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# Recognition of some alternating groups by the order and the set of vanishing elements orders

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**Abstract** For a finite group $G$, an element $g$ is called a vanishing element of $G$ whenever there is an irreducible character $\chi$ in Irr$(G)$ such that $\chi(g) = 0$. We denote by Vo$(G)$ the set of orders of vanishing elements of $G$. In [M.F. Ghasemabadi et al., A new characterization of some finite simple groups, Siberian Mathematical Journal, 2015], the authors put the following conjecture: Let $G$ be a finite group and $M$ be a finite nonabelian simple group. If Vo$(G)$ = Vo$(M)$ and $|G| = |M|$, then $G \cong M$.

In this paper, we prove that if $G$ is a finite group such that $|G| = |A_n|$ and Vo$(G)$ = Vo$(A_n)$, where $A_n$ is an alternating group and $5 \leq n \leq 9$, then $G$ is isomorphic to $A_n$. In particular, the above conjecture holds for these simple groups.

## 1 Introduction

For a finite group $G$, the set of irreducible characters of $G$ is denoted by Irr$(G)$. Also an element $g \in G$ is called a vanishing element whenever there exists an irreducible character $\chi$ in Irr$(G)$ such that $\chi(g) = 0$. The set of vanishing elements of $G$ and their orders are denoted by Van$(G)$ and Vo$(G)$, respectively. Also we denote by $\pi(G)$ and $\pi_e(G)$, the set of prime divisors of the order of $G$ and the set of element orders of $G$, respectively.

We will recall the required definitions as follows: Given a finite set of positive integers $X$, the graph $\Pi(X)$ is defined as the simple undirected graph whose vertices are the primes $p$ such that there exists an element of $X$ divisible by $p$, and two distinct vertices $p$ and $q$ are adjacent if and only if there exists an element of $X$ divisible by $p$ and $q$. For a finite group $G$, the graph $\Pi(\pi_e(G))$, which we denote by GK$(G)$, is also known as the *prime graph* (or *Gruenbergâ ĂŞKegel graph*) of $G$. Also, the prime graph $\Pi(\mathrm{Vo}(G))$, which in this paper we denote by $\Gamma(G)$, is called the vanishing prime graph of $G$. Note that by $n(\Gamma)$ we mean the number of connected components of a graph $\Gamma$.

For $p \in \pi(G)$, an irreducible character $\chi$ of $G$ is said to be of $p$-defect zero if $p$ does not divide $|G|/\chi(1)$. We know that if $\chi \in$ Irr$(G)$ is of $p$-defect zero, then for every element $g \in G$ such that $p$ divides the order of $g$, we have $\chi(g) = 0$. (see Theorem 8.17 in [5]). All further unexplained notation is standard and can be found, for instance, in [1].

In [4], the author put the following conjecture:

**Conjecture 1.1.** Let $G$ be a finite group and let $M$ be a finite nonabelian simple group. If Vo$(G)$ = Vo$(M)$ and $|G| = |M|$, then $G \cong M$.

In [4], the above conjecture is proved for some finite simple groups. Also in [4, 7], it is proved that the alternating groups $A_5$, $A_6$ ($\cong L_2(9)$) and $A_7$ are characterizable by the set of orders of vanishing elements. There are many results about the order of vanishing elements (for example see the references of [4]). In this paper we prove that the simple group $A_n$ is characterizable by its order and vanishing prime graph for $7 \leq n \leq 8$. In particular, we get that Conjecture 1.1 holds for these simple groups.

## 2 Main Results

**Lemma 2.1.** *Let $G$ be a finite group and let $p$ be a prime number which belong to the vertex set of vanishing graph of $G$. If $|G|_p = p$, then $G$ has an irreducible character of $p$-defect zero.*

*Proof.* Since $p$ is a prime number which belongs to the vertex set of vanishing graph of $G$, there exists an irreducible character $\chi$ and an element $g \in G$ such that $|g| = p$ and $\chi(g) = 0$. Let $\epsilon$ be a complex primitive root of unity. Since $\chi(g)$ is a sum of $\chi(1)$ $p$-th root of unity, we have $\chi(g) = \Sigma_{i=1}^{\chi(1)} \epsilon^{k_i}$ with $0 \leq k_i < p$. Now, $\epsilon$ is a root of the polynomial $h(x) = \Sigma_{i=1}^{\chi(1)} x^{k_i}$. Whence $h(x)$ is divisible by the $p$th cyclotomic polynomial $\Phi_p(x)$. In particular, $p = \Phi_p(1)$ divides $h(1) = \chi(1)$. On the other hand $|G|_p = p$. Hence $p \nmid |G|/\chi(1)$, which implies that $\chi$ is an irreducible character of $p$-defect zero, as desired. □

**Lemma 2.2.** *Let $G$ be a finite group and let $p$ and $q$ be two distinct prime numbers in the vertex set of the vanishing prime graph of $G$, $V(\Gamma(G))$. Also let the following conditions hold:*

*a) $|G|_p = p$, $|G|_q = q$*

*b) there is no edge between $p$ and $q$ in $\Gamma(G)$,*

*c) $p \nmid (q-1)$ and $q \nmid (p-1)$.*

*Then there exists a nonabelian simple group $S$ such that $S \leq G/K \leq \mathrm{Aut}(S)$, where $K = O_{\{p,q\}'}(G)$. Moreover, we have $|S|_p = p$, $|S|_q = q$ and $p$ is not adjacent to $q$ in both graphs $\mathrm{GK}(S)$ and $\Gamma(S)$.*

*Proof.* Let $K = O_{\{p,q\}'}(G)$ be the maximal normal subgroup of $G$ whose order is not divisible by $p$ or $q$. We put $\bar{G} := G/K$. Also let $\bar{M}$ be an arbitrary minimal normal subgroup of $\bar{G}$. By the definition of $K$, we deduce that $\pi(\bar{M}) \cap \{p,q\} \neq \emptyset$. We claim that $\pi(\bar{M})$ contains both prime numbers $p$ and $q$.

Suppose $|\pi(\bar{M}) \cap \{p,q\}| = 1$. So without loss of generality we may assume that the intersection of $\pi(\bar{M})$ and the set $\{p,q\}$ only contains $p$. Let $\bar{P}$ be a Sylow $p$-subgroup of $\bar{M}$. By Frattini argument, $\bar{G} = \bar{M}N_{\bar{G}}(\bar{P})$. Since $\pi(\bar{M}) \cap \{p,q\} = \{p\}$, we get that $q \mid |N_{\bar{G}}(\bar{P})|$. So $\bar{G}$ contains a subgroup $\bar{P} \rtimes \bar{Q}$, where $\bar{Q}$ is a Sylow $q$-subgroup of $N_{\bar{G}}(\bar{P})$. On the other hand by the assumption, there is no edge between $p$ and $q$ in $\Gamma(G)$ (and so in $\Gamma(\bar{G})$). Also by the assumption, $|G|_p = p$, $|G|_q = q$. So by Lemma 2.1 and Theorem 8.17 in [5], in the prime graph of $G$, $\mathrm{GK}(G)$, $p$ and $q$ are nonadjacent. This implies that the subgroup $\bar{P} \rtimes \bar{Q}$ is a Frobenius groups of order $pq$. Thus by the properties of Frobenius group, we conclude that $q \mid (p-1)$, which contradicts to our assumptions (Condition (c)).

Therefore, by the above discussion, we get that $\pi(\bar{M})$ contains both prime numbers $p$ and $q$. On the other hand, since $\bar{M}$ is a minimal normal subgroup of $\bar{G}$, there are some isomorphic nonabelian simple groups $S_1, \ldots, S_k$ such that $\bar{M} = S_1 \times \cdots \times S_k$. We know that $\{p,q\} \subseteq \pi(\bar{M})$, $|G|_p = p$ and $|G|_q = q$. Then, obviously, $k = 1$ and so $\bar{M}$ is isomorphic to a nonabelian simple group $S$.

Now we remark that $\bar{M}$ was assumed to be an arbitrary minimal normal subgroup of $\bar{G}$. So by $|\bar{G}|_p = |\bar{M}|_p = p$, we get that $\bar{M}$ is the unique minimal normal subgroup of $\bar{G}$. Also since $\bar{M}$ is a nonabelian simple group, we conclude that $C_{\bar{G}}(\bar{M}) = 1$. This yields that

$$\bar{M} \leq \bar{G} := \frac{G}{O_{\{p,q\}'}(G)} \leq \mathrm{Aut}(\bar{M}),$$

which completes the proof. □

**Theorem 2.3.** *Let $A_n$ be an alternating group such that $8 \leq n \leq 9$. Also let $G$ be a finite group with the same order and vanishing graph as alternating group $A_n$, i.e. $|G| = |A_n|$ and $\Gamma(G) = \Gamma(A_n)$. Then $G$ is isomorphic to $A_n$.*

*Proof.* First let $L$ be the alternating group $A_n$ where $8 \leq n \leq 9$. So using [1], we get that for prime numbers $p = 5$ and $q = 7$, we have $|L|_p = p$ and $|L|_q = q$ and there is no edge between $p$ and $q$ in the vanishing prime graph of $L$. Let $G$ be a finite group such that

$$|G| = |L| = 2^6 \cdot 3^\beta \cdot 5 \cdot 7,$$

where $\beta \in \{2,4\}$ and $\Gamma(G) = \Gamma(L)$.

Using Lemma 2.2, we get that there exists a nonabelian simple group $S$ such that

$$S \leq \bar{G} := \frac{G}{O_{\{5,7\}'}(G)} \leq \mathrm{Aut}(S).$$

Let $K := O_{\{5,7\}'}(G)$. Since $\pi(G) = \pi(L) = \{2,3,5,7\}$, we get that $\pi(K) \subseteq \{2,3\}$. Also since $\pi(S) \subseteq \pi(G)$, by Lemma 2.2, we conclude that $\pi(S) \subseteq \{2,3,5,7\}$ and $|S|_5 = 5$ and $|S|_7 = 7$.

Now we investigate each possibility for the simple group $S$. We note that in [8], such simple group are listed. So the nonabelian simple group $S$ is isomorphic to $A_7$, $A_8$, $A_9$, $A_{10}$, $S_6(2)$, $O_8^+(2)$, $L_3(2^2)$, $U_4(3)$, $U_3(5)$, $S_4(7)$, $L_2(7^2)$ or $J_2$.

We remark that the order of the simple group $S$ divides the order of $G$. So by considering the order of the above simple groups, we get that $S$ is not isomorphic to $A_{10}$, $S_6(2)$ (for 2-part of $|G|$ and $|S|$), $O_8^+(2)$, $U_4(3)$ (for 3-part of $|G|$ and $|S|$), $U_3(5)$, $S_4(7)$, $L_2(7^2)$ and $J_2$. Hence $S \cong A_7$, $A_8$, $A_9$ or $L_3(2^2)$. In the following we consider the cases $L = A_8$ and $L = A_9$ separately.

**Case 1.** Let $L = A_9$. Let $S \cong A_7$ or $A_8$, i.e. $A_7 \leq G/K \leq S_7$ or $A_8 \leq G/K \leq S_8$. So either $|G| = 2^3 \cdot 3^2 \cdot 5 \cdot 7 \cdot \epsilon \cdot |K|$ or $|G| = 2^6 \cdot 3^2 \cdot 5 \cdot 7 \cdot \epsilon \cdot |K|$, where $\epsilon = 1$ or 2. On the other hand by the assumption $|G| = |L| = 2^6 \cdot 3^4 \cdot 5 \cdot 7$. This implies that $|K|_3 = 3^2$. We note that $7 \in \pi(S)$ and 3 and 7 are nonadjacent in $\Gamma(G)$. On the other hand by Lemma 2.1, every element $g \in G$ such that 7 divides the order of $g$, is a vanishing element of $G$ and so $|g|$ belongs to $\mathrm{Vo}(G)$. This shows that $G$ does not contain any element order $3 \cdot 7$. Let $U$ be a Sylow 7-subgroup of $G$ and $K_3$ be a Sylow 3-subgroup of $K$. Thus by Frattini argument we get that $K_3 \rtimes U$ is a $\{3,7\}$-subgroup of $G$. Also by the previous discussion we get that $K_3 \rtimes U$ is a Frobenius group with kernel $K_3$. So $|U| \mid (|K_3| - 1)$ and so $7 \mid (3^2 - 1)$, a contradiction. Also if $S \cong L_3(2^2)$, then we deduce that $|K_3| = 3$ or 9 and so similarly, we get a contradiction. Hence $S$ is not isomorphic to any simple group, except $A_9$. Therefore, $A_9 \leq G/K \leq S_9$, which by the order of $|G|$ we get that $K = 1$ and so $G \cong A_9$.

**Case 2.** Let $L = A_8$. Obviously, $S$ is not isomorphic to $A_9$ (3-part of $|S|$). Let $S \cong A_7$, i.e. $A_7 \leq G/K \leq S_7$. This implies that $K$ is a 2-group, since $|G| = |A_8|$. We remark that by the assumption in $\Gamma(G)$, 5 and 3 are adjacent. This mean that $G$ has an element of order $3 \cdot 5$, while in $A_7$ there is no element of order $3 \cdot 5$. Thus since $K$ is a 2-group, we get a contradiction. Let $S \cong L_3(2^2)$. In this case, we have $|G| = |L| = |S|$. Hence we get that $G \cong L_3(2^2)$ and so $\Gamma(A_8) = \Gamma(L_3(2^2))$, which is a contradiction by [1]. Therefore $S \cong A_8$ and so similar to the above case, we conclude that $G$ is isomorphic to $A_8$, which completes the proof. □

**Corollary 2.4.** *Let $G$ be a finite group such that $|G| = |A_n|$ and $\mathrm{Vo}(G) = \mathrm{Vo}(A_n)$, where $5 \leq n \leq 9$. Then $G \cong A_n$, i.e. Conjecture 1.1, holds for these simple groups.*

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