Palestine Journal of Mathematics

Vol. 6(1)(2017) , 68–75

Evaluation of Some Novel Integrals Involving Legendre Function of Second Kind Using Hypergeometric Approach

M.I.Qureshi[a], M.Shadab[b,*] and M.S.Baboo[c]

Communicated by Jose Luis Lopez-Bonilla

MSC 2010 Classifications: 33B99, 33C05, 33C20, 33C45, 33C47.

Keywords and phrases: Heine's integral; Generalized Neumann's integral; Pfaff-Kummer transformation formula;Analytic continuation formula.

Abstract. In this paper some novel integrals with suitable restrictions:

$$\underbrace{\int_x^\infty \cdots \int_x^\infty}_{(n+1)} (x^2-1)^{-n-1} \underbrace{dx \ldots dx}_{(n+1)},\ \int_{-1}^{+1} \frac{y^m P_n(y)dy}{(x-y)},\ \int_{-1}^{+1} \frac{y^{n+1} P_n(y)dy}{(x-y)} \text{ and } \int_0^\infty \frac{d\theta}{\{x+\sqrt{(x^2-1)}\cosh\theta\}^{n+1}}$$

are evaluated in terms of Legendre's function of second kind, using a systematic hypergeometric approach.Such different approach for the evaluation of these integrals is not recorded earlier in the literature of special functions.

Introduction, Definitions and Preliminaries:

In the usual notation, let $\mathbb{R}$ and $\mathbb{C}$ denote the sets of real and complex numbers respectively. Also let

$$\mathbb{N}_0 = \mathbb{N} \cup \{0\}\ , \qquad \mathbb{N} = \{1,2,3,\ldots\} = \mathbb{N}_0 \setminus \{0\}$$

and

$$\mathbb{Z}_0^- = \{0,-1,-2,\ldots\} = \mathbb{Z}^- \cup \{0\}\ , \qquad \mathbb{Z}^- = \{-1,-2,-3,\ldots\}$$

and $\mathbb{Z} = \mathbb{Z}_0^- \cup \mathbb{N}$ being the set of integers.

The *generalized hypergeometric function* ${}_pF_q$ with p numerator parameters $\alpha_1, \alpha_2, \ldots, \alpha_p$ and q denominator parameters $\beta_1, \beta_2, \ldots, \beta_q$, is defined by

$${}_pF_q\left[\begin{array}{c} \alpha_1, \alpha_2, \ldots, \alpha_p; \\ \\ \beta_1, \beta_2, \ldots, \beta_q; \end{array} z\right] = \sum_{n=0}^{\infty} \frac{\prod_{j=1}^{p} (\alpha_j)_n}{\prod_{j=1}^{q} (\beta_j)_n} \frac{z^n}{n!} \tag{0.1}$$

$$\big(p,q \in \mathbb{N}_0;\ p \leqq q+1\ ;\ p \leqq q \text{ and } |z| < \infty;$$

$$p = q+1 \text{ and } |z| < 1;\ p = q+1, |z| = 1 \text{ and } \Re(\omega) > 0\big)$$

where

$$\omega := \sum_{j=1}^{q} \beta_j - \sum_{j=1}^{p} \alpha_j$$

$$\big(\alpha_j \in \mathbb{C}\ (j = 1,2,\ldots,p); \beta_j \in \mathbb{C} \setminus \mathbb{Z}_0^-\ (j = 1,2,\ldots,q)\big).$$

The widely-used Pochhammer symbol $(\lambda)_\nu$ $(\lambda, \nu \in \mathbb{C})$ is defined by

$$(\lambda)_\nu := \frac{\Gamma(\lambda+\nu)}{\Gamma(\lambda)} = \begin{cases} 1 & (\nu = 0; \lambda \in \mathbb{C} \setminus \{0\}) \\ \lambda(\lambda+1)\ldots(\lambda+n-1) & (\nu = n \in \mathbb{N}; \lambda \in \mathbb{C}), \end{cases} \tag{0.2}$$

it is being understood *conventionally* that $(0)_0 = 1$ and assumed *tacitly* that the Γ quotient exists.

$$\int_0^\infty e^{-st} t^{\alpha-1} \mathrm{d}t = \frac{\Gamma(\alpha)}{s^\alpha} \tag{0.3}$$

$$\Big(\Re(s) > 0,\ 0 < \Re(\alpha) < \infty \qquad \text{or} \qquad \Re(s) = 0,\ 0 < \Re(\alpha) < 1\Big).$$

Legendre's duplication formula is given by

$$\sqrt{(\pi)}\Gamma(2z) = 2^{2z-1}\Gamma(z)\Gamma\Big(z+\frac{1}{2}\Big) \qquad \big(2z \in \mathbb{C}\backslash\mathbb{Z}_0^-\big). \tag{0.4}$$

Special case of equation (0.4):

$$\frac{1}{\Gamma(\frac{n+1}{2})} = \frac{2^n\Gamma(\frac{n+2}{2})}{\sqrt{\pi}\,\Gamma(n+1)}. \tag{0.5}$$

If $\Re(m) > -1$ and $\Re(n) > -1$ then

$$\int_0^{\frac{\pi}{2}} \sin^m\theta\cos^n\theta d\theta = \frac{\Gamma(\frac{m+1}{2})\Gamma(\frac{n+1}{2})}{2\Gamma(\frac{m+n+2}{2})}. \tag{0.6}$$

Rodrigue's formula for Legendre's polynomial of first kind :

$$P_n(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}(x^2-1)^n. \tag{0.7}$$

$$\frac{d^n}{dx^n}x^\nu = \nu(\nu-1).........(\nu-n+1)x^{\nu-n}. \tag{0.8}$$

$$\int_{-1}^{+1} x^m P_n(x)dx = 0 \quad if \ \ m = 0,1,2,3,\ldots,(n-1). \tag{0.9}$$

$$\left[\frac{d^m}{dx^m}(x^2-1)^n\right]_{x=\pm1} = 0 \quad if \ \ m = 0,1,2,3,\ldots,(n-1). \tag{0.10}$$

Decomposition of infinite series:

$$\sum_{r=0}^\infty \phi(r) = \sum_{r=0}^\infty \phi(2r) + \sum_{r=0}^\infty \phi(2r+1), \tag{0.11}$$

provided that involved series are convergent.

Property of definite integral:

$$\int_{-a}^{+a} f(x)dx = \begin{cases} 2\int_0^a f(x)dx & (f(-x) = f(x)) \\ 0 & (f(-x) = -f(x)) \end{cases}. \tag{0.12}$$

Pfaff-Kummer transformation formula:

$${}_2F_1\left[\begin{matrix} a,b; \\ d; \end{matrix}\ z\right] = (1-z)^{-a}{}_2F_1\left[\begin{matrix} a,d-b; \\ d; \end{matrix}\ \frac{-z}{1-z}\right] \tag{0.13}$$

where $|arg(1-z)|<\pi$ and $d\in\mathbb{C}\backslash\mathbb{Z}_0^-$.

$$ {}_3F_2\left[\begin{array}{c} \frac{n+3}{2},\frac{n+4}{2},1; \\ \frac{2n+5}{2},2; \end{array} z\right] = \frac{2(2n+3)}{(n+1)(n+2)z}\left\{{}_2F_1\left[\begin{array}{c} \frac{n+1}{2},\frac{n+2}{2}; \\ \frac{2n+3}{2}; \end{array} z\right]-1\right\}, \tag{0.14} $$

The equation (0.14) can be derive easily by expanding ${}_2F_1\left[\begin{array}{c} \frac{n+1}{2},\frac{n+2}{2}; \\ \frac{2n+3}{2}; \end{array} z\right]$.

Analytic continuation formula [1, p.108(2.10.1)]:

$$ \begin{aligned} {}_2F_1\left[\begin{array}{c} a,b; \\ c; \end{array} z\right] &= \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}{}_2F_1\left[\begin{array}{c} a,b; \\ 1+a+b-c; \end{array} 1-z\right] \\ &+ \frac{\Gamma(c)\Gamma(a+b-c)}{\Gamma(a)\Gamma(b)}(1-z)^{c-a-b}{}_2F_1\left[\begin{array}{c} c-a,c-b; \\ c-a-b+1; \end{array} 1-z\right], \end{aligned} \tag{0.15} $$

where $|arg(1-z)|<\pi, |arg(z)|<\pi$ and $a+b-c\neq 0,\pm1,\pm2,\pm3,\ldots$
Above formula holds for all values of a,b,c for which the gamma functions of the numerators are finite and for all values of z for which the series involved converge.

Legendre's function of second kind of order n [2, p.182-equation-4]:

$$ 2^n x^{n+1}\left(\frac{3}{2}\right)_n Q_n(x) = n!\,{}_2F_1\left[\begin{array}{c} \frac{n+1}{2},\frac{n+2}{2}; \\ \frac{2n+3}{2}; \end{array} \frac{1}{x^2}\right]. \tag{0.16} $$

Using Pfaff-Kummer transformation formula(0.13) in equation (0.16), we get

$$ Q_n(x) = \frac{n!}{2^n(\frac{3}{2})_n(\sqrt{x^2-1})^{n+1}}{}_2F_1\left[\begin{array}{c} \frac{n+1}{2},\frac{n+1}{2}; \\ \frac{2n+3}{2}; \end{array} \frac{1}{1-x^2}\right]. \tag{0.17} $$

Further using analytic continuation formula (0.15) in equation (0.17) and applying the result (0.5),we get

$$ \begin{aligned} Q_n(x) &= \frac{2^{n-1}}{n!(\sqrt{x^2-1})^{n+1}}\left\{\left(\Gamma\left(\frac{n+1}{2}\right)\right)^2{}_2F_1\left[\begin{array}{c} \frac{n+1}{2},\frac{n+1}{2}; \\ \frac{1}{2}; \end{array} \frac{-x^2}{1-x^2}\right]\right. \\ &- \left.\left(\frac{2x}{\sqrt{x^2-1}}\right)\left(\Gamma\left(\frac{n+2}{2}\right)\right)^2{}_2F_1\left[\begin{array}{c} \frac{n+2}{2},\frac{n+2}{2}; \\ \frac{3}{2}; \end{array} \frac{-x^2}{1-x^2}\right]\right\}. \end{aligned} \tag{0.18} $$

When n, r are non-negative integers and using the integral (0.6) then we can obtain

$$ \begin{aligned} \int_{-\infty}^{\infty}\frac{d\theta}{\{\cosh(\theta)\}^{(n+r+1)}} &= 2^{n+r+1}\int_{-\infty}^{+\infty}\frac{d\theta}{\{e^{-\theta}(1+e^{2\theta})\}^{(n+r+1)}} \\ &= 2^{n+r+1}\int_0^{\frac{\pi}{2}}\sin^{n+r}(t)\cos^{n+r}(t)dt \\ &= \frac{2^{n+r}\{\Gamma(\frac{n+r+1}{2})\}^2}{n!\,(1+n)_r}. \end{aligned} \tag{0.19} $$

$$ (n+2r+1)_{n+1} = \frac{2^{2n}\,(\frac{3}{2})_n\,(\frac{2n+3}{2})_r\,(n+1)_r}{(\frac{n+1}{2})_r\,(\frac{n+2}{2})_r}. \tag{0.20} $$

1 First integral : Evaluation of the (n+1)-ple integral:

$$I_1 = \underbrace{\int_x^\infty \dots \int_x^\infty}_{(n+1)} (x^2-1)^{-n-1} \underbrace{dx \dots dx}_{(n+1)} = \frac{Q_n(x)}{n!\, 2^n} \tag{1.1}$$

where $x > 1$.

Derivation: Consider the single integral

$$\begin{aligned}
\int_x^\infty (x^2-1)^{-n-1} dx &= \int_x^\infty (x^2)^{-n-1} \left[1 - \frac{1}{x^2}\right]^{-n-1} dx \\
&= \int_x^\infty \Big(\sum_{r=0}^\infty \frac{(n+1)_r}{r! x^{2r+2n+2}} \Big) dx = \sum_{r=0}^\infty \frac{(n+1)_r}{r!} \int_x^\infty x^{-2n-2r-2} dx \\
&= \sum_{r=0}^\infty \frac{(n+1)_r}{r!\, (2n+2r+1)\, x^{2n+2r+1}}
\end{aligned}$$

$$\int_x^\infty (x^2-1)^{-n-1} dx = \sum_{r=0}^\infty \frac{(n+1)_r}{r!\, (2n+2r+1)\, x^{2n+2r+1}} \tag{1.2}$$

Similarly we can obtain double integral in the following form

$$\int_x^\infty \int_x^\infty (x^2-1)^{-n-1} dx dx = \sum_{r=0}^\infty \frac{(n+1)_r}{r!\, (2n+2r+1)\, (2n+2r)\, x^{2n+2r}} \tag{1.3}$$

Therefore

$$\begin{aligned}
I_1 &= \underbrace{\int_x^\infty \dots \int_x^\infty}_{(n+1)} (x^2-1)^{-n-1} \underbrace{dx \dots dx}_{(n+1)} \\
&= \sum_{r=0}^\infty \frac{(n+1)_r}{r!(2n+2r+1)\, (2n+2r) \dots (n+2r+2)(n+2r+1)\, x^{n+2r+1}} \\
&= \sum_{r=0}^\infty \frac{(n+1)_r}{r!(n+2r+1)(n+2r+2)(n+2r+3)\dots\, (2n+2r)(2n+2r+1)\, x^{n+2r+1}} \\
&= \sum_{r=0}^\infty \frac{(n+1)_r}{r!(n+2r+1)_{n+1}\, x^{n+2r+1}}
\end{aligned} \tag{1.4}$$

Now applying the formula (0.20) in equation (1.4),we get

$$\begin{aligned}
I_1 &= \frac{1}{2^{2n}\, x^{n+1}\, (\frac{3}{2})_n} \sum_{r=0}^\infty \frac{(\frac{n+1}{2})_r\, (\frac{n+2}{2})_r}{r! (\frac{2n+3}{2})_r\, x^{2r}} \\
&= \frac{1}{2^{2n}\, x^{n+1}\, (\frac{3}{2})_n} {}_2F_1 \left[\begin{matrix} \frac{n+1}{2}, \frac{n+2}{2}; & \frac{1}{x^2} \\ n+\frac{3}{2}; & \end{matrix} \right] = \frac{Q_n(x)}{n!\, 2^n}.
\end{aligned}$$

2 Second integral : Generalization of Neumann's Integral

$$I_2 = \int_{-1}^{+1} \frac{y^m P_n(y) dy}{(x-y)} = 2x^m Q_n(x) \tag{2.1}$$

where $m \leq n$, $|y| \leq 1$, $x > 1$ and m, n are non-negative integers.

Derivation: Case I: If m=n then

$$I_3 = \int_{-1}^{+1} \frac{y^n P_n(y)dy}{(x-y)} = \int_{-1}^{+1} \frac{y^n}{x}\Big(1-\frac{y}{x}\Big)^{-1} P_n(y)dy \qquad ,\Big(\Big|\frac{y}{x}\Big| < 1\Big)$$

$$= \frac{1}{x}\int_{-1}^{+1} y^n {}_1F_0\left[\begin{array}{cc} 1; & \frac{y}{x} \\ -; & \end{array}\right] P_n(y)dy = \frac{1}{2^n n!x}\sum_{r=0}^{\infty}\frac{1}{x^r}\int_{-1}^{+1} y^{n+r}\Big(\frac{d^n}{dy^n}(y^2-1)^n\Big)dy$$

$$= \frac{1}{2^n n!x}\sum_{r=0}^{\infty}\frac{(-1)(n+r)}{x^r}\int_{-1}^{+1} y^{n+r-1}\Big(\frac{d^{n-1}}{dy^{n-1}}(y^2-1)^n\Big)dy$$

$\vdots$

$$I_3 = \frac{1}{2^n n!x}\sum_{r=0}^{\infty}\frac{(r+1)_n}{x^r}\int_{-1}^{+1} y^r(1-y^2)^n dy \tag{2.2}$$

Applying the series identity (0.11) in equation (2.2), using the definite integral property (0.12) and special integral (0.6) ,we get

$$I_3 = \frac{1}{2^n n!x}\sum_{r=0}^{\infty}\frac{(2r+1)_n}{x^{2r}}\int_{-1}^{+1} y^{2r}(1-y^2)^n dy = \frac{1}{2^{n-1}n!x}\sum_{r=0}^{\infty}\frac{(2r+1)_n}{x^{2r}}\int_0^{\frac{\pi}{2}} \sin^{2r}\theta\cos^{2n+1}\theta d\theta$$

$$= \frac{n!}{2^{n-1}(\frac{3}{2})_n x}\sum_{r=0}^{\infty}\frac{(\frac{n+1}{2})_r(\frac{n+2}{2})_r}{(\frac{2n+3}{2})_r r! x^{2r}} = \frac{n!}{2^{n-1}(\frac{3}{2})_n x}{}_2F_1\left[\begin{array}{cc} \frac{n+1}{2},\frac{n+2}{2}; & \frac{1}{x^2} \\ \frac{2n+3}{2}; & \end{array}\right]$$

Therefore

$$I_3 = 2x^n Q_n(x). \tag{2.3}$$

Case II: If $m = n-1$ then

$$I_4 = \int_{-1}^{+1} \frac{y^{n-1} P_n(y)dy}{(x-y)} = \int_{-1}^{+1} \frac{y^{n-1}}{x}\Big(1-\frac{y}{x}\Big)^{-1} P_n(y)dy \qquad ,\Big(\Big|\frac{y}{x}\Big| < 1\Big)$$

$$= \frac{1}{x}\int_{-1}^{+1} y^{n-1} {}_1F_0\left[\begin{array}{cc} 1; & \frac{y}{x} \\ -; & \end{array}\right] P_n(y)dy = \frac{1}{2^n n!x}\sum_{r=0}^{\infty}\frac{1}{x^r}\int_{-1}^{+1} y^{n+r-1}\Big(\frac{d^n}{dy^n}(y^2-1)^n\Big)dy$$

$$= \frac{1}{2^n n!x}\sum_{r=0}^{\infty}\frac{(-1)(n+r-1)}{x^r}\int_{-1}^{+1} y^{n+r-2}\Big(\frac{d^{n-1}}{dy^{n-1}}(y^2-1)^n\Big)dy$$

$\vdots$

$$= \frac{1}{2^n n!x}\sum_{r=0}^{\infty}\frac{(r)_n}{x^r}\int_{-1}^{+1} y^{r-1}(1-y^2)^n dy = \frac{1}{2^n n!x}\sum_{r=0}^{\infty}\frac{(2r+1)_n}{x^{2r+1}}\int_{-1}^{+1} y^{2r}(1-y^2)^n dy$$

$$= \frac{1}{2^{n-1}n!x}\sum_{r=0}^{\infty}\frac{(2r+1)_n}{x^{2r+1}}\int_0^{\frac{\pi}{2}} \sin^{2r}\theta\cos^{2n+1}\theta d\theta = \frac{n!}{2^{n-1}(\frac{3}{2})_n x^2}\sum_{r=0}^{\infty}\frac{(\frac{n+1}{2})_r(\frac{n+2}{2})_r}{(\frac{2n+3}{2})_r r! x^{2r}}$$

$$= \frac{n!}{2^{n-1}(\frac{3}{2})_n x^2}{}_2F_1\left[\begin{array}{cc} \frac{n+1}{2},\frac{n+2}{2}; & \frac{1}{x^2} \\ \frac{2n+3}{2}; & \end{array}\right]$$

Therefore

$$I_4 = 2x^{n-1}Q_n(x). \tag{2.4}$$

Case III: When $m = 0$, we get Neumann's integral

$$I_5 = \int_{-1}^{+1} \frac{P_n(y)dy}{(x-y)} = 2Q_n(x). \tag{2.5}$$

3 Third integral : Further generalization of Neumann's integral

$$I_6 = \int_{-1}^{+1} \frac{y^{n+1}P_n(y)dy}{(x-y)} = 2x^{n+1}Q_n(x) - \frac{2^{n+1}(n!)^2}{(2n+1)!} \tag{3.1}$$

where $x > 1$ and $|y| \leq 1$.
Derivation :

$$\begin{aligned}
I_6 &= \int_{-1}^{+1} y^{n+1}\frac{1}{x}\Big(1-\frac{y}{x}\Big)^{-1} P_n(y)dy = \frac{1}{x}\int_{-1}^{+1} y^{n+1}\Big(\sum_{r=0}^{\infty}\frac{(1)_r(\frac{y}{x})^r}{r!}\Big)P_n(y)dy \\
&= \frac{1}{x}\sum_{r=0}^{\infty}\frac{1}{x^r}\int_{-1}^{+1} y^{n+r+1}P_n(y)dy = \frac{1}{2^n n! x}\sum_{r=0}^{\infty}\frac{1}{x^r}\int_{-1}^{+1} y^{n+r+1}\Big(\frac{d^n}{dy^n}(y^2-1)^n\Big)dy \\
&= \frac{1}{2^n n! x}\sum_{r=0}^{\infty}\frac{(-1)(n+r+1)}{x^r}\int_{-1}^{+1} y^{n+r}\Big(\frac{d^{n-1}}{dy^{n-1}}(y^2-1)^n\Big)dy \\
&= \frac{1}{2^n n! x}\sum_{r=0}^{\infty}\frac{(-1)^2(n+r+1)(n+r)}{x^r}\int_{-1}^{+1} y^{n+r-1}\Big(\frac{d^{n-2}}{dy^{n-2}}(y^2-1)^n\Big)dy \\
&\;\vdots
\end{aligned}$$

$$I_6 = \frac{1}{2^n n! x}\sum_{r=0}^{\infty}\frac{(r+2)_n}{x^r}\int_{-1}^{+1} y^{r+1}(1-y^2)^n dy \tag{3.2}$$

Now applying the series identity (0.11) in equation (3.2),using the definite integral property (0.12) and special integral (0.6),we get

$$\begin{aligned}
I_6 &= \frac{1}{2^n n! x}\sum_{r=0}^{\infty}\frac{(n+2)!(n+3)_{2r}}{(3)_{2r}x^{2r+1}}\int_0^1 y^{2r+2}(1-y^2)^n dy \\
&= \frac{1}{2^n n! x}\sum_{r=0}^{\infty}\frac{(n+2)!(n+3)_{2r}}{(3)_{2r}x^{2r+1}}\int_0^{\frac{\pi}{2}} \sin^{2r+2}\theta\cos^{2n+1}\theta d\theta \\
&= \frac{(n+2)!}{3(2)^n x^2(\frac{5}{2})_n}\,{}_3F_2\left[\begin{matrix}\frac{n+3}{2},\frac{n+4}{2},1; & \frac{1}{x^2}\\ \frac{5+2n}{2},2; & \end{matrix}\right]
\end{aligned} \tag{3.3}$$

Using the formula (0.14) in equation (3.3) and applying the definition of $Q_n(x)$, we get (3.1).

4 Fourth integral : Heine's integral

$$I_7 = \int_0^\infty \frac{d\theta}{\{x+\sqrt{(x^2-1)}\cosh(\theta)\}^{n+1}} \quad = \quad Q_n(x) \tag{4.1}$$

where $|x|>1$.
Derivation:
Since the integrand of the integral I_7 is an even function of θ, therefore in the view of definite integral property (0.12), we can write

$$\begin{aligned}
2I_7 &= \int_{-\infty}^{+\infty} \frac{d\theta}{\{x+\sqrt{(x^2-1)}\cosh(\theta)\}^{n+1}} \\
&= \int_{-\infty}^{+\infty} \frac{1}{\{\sqrt{(x^2-1)}\cosh(\theta)\}^{n+1}} \left[1+\frac{x}{\sqrt{(x^2-1)}\cosh(\theta)}\right]^{-n-1} d\theta \\
&= \frac{1}{(\sqrt{x^2-1})^{n+1}} \sum_{r=0}^{\infty} \frac{(n+1)_r\left(\frac{-x}{\sqrt{x^2-1}}\right)^r}{r!} \int_{-\infty}^{\infty} \frac{d\theta}{\{\cosh(\theta)\}^{(n+r+1)}} \qquad (4.2)
\end{aligned}$$

Applying the integral (0.19) in equation (4.2),we get

$$2I_7 \quad = \quad \frac{2^n}{n!(\sqrt{x^2-1})^{n+1}} \sum_{r=0}^{\infty} \frac{\{\Gamma(\frac{n+r+1}{2})\}^2\left(\frac{-2x}{\sqrt{x^2-1}}\right)^r}{r!} \tag{4.3}$$

Applying the series identity (0.11) in equation (4.3),after simplification we get

$$\begin{aligned}
2I_7 &= \frac{2^n}{n!(\sqrt{x^2-1})^{n+1}} \left\{ \left(\Gamma\left(\frac{n+1}{2}\right)\right)^2 \sum_{r=0}^{\infty} \frac{(\frac{n+1}{2})_r(\frac{n+1}{2})_r(\frac{-x^2}{1-x^2})^r}{(\frac{1}{2})_r r!} \right. \\
&- \left. \left(\frac{2x}{\sqrt{x^2-1}}\right)\left(\Gamma\left(\frac{n+2}{2}\right)\right)^2 \sum_{r=0}^{\infty} \frac{(\frac{n+2}{2})_r(\frac{n+2}{2})_r(\frac{-x^2}{1-x^2})^r}{(\frac{3}{2})_r r!} \right\} \\
&= \frac{2^n}{n!(\sqrt{x^2-1})^{n+1}} \left\{ \left(\Gamma\left(\frac{n+1}{2}\right)\right)^2 {}_2F_1\left[\begin{array}{c} \frac{n+1}{2}, \frac{n+1}{2}; \\ \frac{1}{2}; \end{array} \frac{-x^2}{1-x^2}\right] \right. \\
&- \left. \left(\frac{2x}{\sqrt{x^2-1}}\right)\left(\Gamma\left(\frac{n+2}{2}\right)\right)^2 {}_2F_1\left[\begin{array}{c} \frac{n+2}{2}, \frac{n+2}{2}; \\ \frac{3}{2}; \end{array} \frac{-x^2}{1-x^2}\right] \right\} \qquad (4.4)
\end{aligned}$$

Now using the formula (0.18) in equation (4.4),we get

$$2I_7 = 2Q_n(x). \tag{4.5}$$

References

[1] A. Erdélyi, W. Magnus, F. Oberhettinger and F.G. Tricomi, *Higher Transcendental Functions*, **Vol.I** (Bateman Manuscript Project),McGraw-Hill,Book Co. Inc.,New York,Toronto and London,(1953).

[2] E.D. Rainville, *Special Functions*, The Macmillan Co. Inc., New York (1960). Reprinted by Chelsea Publ. Co. Bronx, NewYork(1971).

Author information

M.I.Qureshi[a], M.Shadab[b,*] and M.S.Baboo[c],
[a,b]Department of Applied Sciences and Humanities, Faculty of Engineering and Technology, Jamia Millia Islamia (A Central University), New Delhi-110025,
[c]School of Basic Sciences and Research,Sharda University, Greater Noida,Uttar Pradesh 201306, India.
E-mail: miqureshi_delhi@yahoo.co.in[a], shadabmohd786@gmail.com(Corresponding author)[b,*], mesub007@gmail.com[c].

Received: November 23, 2015.

Accepted: January 19, 2016.