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Recognition of some alternating groups by the order and the set of vanishing elements orders

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Abstract For a finite group G, an element g is called a vanishing element of G whenever there is an irreducible character χ in Irr(G) such that $\chi(g) = 0$. We denote by Vo(G) the set of orders of vanishing elements of G. In [M.F. Ghasemabadi et al., A new characterization of some finite simple groups, Siberian Mathematical Journal, 2015], the authors put the following conjecture: Let G be a finite group and M be a finite nonabelian simple group. If Vo(G) = Vo(M) and $|G| = |M|$, then $G \cong M$.

In this paper, we prove that if G is a finite group such that $|G| = |A_n|$ and Vo(G) = Vo(A_n), where A_n is an alternating group and $5 \le n \le 9$, then G is isomorphic to A_n. In particular, the above conjecture holds for these simple groups.

1 Introduction

For a finite group G, the set of irreducible characters of G is denoted by Irr(G). Also an element $g \in G$ is called a vanishing element whenever there exists an irreducible character χ in Irr(G) such that $\chi(g) = 0$. The set of vanishing elements of G and their orders are denoted by Van(G) and Vo(G), respectively. Also we denote by $\pi(G)$ and $\pi_e(G)$, the set of prime divisors of the order of G and the set of element orders of G, respectively.

We will recall the required definitions as follows: Given a finite set of positive integers X, the graph $\Pi(X)$ is defined as the simple undirected graph whose vertices are the primes p such that there exists an element of X divisible by p, and two distinct vertices p and q are adjacent if and only if there exists an element of X divisible by p and q. For a finite group G, the graph $\Pi(\pi_e(G))$, which we denote by GK(G), is also known as the *prime graph* (or *GruenbergâĂŞKegel graph*) of G. Also, the prime graph Π(Vo(G)), which in this paper we denote by $\Gamma(G)$, is called the vanishing prime graph of G. Note that by $n(\Gamma)$ we mean the number of connected components of a graph Γ.

For $p \in \pi(G)$, an irreducible character χ of G is said to be of p-defect zero if p does not divide $|G|/\chi(1)$. We know that if $\chi \in$ Irr(G) is of p-defect zero, then for every element $g \in G$ such that p divides the order of g, we have $\chi(g) = 0$. (see Theorem 8.17 in [5]). All further unexplained notation is standard and can be found, for instance, in [1].

In [4], the author put the following conjecture:

Conjecture 1.1. Let G be a finite group and let M be a finite nonabelian simple group. If Vo(G) = Vo(M) and $|G| = |M|$, then $G \cong M$.

In [4], the above conjecture is proved for some finite simple groups. Also in [4, 7], it is proved that the alternating groups A_5, A_6 ($\cong L_2(9)$) and A_7 are characterizable by the set of orders of vanishing elements. There are many results about the order of vanishing elements (for example see the references of [4]). In this paper we prove that the simple group A_n is characterizable by its order and vanishing prime graph for $7 \le n \le 8$. In particular, we get that Conjecture 1.1 holds for these simple groups.

2 Main Results

Lemma 2.1. *Let G be a finite group and let p be a prime number which belong to the vertex set of vanishing graph of G. If $|G|_p = p$, then G has an irreducible character of p-defect zero.*

Proof. Since p is a prime number which belongs to the vertex set of vanishing graph of G, there exists an irreducible character χ and an element $g \in G$ such that $|g| = p$ and $\chi(g) = 0$. Let ϵ be a complex primitive root of unity. Since $\chi(g)$ is a sum of $\chi(1)$ p-th root of unity, we have $\chi(g) = \Sigma_{i=1}^{\chi(1)} \epsilon^{k_i}$ with $0 \leq k_i < p$. Now, ϵ is a root of the polynomial $h(x) = \Sigma_{i=1}^{\chi(1)} x^{k_i}$. Whence $h(x)$ is divisible by the pth cyclotomic polynomial $\Phi_p(x)$. In particular, $p = \Phi_p(1)$ divides $h(1) = \chi(1)$. On the other hand $|G|_p = p$. Hence $p \nmid |G|/\chi(1)$, which implies that χ is an irreducible character of p-defect zero, as desired. □

Lemma 2.2. *Let G be a finite group and let p and q be two distinct prime numbers in the vertex set of the vanishing prime graph of G, $V(\Gamma(G))$. Also let the following conditions hold:*

a) $|G|_p = p$, $|G|_q = q$

b) there is no edge between p and q in $\Gamma(G)$,

c) $p \nmid (q-1)$ and $q \nmid (p-1)$.

Then there exists a nonabelian simple group S such that $S \leq G/K \leq \mathrm{Aut}(S)$, where $K = O_{\{p,q\}'}(G)$. Moreover, we have $|S|_p = p$, $|S|_q = q$ and p is not adjacent to q in both graphs $\mathrm{GK}(S)$ and $\Gamma(S)$.

Proof. Let $K = O_{\{p,q\}'}(G)$ be the maximal normal subgroup of G whose order is not divisible by p or q. We put $\bar{G} := G/K$. Also let $\bar{M}$ be an arbitrary minimal normal subgroup of $\bar{G}$. By the definition of K, we deduce that $\pi(\bar{M}) \cap \{p, q\} \neq \emptyset$. We claim that $\pi(\bar{M})$ contains both prime numbers p and q.

Suppose $|\pi(\bar{M}) \cap \{p, q\}| = 1$. So without loss of generality we may assume that the intersection of $\pi(\bar{M})$ and the set $\{p, q\}$ only contains p. Let $\bar{P}$ be a Sylow p-subgroup of $\bar{M}$. By Frattini argument, $\bar{G} = \bar{M}N_{\bar{G}}(\bar{P})$. Since $\pi(\bar{M}) \cap \{p, q\} = \{p\}$, we get that $q \mid |N_{\bar{G}}(\bar{P})|$. So $\bar{G}$ contains a subgroup $\bar{P} \rtimes \bar{Q}$, where $\bar{Q}$ is a Sylow q-subgroup of $N_{\bar{G}}(\bar{P})$. On the other hand by the assumption, there is no edge between p and q in $\Gamma(G)$ (and so in $\Gamma(\bar{G})$). Also by the assumption, $|G|_p = p$, $|G|_q = q$. So by Lemma 2.1 and Theorem 8.17 in [5], in the prime graph of G, $\mathrm{GK}(G)$, p and q are nonadjacent. This implies that the subgroup $\bar{P} \rtimes \bar{Q}$ is a Frobenius groups of order pq. Thus by the properties of Frobenius group, we conclude that $q \mid (p-1)$, which contradicts to our assumptions (Condition (c)).

Therefore, by the above discussion, we get that $\pi(\bar{M})$ contains both prime numbers p and q. On the other hand, since $\bar{M}$ is a minimal normal subgroup of $\bar{G}$, there are some isomorphic nonabelian simple groups $S_1, \ldots, S_k$ such that $\bar{M} = S_1 \times \cdots \times S_k$. We know that $\{p, q\} \subseteq \pi(\bar{M})$, $|G|_p = p$ and $|G|_q = q$. Then, obviously, $k = 1$ and so $\bar{M}$ is isomorphic to a nonabelian simple group S.

Now we remark that $\bar{M}$ was assumed to be an arbitrary minimal normal subgroup of $\bar{G}$. So by $|\bar{G}|_p = |\bar{M}|_p = p$, we get that $\bar{M}$ is the unique minimal normal subgroup of $\bar{G}$. Also since $\bar{M}$ is a nonabelian simple group, we conclude that $C_{\bar{G}}(\bar{M}) = 1$. This yields that

$$\bar{M} \leq \bar{G} := \frac{G}{O_{\{p,q\}'}(G)} \leq \mathrm{Aut}(\bar{M}),$$

which completes the proof. □

Theorem 2.3. *Let A_n be an alternating group such that $8 \leq n \leq 9$. Also let G be a finite group with the same order and vanishing graph as alternating group A_n, i.e. $|G| = |A_n|$ and $\Gamma(G) = \Gamma(A_n)$. Then G is isomorphic to A_n.*

Proof. First let L be the alternating group A_n where $8 \leq n \leq 9$. So using [1], we get that for prime numbers $p = 5$ and $q = 7$, we have $|L|_p = p$ and $|L|_q = q$ and there is no edge between p and q in the vanishing prime graph of L. Let G be a finite group such that

$$|G| = |L| = 2^6 \cdot 3^\beta \cdot 5 \cdot 7,$$

where $\beta \in \{2, 4\}$ and $\Gamma(G) = \Gamma(L)$.

Using Lemma 2.2, we get that there exists a nonabelian simple group S such that

$$S \leq \bar{G} := \frac{G}{O_{\{5,7\}'}(G)} \leq \mathrm{Aut}(S).$$

Let $K := O_{\{5,7\}'}(G)$. Since $\pi(G) = \pi(L) = \{2,3,5,7\}$, we get that $\pi(K) \subseteq \{2,3\}$. Also since $\pi(S) \subseteq \pi(G)$, by Lemma 2.2, we conclude that $\pi(S) \subseteq \{2,3,5,7\}$ and $|S|_5 = 5$ and $|S|_7 = 7$.

Now we investigate each possibility for the simple group S. We note that in [8], such simple group are listed. So the nonabelian simple group S is isomorphic to A_7, A_8, A_9, A_{10}, $S_6(2)$, $O_8^+(2)$, $L_3(2^2)$, $U_4(3)$, $U_3(5)$, $S_4(7)$, $L_2(7^2)$ or J_2.

We remark that the order of the simple group S divides the order of G. So by considering the order of the above simple groups, we get that S is not isomorphic to A_{10}, $S_6(2)$ (for 2-part of $|G|$ and $|S|$), $O_8^+(2)$, $U_4(3)$ (for 3-part of $|G|$ and $|S|$), $U_3(5)$, $S_4(7)$, $L_2(7^2)$ and J_2. Hence $S \cong A_7$, A_8, A_9 or $L_3(2^2)$. In the following we consider the cases $L = A_8$ and $L = A_9$ separately.

Case 1. Let $L = A_9$. Let $S \cong A_7$ or A_8, i.e. $A_7 \leq G/K \leq S_7$ or $A_8 \leq G/K \leq S_8$. So either $|G| = 2^3 \cdot 3^2 \cdot 5 \cdot 7 \cdot \epsilon \cdot |K|$ or $|G| = 2^6 \cdot 3^2 \cdot 5 \cdot 7 \cdot \epsilon \cdot |K|$, where $\epsilon = 1$ or 2. On the other hand by the assumption $|G| = |L| = 2^6 \cdot 3^4 \cdot 5 \cdot 7$. This implies that $|K|_3 = 3^2$. We note that $7 \in \pi(S)$ and 3 and 7 are nonadjacent in $\Gamma(G)$. On the other hand by Lemma 2.1, every element $g \in G$ such that 7 divides the order of g, is a vanishing element of G and so $|g|$ belongs to $\mathrm{Vo}(G)$. This shows that G does not contain any element order $3 \cdot 7$. Let U be a Sylow 7-subgroup of G and K_3 be a Sylow 3-subgroup of K. Thus by Frattini argument we get that $K_3 \rtimes U$ is a $\{3,7\}$-subgroup of G. Also by the previous discussion we get that $K_3 \rtimes U$ is a Frobenius group with kernel K_3. So $|U| \mid (|K_3| - 1)$ and so $7 \mid (3^2 - 1)$, a contradiction. Also if $S \cong L_3(2^2)$, then we deduce that $|K_3| = 3$ or 9 and so similarly, we get a contradiction. Hence S is not isomorphic to any simple group, except A_9. Therefore, $A_9 \leq G/K \leq S_9$, which by the order of $|G|$ we get that $K = 1$ and so $G \cong A_9$.

Case 2. Let $L = A_8$. Obviously, S is not isomorphic to A_9 (3-part of $|S|$). Let $S \cong A_7$, i.e. $A_7 \leq G/K \leq S_7$. This implies that K is a 2-group, since $|G| = |A_8|$. We remark that by the assumption in $\Gamma(G)$, 5 and 3 are adjacent. This mean that G has an element of order $3 \cdot 5$, while in A_7 there is no element of order $3 \cdot 5$. Thus since K is a 2-group, we get a contradiction. Let $S \cong L_3(2^2)$. In this case, we have $|G| = |L| = |S|$. Hence we get that $G \cong L_3(2^2)$ and so $\Gamma(A_8) = \Gamma(L_3(2^2))$, which is a contradiction by [1]. Therefore $S \cong A_8$ and so similar to the above case, we conclude that G is isomorphic to A_8, which completes the proof. $\square$

Corollary 2.4. *Let G be a finite group such that $|G| = |A_n|$ and $\mathrm{Vo}(G) = \mathrm{Vo}(A_n)$, where $5 \leq n \leq 9$. Then $G \cong A_n$, i.e. Conjecture 1.1, holds for these simple groups.*

References

[1] J. H. Conway, R. T. Curtis, S. P. Norton, R. A. Park and R. A. Wilson, Atlas of Finite Groups, Clarendon Press (Oxford), (1985).

[2] S. Dolfi, E. Pacifici, L. Sanus and P. Spiga, On the vanishing prime graph of solvable groups, *J. Group Theory* **13**, 189–206 (2010).

[3] S. Dolfi, E. Pacifici, L. Sanus and P. Spiga, On the orders of zeros of irreducible characters, *J. Algebra* **321**, 345–352 (2009).

[4] M.F. Ghasemabadi et al., A new characterization of some finite simple groups, *Siberian Mathematical Journal*, (2015)

[5] I. M. Isaacs, Characters of Finite Groups, Academic Prees, New York, (1976).

[6] A. V. Vasiliev, M. A. Grechkoseeva and V. D. Mazurov, Characterization of the finite simple groups by spectrum and order, *Algebra and Logic* **48**, 385–409 (2009).

[7] J. Zhang, Z. Li and C. Shao, Finite groups whose irreducible characters vanish only on elements of prime power order, *Int. Electron. J. Algebra* **9**, 114–123 (2011).

[8] R. Kogani-Moghadam and A. R. Moghaddamfar, Groups with the same order and degree pattern, *Sci. China Math.* **55**, 701–720 (2012).

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