Y^2 - x) = K[y] = K + Ky + Ky^2$, where $y := Y + (Y^3, Y^2 - x)$ satisfies $y^2 = x \neq 0 = y^3$. By virtue of what was proved in [12, Remark 3.6 (b)], it remains only to show that $|[R, T]| = 3$. Observe that $S = \{0, 1, x, 1 + x\}$ and $T = \{0, 1, y, 1 + y, x, 1 + x, x + y, 1 + x + y\}$. To complete a proof of the assertion concerning (xiii), it suffices to show that there is no ring $A \in [K, T] \setminus \{K, S, T\}$. By considering its dimension as a K-vector space (or by using Lagrange's Theorem from elementary group theory), we see that any such A would have cardinality 4 and would consist of $0, 1$ and two other elements. Since any such A would need to be closed under addition, there are only two candidates for A, namely, $A_1 := \{0, 1, y, 1 + y\}$ and $A_2 := \{0, 1, x + y, 1 + x + y\}$. However, neither A_1 nor A_2 is closed under multiplication, since $y(1 + y) = y + y^2 = y + x \notin A_1$ and $(x + y)^2 = x^2 + 2xy + y^2 = 0 + 2y^3 + y^2 = 0 + x = x \notin A_2$. The proof is complete. □

To fully answer Ohm's question (when viewed as arising via the juxtaposition of two minimal ring extensions), one should offer a characterization of $|[R, T]| = 3$ for each of the situations where any ambiguity remains, namely, the four situations noted in Theorem 2.9 (c). For the first and second of those situations (i.e., concerning conditions (i) and (ii) from Theorem 2.1), Proposition 2.10 (b), (c) will give a complete answer. For the third situation (concerning condition (v) from Theorem 2.1), the question will be reduced to a problem in field theory in Remark 2.11. Finally, in regard to condition (xiii) from Theorem 2.1, parts (d) and (e) of Proposition 2.10 will give a fuller (but still incomplete) answer, including an answer when the base ring (which may be assumed quasi-local) has an infinite residue field. For the sake of completeness, Proposition 2.10 (a) includes a rather trivial characterization, in the spirit of the literature on "subrings maximal without a given element", of "$|[R, T]| = 3$" which is applicable to *each* of the 13 conditions in the statement of Theorem 2.1.

The following background will be needed in the statement of Proposition 2.10 (b) and the proof of Proposition 2.10 (e). Let A be a ring. Then, as usual, $\mathrm{Max}(A)$ denotes the set of maximal ideals of A. Also, as in [16, Section 33], the *Nagata ring* $A(X)$ is defined to be the ring of fractions $A[X]_\Sigma$, where Σ is the multiplicatively closed subset of $A[X]$ consisting of the polynomials whose coefficients generate A as a ring. Now, let $A \subseteq B$ be rings. By the proof of the Proposition preceding [8, Proposition II.9], the given inclusion of rings and the assignment $X \mapsto X$ induce an injective ring homomorphism $A(X) \to B(X)$, by means of which we view $A(X) \subseteq B(X)$. Also, the *support* of the ring extension $A \subseteq B$ is defined as $\mathrm{Supp}(B/A) := \{P \in \mathrm{Spec}(A) \mid A_P = B_P \text{ canonically}\}$ (that is, such that the canonical injective ring homomorphism $A_P \to B_P$ is an isomorphism); and we put $\mathrm{MSupp}(T/R) := \mathrm{Max}(R) \cap \mathrm{Supp}(T/R)$.

Proposition 2.10. (a) Let $R \subset S$ and $S \subset T$ be minimal ring extensions such that $R \subset T$ satisfies FIP. (In other words, consider data satisfying any of the 13 conditions in the statement of Theorem 2.1.) Then $|[R, T]| = 3$ if and only if there exists $\alpha \in T$ such that $\alpha \in B$ for each $B \in [R, T] \setminus \{R, T\}$. (Necessarily, $S = R[\alpha]$ for any such α.)

(b) Given data that satisfy condition (i) in the statement of Theorem 2.1, let $m := |\mathrm{MSupp}(T/R)|$; and for each $k = 1, \ldots, m$, let Γ_k be the set of antichains of $\mathrm{Supp}(T/R)$ that have cardinality k.

Then $|[R,T]| = 3$ if and only if $\sum_{i=1}^m |\Gamma_i| = 2$.

(c) Given data that satisfy condition (ii) in the statement of Theorem 2.1, then $|[R,T]| = 3$ if and only if, whenever $u \in T$ is not integral over R, then $R[u] = T$.

(d) If data satisfy condition (xiii) in the statement of Theorem 2.1 and R/M is infinite, then $|[R,T]| = 3$.

(e) If data satisfy condition (xiii) in the statement of Theorem 2.1 and $R(X) \subset T(X)$ satisfies FIP, then $|[R,T]| = 3$.

Proof. (a) The elementary proof of this assertion is left to the reader.

(b) Since (R,T) is a normal pair (cf. [23, Theorem 5.6, Chapter I]) and $R \subset T$ satisfies FIP, we can apply [10, Theorem 4.3], with the upshot that $|[R,T]| = 1 + \sum_{i=1}^m |\Gamma_i|$. The assertion is now immediate.

(c) This assertion was established in [12, Remark 4.2 (b)].

(d) By [12, Proposition 3.1 (a), (c), (d)], we can replace $R \subset S \subset T$ with $R_M \subset S_M \subset T_M$; that is, without loss of generality, (R,M) is quasi-local. (Note also that, up to isomorphism, this change of notation has not changed R/M.) Observe that the extension $R \subset T$ is subintegral, since it results from juxtaposing two subintegral (minimal) extensions. Consequently, by [10, Proposition 4.13], the poset $[R,T]$ is linearly ordered by inclusion. Thus, S is comparable with each intermediate ring of $R \subset T$, and so it follows easily from the minimality of $R \subset S$ and $S \subset T$ that $[R,T] = \{R,S,T\}$.

(e) As in the proof of (d), we can assume, without loss of generality, that (R,M) is quasi-local. Consider the induced tower of Nagata rings, $R(X) \subset S(X) \subset T(X)$. By [10, Theorem 3.4], $R(X) \subset S(X)$ and $S(X) \subset T(X)$ are each ramified minimal ring extensions. By [10, Theorem 3.4 (c)] (cf. also [10, Lemma 3.3]), their crucial maximal ideals are $MR(X)$ and $NS(X)$, respectively. As $S(X) = R(X) \otimes_R S$ canonically by [10, Lemma 3.1(e)] and finite intersections commute with the formation of rings of fractions, the hypothesis that $N \cap R = M$ leads to $NS(X) \cap R(X) = MR(X)$. (In detail, with $R(X) = R[X]_\Sigma$, we get as above that $S(X) = S[X]_\Sigma$ canonically, whence $NS(X) \cap R(X) = (NS[X] \cap R[X])_\Sigma = MR[X]_\Sigma = MR(X)$.) It is well known that the residue field of $R(X)$ can be identified as follows: $R(X)/MR(X) = (R/M)(X)$, which is infinite. Therefore, since $R(X) \subset T(X)$ satisfies FIP, we can apply (d) to the above tower of Nagata rings, the upshot being that $|[R(X),T(X)]| = 3$. As in the proof of (d), $R \subset T$ is subintegral, and so [10, Proposition 4.14] can be applied, giving that $|[R,T]| = |[R(X),T(X)]|$, which completes the proof. □

Remark 2.11. (a) The above work in Theorem 2.2, Theorem 2.4 and Proposition 2.10 (b), (c) has determined necessary and sufficient conditions for $|[R,T]| = 3$ when any of the following 11 conditions from the statement of Theorem 2.1 applies: (i), (ii), (iii), (iv), (vi), (vii), (viii), (ix), (x), (xi) and (xii). Unless one accepts the triviality in Proposition 2.10 (a) as a final answer, a more desirable answer is less complete when either of the remaining conditions applies (that is, for conditions (v) and (xiii) from that statement). This remark offers all that we know at this time with respect to a possible characterization of $|[R,T]| = 3$ when one of these two conditions applies.

Suppose that condition (v) applies. Thanks to [12, Theorem 3.1], one can assume that (R,M) is quasi-local; that is, $|[R,T]| = 3$ if and only if $|[R_M,T_M]| = 3$. (In fact, the reduction to the subcase of a quasi-local base ring can similarly be made for any of the scenarios in Theorem 2.1 where both $R \subset S$ and $S \subset T$ are integral (minimal) ring extensions. In particular, this reduction can also be made for condition (xiii).) The question of when $|[R,T]| = 3$, given that case (v) applies, remains open. But we would note that much can be said, thanks to the archetypical subcase arising from a FIP-tower of fields. For instance, as noted in [12, Remark 4.3 (a)], the classical theory of finite fields shows that if $\mathbb{F}_q \subset \mathbb{F}_r$ are finite fields (where $q = p^n$, for some prime number p and some positive integer n, and necessarily, $r = p^m$ for some positive integer m which is divisible by n), then $|[\mathbb{F}_q,\mathbb{F}_r]| = 3$ if and only if $m = s^2n$ for some prime number s. In general, since the hypotheses of (v) imply that $M = N$, the assignment $A \mapsto A/M$ gives an order-isomorphism $[R,T] \to [R/M,T/M]$. Therefore, what remains of our task in regard to (v) can be reduced to studying a FIP-tower of (without loss if generality, infinite) fields. In short, the question in regard to condition (v) has been reduced to a question in field theory, that is, characterizing towers of minimal field extensions $K \subset F \subset L$ such that K is infinite and $|[K,L]| = 3$.

Suppose, finally, that condition (xiii) applies. As noted above, one can assume that (R, M) is quasi-local. Further study of this situation may be aided by noting that [12, Proposition 3.5 (b)] can be sharpened; indeed, the ring therein denoted by "A" is actually T, since (iii) implies that $R \subset T$ is subintegral. Beyond that observation, the question of when $|[R, T]| = 3$, given that case (xiii) applies, remains open. But we would note that Proposition 2.10 (d), (e) did settle the subcase where R/M is infinite (and its subcase where $R(X) \subset T(X)$ satisfies FIP); and that parts (b) and (c) below will provide additional examples which may help one to formulate an eventual characterization.

We wish to stress that unless one is content with the triviality in Proposition 2.10 (a) or with the reduction to a problem in field theory that was noted two paragraphs ago, Ohm's question remains open for data satisfying either condition (v) or condition (xiii) from the statement of Theorem 2.1.

(b) Among the statements of the conditions in Theorem 2.1, those of conditions (xii) and (xiii) are especially cumbersome (because of their reference to the statement of [12, Proposition 3.5]). Consequently, it is perhaps not surprising that the analysis of the "$|[R, T]| = 3$" issue for condition (xiii) that was just given in (a) is less complete than for the analysis for any of the other 12 conditions. Nevertheless, as we showed via the example in the final paragraph of the proof of Theorem 2.9, it is not difficult to provide data that satisfy condition (xiii) and "$|[R, T]| = 3$". We next give a companion for that example.

Let K be a field and let $T := K[X]/(X^3)$. Viewing $K \subset T$ as usual, we have $T = K[u] = K \oplus Ku \oplus Ku^2$, where $u := X + (X^3)$ satisfies $u^3 = 0 \neq u^2$. It is convenient to put $R := K$. It was shown in [15, Proposition 3.5] that $|[R, T]| = 3$. Next, consider $S := K[u^2] = K \oplus Ku^2 \cong K[Y]/(Y^2)$. It is clear that $R \subset S$ is a(n integral) ramified (minimal ring) extension, necessarily with crucial maximal ideal $M := 0$. Note that $N := (S : T) = Ku^2$, which is the unique maximal ideal of S; that $S/N \cong K$; and that $T/N = K \oplus Kv$, where $v := u + N$ satisfies $v^2 = 0 \neq v$. Consequently, $S \subset T$ is also a ramified extension, with crucial maximal ideal N. Since $N \cap R = M$ and $|[R, T]| = 3$, the assembled data satisfy condition (xiii) in Theorem 2.1 regardless of whether the field K is finite.

(c) It was mentioned in the Introduction of [12] that [15, Remarks 4.15 (a)] gives an example of integral minimal ring extensions $R \subset S$ and $S \subset T$ such that $|[R, T]| = 3$. In fact, the discussion in [15, Remarks 4.15 (a)] is extremely terse. It may be helpful to develop an explicit example along the lines indicated in [15, Remarks 4.15 (a)]. We close by giving what is perhaps the simplest such example. It turns out to be *another* companion for the example in the final paragraph of the proof of Theorem 2.9; that is, another example that satisfies condition (xiii) in Theorem 2.1 and $|[R, T]| = 3$. As the somewhat intricate example given below is based on some material [15, pages 72-75] from an unpublished doctoral dissertation, we will provide extensive details (along with appropriate references).

Take $R := \mathbb{Z}/4\mathbb{Z} = \{\overline{0}, \overline{1}, \overline{2}, \overline{3}\}$, where as usual, $\overline{a}$ denotes $a + 4\mathbb{Z}$ for all $a \in \mathbb{Z}$. Next, take $T := R[X]/(X^2)$. Note that $t := X + (X^2) \in T$ satisfies $t^2 = 0 \neq t$ and $T = R[t] = R + Rt$ is

$$\{\overline{0}, \overline{1}, \overline{2}, \overline{3}, t, \overline{2}t, \overline{3}t, \overline{1} + t, \overline{1} + \overline{2}t, \overline{1} + \overline{3}t, \overline{2} + t, \overline{2} + \overline{2}t, \overline{2} + \overline{3}t, \overline{3} + t, \overline{3} + \overline{2}t, \overline{3} + \overline{3}t\}$$

which has cardinality 16. It is easy to check that $(R : T) = 0$. Therefore, [15, Proposition 4.12] gives a bijection between $[R, T]$ and the set of ideals of the ring $R/(R : T)$ $(\cong R)$. Thus, $|[R, T]| = |\{0, 2R, R\}| = 3$. Of course, $R \subset T$ is an integral extension. Hence, to complete a proof of the assertions, it suffices to find a ring $S \in [R, T]$ such that $R \subset S$ is a ramified (integral minimal ring) extension with crucial maximal ideal M, $S \subset T$ is a ramified extension with crucial maximal ideal N, and $N \cap R = M$. We will show that $S := R + R2t$ has all the desired properties.

Observe that $S = \{\overline{0}, \overline{1}, \overline{2}, \overline{3}, \overline{2}t, \overline{1} + \overline{2}t, \overline{2} + \overline{2}t, \overline{3} + \overline{2}t, \}$, which has cardinality 8. Also, since $2(2t) = 4t = \overline{0}$, it follows that $(R : S) = \{\overline{0}, \overline{2}\}$, the unique maximal ideal of R, which we denote by M; and $R/M = \{\overline{0} + M, \overline{1} + M\} \cong \mathbb{F}_2$. Moreover,

$$S/M = (R + R2t)/M = R/M + (R/M + \overline{2}t) = (R/M)[\overline{2}t],$$

where $\overline{2}t \in S/M$ is a (nonzero) nilpotent element having nilpotency index 2. Consequently, $S/M \cong (R/M)[Y]/(Y^2)$ (where Y denotes a new indeterminate which is algebraically independent of X), whence $R \subset S$ is indeed a ramified extension with crucial maximal ideal M.

Next, observe that the set of nonunits of S is $N := \{\overline{0}, \overline{2}, \overline{2}t, \overline{2} + \overline{2}t\}$ and that N forms an ideal of S. Consequently, S is quasi-local, with unique maximal ideal N. Clearly, $N \cap R = M$. Also, one checks easily that the conductor $(S : T) = N$. In addition, $S/N = \{N, \overline{1} + N\} \cong \mathbb{F}_2$ (since $R \subset S$ is ramified). Therefore, to show that $S \subset T$ is a ramified extension with crucial maximal ideal N, it suffices to prove that $T/N \cong S/N \times S/N$ ($\cong \mathbb{F}_2 \times \mathbb{F}_2$). But in fact,

$$T/N = \{N, \overline{1} + N, t + N, \overline{1} + t + N\} = (S/N)[t + N],$$

where $t + N \in T/N$ is a (nonzero) nilpotent element having nilpotency index 2. As above, it follows that $S \subset T$ is a ramified minimal ring extension with crucial maximal ideal N. This completes our verification that the above data constructed as indicated by Gilbert [15, Remarks 4.15 (a)] satisfies both condition (xiii) in Theorem 2.1 and $|[R, T]| = 3$.

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