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Bounds for the initial coefficients of a certain subclass of bi-univalent functions of complex order

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Abstract The object of the present paper is to examine some properties of the coefficients of a certain class of analytic functions defined on a region of the complex plane. The first of these properties is upper bound estimates for the coefficients of this function class. Here, we obtain upper bound estimates for the initial three coefficients $|a_2|, |a_3|$ and $|a_4|$ of the functions belonging to the examined class. Moreover, we are investigating the Fekete-Szegö problem for this function class.

1 Introduction and preliminaries

Let $\mathcal{A}$ be the class of the functions in the form

$$f(z) = z + a_2 z^2 + a_3 z^3 + \cdots = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1.1}$$

which are analytic in the open unit disk $\mathbf{U} = \{z \in \mathbb{C} : |z| < 1\}$.
It is well-known that a function $f : \mathbb{C} \to \mathbb{C}$ is said to be univalent if the following condition is satisfied: $z_1 = z_2$ if $f(z_1) = f(z_2)$ or $f(z_1) \neq f(z_2)$ if $z_1 \neq z_2$.
We denote by $\mathcal{S}$ the subclass of $\mathcal{A}$ consisting of univalent functions in $\mathbf{U}$. Some of the important subclass of $\mathcal{S}$ is the class $\aleph(\alpha, \beta)$ that is defined as follows

$$\aleph(\alpha, \beta) = \left\{ f \in \mathcal{S} : |\arg [f'(z) + \beta z f''(z)]| < \frac{\pi}{2}\alpha, z \in \mathbf{U} \right\}, \alpha \in (0, 1), \beta \geq 0$$

It is well-known that (see, for example, [7]) every function $f \in \mathcal{S}$ has an inverse f^{-1}, which is defined by

$$f^{-1}(f(z)) = z, z \in \mathbf{U}, f(f^{-1}(w)) = w, w \in D = \{w : |w| < r_0(f)\}, r_0(f) \geq \frac{1}{4},$$

where $f^{-1}(w) = w - a_2 w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2 a_3 + a_4)w^4 + \cdots, w \in D$.
A function $f \in A$ is said to be bi-univalent in $\mathbf{U}$ if both f and f^{-1} are univalent. Let Σ denote the class of bi-univalent functions in $\mathbf{U}$ given (1.1).
In 1967, Brannan and Clunie [1] conjectured that $|a_2| < \sqrt{2}$ for each function $f \in \Sigma$. In 1984, Tan [33] obtained the bound for $|a_2|$, namely, that $|a_2| < 1.485$, which is the best known estimate for functions in the class Σ. In 1985, Kedzierawski [15] proved the Brannan-Clunie conjecture for bi-starlike functions. Brannan and Taha [2] obtained bound estimates on the initial coefficients $|a_2|$ and $|a_3|$ for the functions in the classes of bi-starlike functions of order α and bi-convex functions of order α.
In recent years, the pioneering work by Srivastava et al. [24] actually revived the study of bi-univalent functions. In fact, ever since publication of their widely-cited paper [24], several results on coefficient bound estimates for the initial coefficients $|a_2|, |a_3|$ and $|a_4|$ were proved for various subclasses of Σ (see, for example, [4, 5, 9, 12, 17, 21, 23, 25, 26, 27, 28, 29, 30, 31, 32, 34, 35, 36]).
We want to briefly mention some of these works. By Srivastava et all. [31], two general subclasses $H_\Sigma(\tau, \mu, \gamma; \alpha)$ and $H_\Sigma(\tau, \mu, \gamma; \beta)$ of analytic and bi-univalent functions were introduced

and estimates of the first coefficients in the series expansion of the functions in these classses were given. The results obtained by authors generalize many earlier results in this field. By Srivastava and Bansal [27], a subclass $\Sigma(\tau,\gamma,\varphi)$ of analytic and bi-univalent functions is defined as follows: Let $\gamma\in[0,1]$ and $\tau\in\mathbb{C}-\{0\}$. A function $f\in\Sigma$ is said to be in the class $\Sigma(\tau,\gamma,\varphi)$ if each of the following subordination conditions holds true:

$$1+\frac{1}{\tau}\left(f'(z)+\gamma zf''(z)-1\right)\prec\varphi(z)\,, z\in U,$$

$$1+\frac{1}{\tau}\left(g'(w)+\gamma wg''(w)-1\right)\prec\varphi(w)\,, w\in U,$$

where $\varphi(z)$ is an analytic function in U and $g(w)=f^{-1}(w)$; that is, $g(f(z))=z$ for all $z\in U$. In this study, coefficient problem in the defined class is examined.

By Srivastava et all. [30], two new subclasses of Σ_B meromorfphically bi-univalent function class were introduced and estimates for the coefficients $|b_0|$ and $|b_1|$ were given. The study Srivastava et all. [26] is on coefficient problems for $H^{\alpha}_{\Sigma,m}$ and $H_{\Sigma,m}(\beta)$ $m-$ fold symmetric bi-univalent function classes. In the same study, coefficient problem for the inverse function was also investigated. In the study [32], introduced and investigated the Fekete-Szegö functional associated with a new subclass of analytic functions, which defined by using the principle of quasi-subordination between analytic functions. Some sufficient conditions for the functions belonging to this class are also derived.

Recently, Deniz [6] and Kumar et al. [18] both extended and improved the results of Brannan and Taha [2] by generalizing their classes by means of the principle of subordination between analytic functions.

Despite the numerous studies mentioned above, the problem of estimating the coefficients $|a_n|$ $(n=2,3,...)$ for the general class functions Σ is still open (in this connection see, also [28]).

One of the important tools in the theory of analytic functions is the functional $H_2(1)=a_3-a_2^2$ which is known as the Fekete-Szegö functional and one usually considers the further generalized functional $a_3-\mu a_2^2$, where μ is some real number (see [8]). Estimating for the upper bound of $\left|a_3-\mu a_2^2\right|$ is known as the Fekete-Szegö problem.

The will-known result due to them states that if $f\in\mathcal{A}$, then

$$|a_3-\mu {a_2}^2|\leq\begin{cases}3-4\mu, & if\ \ \mu\in(-\infty,0]\,,\\ 1+exp\left(\frac{-2\mu}{1-\mu}\right), & if\ \ \mu\in[0,1)\,,\\ 4\mu-3, & if\ \ \mu\in[1,+\infty)\,.\end{cases}$$

Furthermore, Hummel [13, 14] obtained sharp estimates for $|a_3-\mu {a_2}^2|$ when f is a convex function.

In 1969, Keogh and Merkes [16] solved the Fekete-Szegö problem for the class close-to-convex, starlike and convex functions.

Someone can see the Fekete-Szegö problem for the classes of starlike functions of order α and convex functions of order α at special cases in the paper of Orhan et al. [20]. On the other hand, recently, Çağlar and Aslan (see [3]) have obtained Fekete-Szegö inequality for a subclass of bi-univalent functions. Also, Zaprawa (see [37, 38]) have studied on Fekete-Szegö problem for some subclasses of bi-univalent functions. In special cases, he studied the Fekete-Szegö problem for the subclasses bi-starlike functions of order α and bi-convex functions of order α.

Motivated by the aforementioned works, we define a new subclass of bi-univalent functions Σ as follows.

Definition 1.1. A function $f\in\Sigma$ given by (1.1) is said to be in the class $\aleph_\Sigma(\alpha,\beta,\tau)$ if the following conditions are satisfied

$$\left|\arg\left\{1+\frac{1}{\tau}\left[f'(z)+\beta zf''(z)-1\right]\right\}\right|<\frac{\pi}{2}\alpha, z\in\mathbf{U}, \tau\in\mathbb{C}^*=\mathbb{C}-\{0\}\,,\alpha\in(0,1)\,,\beta\geq0$$

and

$$\left|\arg\left\{1+\frac{1}{\tau}\left[g'(w)+\beta wg''(w)-1\right]\right\}\right|<\frac{\pi}{2}\alpha, w\in D, \tau\in\mathbb{C}^*=\mathbb{C}-\{0\}\,,\alpha\in(0,1)\,,\beta\geq0$$

where the function $g=f^{-1}$.

Remark 1.2. Choose $\tau = 1$ in Definition 1.1, we have function class $\aleph_\Sigma(\alpha,\beta,1) = H_\Sigma(\alpha,\beta)$, $\alpha \in (0,1)\,, \beta \geq 0$; that is,

$$f \in H_\Sigma(\alpha,\beta) \Leftrightarrow \left|\arg\left(f'(z) + \beta z f''(z)\right)\right| < \frac{\pi}{2}\alpha, z \in \mathbf{U}$$

and

$$\left|\arg\left(g'(w) + \beta w g''(w)\right)\right| < \frac{\pi}{2}\alpha, w \in D,$$

where $g = f^{-1}$.

Remark 1.3. Choose $\beta = 0$ in Definition 1.1, we have function class $\aleph_\Sigma(\alpha,0,\tau)$, $\alpha \in (0,1)\,,$ $\tau \in \mathbb{C}^* = \mathbb{C} - \{0\}$; that is,

$$f \in \aleph_\Sigma(\alpha,0,\tau) \Leftrightarrow \left|\arg\left\{1 + \frac{1}{\tau}\left[f'(z) - 1\right]\right\}\right| < \frac{\pi}{2}\alpha, z \in \mathbf{U}$$

and

$$\left|\arg\left\{1 + \frac{1}{\tau}\left[g'(w) - 1\right]\right\}\right| < \frac{\pi}{2}\alpha, w \in D,$$

where $g = f^{-1}$.

Remark 1.4. Choose $\beta = 0, \tau = 1$ Definition 1.1, we have function class $\aleph_\Sigma(\alpha,0,1) = \aleph_\Sigma(\alpha,0)$, $\alpha \in (0,1)$; that is,

$$f \in \aleph_\Sigma(\alpha,0) \Leftrightarrow \left|\arg\left(f'(z)\right)\right| < \frac{\pi}{2}\alpha, z \in \mathbf{U}$$

and

$$\left|\arg\left(g'(w)\right)\right| < \frac{\pi}{2}\alpha, w \in D,$$

where $g = f^{-1}$.

Remark 1.5. Choose $\beta = 1$ in Definition 1.1, we have function class $\aleph_\Sigma(\alpha,1,\tau)$, $\alpha \in (0,1)\,,$ $\tau \in \mathbb{C}^* = \mathbb{C} - \{0\}$; that is,

$$f \in \aleph_\Sigma(\alpha,1,\tau) \Leftrightarrow \left|\arg\left\{1 + \frac{1}{\tau}\left[f'(z) + zf''(z) - 1\right]\right\}\right| < \frac{\pi}{2}\alpha, z \in \mathbf{U}$$

and

$$\left|\arg\left\{1 + \frac{1}{\tau}\left[g'(w) + wg''(w) - 1\right]\right\}\right| < \frac{\pi}{2}\alpha, w \in D,$$

where $g = f^{-1}$.

Remark 1.6. Choose $\beta = 1, \tau = 1$ in Definition 1.1, we have function class

$$\aleph_\Sigma(\alpha,1,1) = \aleph_\Sigma(\alpha),\ \alpha \in (0,1)\,;$$

that is,

$$f \in \aleph_\Sigma(\alpha,1) \Leftrightarrow \left|\arg\left(f'(z) + zf''(z)\right)\right| < \frac{\pi}{2}\alpha, z \in \mathbf{U}$$

and

$$\left|\arg\left(g'(w) + wg''(w)\right)\right| < \frac{\pi}{2}\alpha, w \in D,$$

where $g = f^{-1}$.

Recently, Frasin [10] investigated the subclass $\aleph_\Sigma(\alpha,\beta,1) = H_\Sigma(\alpha,\beta), \alpha \in (0,1)\,, \beta > 0$ with condition

$$2(1-\alpha)\sum_{n=1}^{+\infty}\frac{(-1)^{n-1}}{\beta n + 1} \leq 1.$$

He found bound estimates for two first coefficients of the functions belonging to this class.
The object of the present paper is to find the upper bound estimates for the initial three coefficients $|a_2|\,, |a_3|$ and $|a_4|$ of the functions belonging to the class $\aleph_\Sigma(\alpha,\beta,\tau)$ and its special cases.
In this paper, the Fekete-Szegö problem for this function class is also investigated.
To prove our main results, we need require the following lemmas.

Lemma 1.7 (See, for example, [22]). *If $p \in \mathbf{P}$, then the estimates $|p_n| \leq 2, n = 1, 2, 3, ...$ are sharp, where $\mathbf{P}$ is the family of all functions p, analytic in $\mathbf{U}$ for which $p(0) = 1$ and* Re $(p(z)) > 0$ *($z \in \mathbf{U}$), and*

$$p(z) = 1 + p_1 z + p_2 z^2 + \cdots, \; z \in \mathbf{U}. \tag{1.2}$$

Lemma 1.8 (See, for example, [11]). *If the function $p \in \mathbf{P}$ is given by the series (1.2), then*

$$2p_2 = p_1^2 + \left(4 - p_1^2\right) x,$$

$$4p_3 = p_1^3 + 2\left(4 - p_1^2\right) p_1 x - \left(4 - p_1^2\right) p_1 x^2 + 2\left(4 - p_1^2\right)\left(1 - |x|^2\right) z$$

for some x and z with $|x| \leq 1$ and $|z| \leq 1$.

2 Coefficient bound estimates for the function class $\aleph_\Sigma(\alpha, \beta, \tau)$

In this section, we prove the following theorem on upper bound estimates for the initial three coefficients of the function class $\aleph_\Sigma(\alpha, \beta, \tau)$.

Theorem 2.1. *Let the function $f(z)$ given by (1.1) be in the class $\aleph_\Sigma(\alpha, \beta, \tau)$, $\alpha \in (0, 1)$, $\beta \in [0, 1]$, $\tau \in \mathbb{C}^* = \mathbb{C} - \{0\}$. Then,*

$$|a_2| \leq \frac{\alpha\,|\tau|}{1 + \beta}$$

and

$$|a_3| \leq \begin{cases} \frac{2\alpha|\tau|}{3(1+2\beta)}, & if \quad |\tau| \in (0, \tau_0), \\ \frac{\alpha^2|\tau|^2}{(1+\beta)^2}, & if \quad |\tau| \in [\tau_0, +\infty), \end{cases}$$

where $\tau_0 = \frac{2(1+\beta)^2}{3\alpha(1+2\beta)}$.
Also,

$$|a_4| \leq \frac{\alpha\,|\tau|}{2\,(1 + 3\beta)}.$$

Proof. Let $f \in \aleph_\Sigma(\alpha, \beta, \tau), \alpha \in (0, 1), \beta \in [0, 1], \tau \in \mathbb{C}^* = \mathbb{C} - \{0\}$ and $g = f^{-1}$. Then,

$$1 + \frac{1}{\tau}\left[f'(z) + \beta z f''(z) - 1\right] = [p(z)]^\alpha \tag{2.1}$$

and

$$1 + \frac{1}{\tau}\left[g'(w) + \beta w g''(w) - 1\right] = [q(w)]^\alpha, \tag{2.2}$$

where functions $p(z) = 1 + p_1 z + p_2 z^2 + \cdots$ and $q(w) = 1 + q_1 w + q_2 w^2 + \cdots$ are in the class $\mathbf{P}$.
Equating the coefficients in (2.1) and (2.2), we get

$$2(1 + \beta)a_2 = \alpha\tau p_1,$$

$$3(1 + 2\beta)a_3 = \alpha\tau p_2 + \frac{\alpha(\alpha - 1)\tau}{2} p_1^2,$$

$$4(1 + 3\beta)a_4 = \alpha\tau p_3. \tag{2.3}$$

and

$$-2(1 + \beta)a_2 = \alpha\tau q_1,$$

$$3(1 + 2\beta)\left(2a_2^2 - a_3\right) = \alpha\tau q_2 + \frac{\alpha(\alpha - 1)\tau}{2} q_1^2,$$

$$-4(1 + 3\beta)\left(5a_2^3 - 5a_2 a_3 + a_4\right) = \alpha\tau q_3. \tag{2.4}$$

From the first equality of (2.3) and (2.4), we find that

$$\frac{\alpha\tau}{2(1+\beta)}p_1 = a_2 = -\frac{\alpha\tau}{2(1+\beta)}q_1. \tag{2.5}$$

Also, from the second equality of (2.3) and (2.4), considering (2.5), we get

$$a_3 = a_2^2 + \frac{\alpha\tau}{6(1+2\beta)}\left(p_2 - q_2\right) = \frac{\alpha^2\tau^2}{4(1+\beta)^2}p_1^2 + \frac{\alpha\tau}{6(1+2\beta)}\left(p_2 - q_2\right). \tag{2.6}$$

Subtracting the third equality of (2.4) from the third equality of (2.3) and considering (2.5), we can easily obtain that

$$a_4 = \frac{5\alpha^2\tau^2}{24(1+\beta)(1+2\beta)}p_1\left(p_2 - q_2\right) + \frac{\alpha\tau}{8(1+3\beta)}\left(p_3 - q_3\right). \tag{2.7}$$

In view of Lemma 1.8, since (see (2.5)) $p_1 = -q_1$, we can write

$$\left.\begin{aligned} 2p_2 &= p_1^2 + \left(4 - p_1^2\right)x, \\ 2q_2 &= q_1^2 + \left(4 - q_1^2\right)y \end{aligned}\right\} \Rightarrow p_2 - q_2 = \frac{4 - p_1^2}{2}(x - y) \tag{2.8}$$

and

$$\left.\begin{aligned} 4p_3 &= p_1^3 + 2\left(4 - p_1^2\right)p_1x - \left(4 - p_1^2\right)p_1x^2 + 2\left(4 - p_1^2\right)\left(1 - |x|^2\right)z, \\ 4q_3 &= q_1^3 + 2\left(4 - q_1^2\right)q_1y - \left(4 - q_1^2\right)q_1y^2 + 2\left(4 - q_1^2\right)\left(1 - |y|^2\right)w \end{aligned}\right\} \Rightarrow$$

$$p_3 - q_3 = \frac{p_1^3}{2} + \frac{p_1\left(4 - p_1^2\right)}{2}(x+y) - \frac{p_1\left(4 - p_1^2\right)}{4}\left(x^2 + y^2\right) + \frac{4 - p_1^2}{2}\left[\left(1 - |x|^2\right)z - \left(1 - |y|^2\right)w\right] \tag{2.9}$$

for some x, y and z, w with $|x| \leq 1, |y| \leq 1, |z| \leq 1$ and $|w| \leq 1$.
Since $|p_1| \leq 2$, we may assume without any restriction that $t \in [0, 2]$, where $t = |p_1|$.
From (2.5), we easily see that

$$|a_2| \leq \frac{\alpha\,|\tau|}{2(1+\beta)}t,\ t \in [0, 2],$$

so

$$|a_2| \leq \frac{\alpha\,|\tau|}{1+\beta} \tag{2.10}$$

Substituting the expression (2.8) in (2.6) and using triangle inequality, putting $|x| = \xi,\ |y| = \eta$, we can easily obtain that

$$|a_3| \leq C_1(t)\left(\xi + \eta\right) + C_2(t) = F\left(\xi, \eta\right), \tag{2.11}$$

where

$$C_1(t) = \frac{\alpha\,|\tau|\left(4 - t^2\right)}{12\left(1+2\beta\right)} \geq 0,\ C_2(t) = \frac{\alpha^2|\tau|^2}{4(1+\beta)^2}t^2 \geq 0,\ t \in [0, 2].$$

It is clear that the maximum of the function $F(\xi, \eta)$ occurs at $(\xi, \eta) = (1, 1)$. Therefore,

$$\max\left\{F(\xi, \eta) : \xi, \eta \in [0, 1]\right\} = F(1, 1) = 2C_1(t) + C_2(t).$$

Let the function $G : [0, 2] \to \mathbb{R}$ defined by

$$G(t) = 2C_1(t) + C_2(t) \tag{2.12}$$

for fixed value of $\tau \in \mathbb{C}^*$=$\mathbb{C} - \{0\}$.
Substituting the value $C_1(t)$ and $C_2(t)$ in (2.12), we get

$$G(t) = A\left(\alpha, \beta; \tau\right)t^2 + B\left(\alpha, \beta; \tau\right),$$

where

$$A(\alpha,\beta;\tau)=\frac{\alpha^2|\tau|}{4(1+\beta)^2}\left[|\tau|-\frac{2(1+\beta)^2}{3\alpha(1+2\beta)}\right] \text{ and } B(\alpha,\beta;\tau)=\frac{2\alpha|\tau|}{3(1+2\beta)}.$$

Now, we must investigate the maximum of the function $G(t)$ in the interval $[0,2]$.
By simple computation, we have

$$G'(t)=2A(\alpha,\beta;\tau)\,t.$$

It is clear that $G'(t)<0$ if $A(\alpha,\beta;\tau)<0$; that is, the function $G(t)$ is a strictly decreasing function if $|\tau|\in(0,\tau_0)$, where $\tau_0=\frac{2(1+\beta)^2}{3\alpha(1+2\beta)}$.
Therefore,

$$\max\{G(t):t\in[0,2]\}=G(0)=2C_1(0)=\frac{2\alpha|\tau|}{3(1+2\beta)}. \tag{2.13}$$

Also, $G'(t)\geq 0$ if $|\tau|\geq\tau_0$; that is, $G(t)$ is an increasing function for $|\tau|\geq\tau_0$. Hence,

$$\max\{G(t):t\in[0,2]\}=G(2)=C_2(2)=\frac{\alpha^2|\tau|^2}{(1+\beta)^2}. \tag{2.14}$$

Thus, from (2.11) and (2.14), we obtain

$$|a_3|\leq\begin{cases}\frac{2\alpha|\tau|}{3(1+2\beta)}, & if \quad |\tau|\in(0,\tau_0),\\ \frac{\alpha^2|\tau|^2}{(1+\beta)^2}, & if \quad |\tau|\in[\tau_0,+\infty),\end{cases} \tag{2.15}$$

where $\tau_0=\frac{2(1+\beta)^2}{3\alpha(1+2\beta)}$.
Substituting the expressions (2.8) and (2.9) in (2.7) and using triangle inequality, putting $|x|=\zeta$ $|y|=\varsigma$, we can easily obtain that

$$|a_4|\leq c_1(t)\left(\zeta^2+\varsigma^2\right)+c_2(t)(\zeta+\varsigma)+c_3(t)=\phi(\zeta,\varsigma),\ \zeta,\varsigma\in[0,1], \tag{2.16}$$

where

$$c_1(t)=\frac{\alpha|\tau|\left(4-t^2\right)(t-2)}{32(1+3\beta)}\leq 0,$$

$$c_2(t)=\frac{\alpha|\tau|\left(4-t^2\right)t\left[5\alpha|\tau|(1+3\beta)+3(1+\beta)(1+2\beta)\right]}{48(1+\beta)(1+2\beta)(1+3\beta)}\geq 0$$

and

$$c_3(t)=\frac{\alpha|\tau|}{16(1+3\beta)}t^3+\frac{\alpha|\tau|\left(4-t^2\right)}{8(1+3\beta)}\geq 0,\ t\in[0,2].$$

Thus, we write

$$|a_4|\leq\min\{\max\{\phi(\zeta,\varsigma,t):\ \zeta,\varsigma\in[0,1]\}:\ t\in[0,2]\}. \tag{2.17}$$

Firstly, we will examine the maximum of the function $\phi(\zeta,\varsigma)$ taking into account the sign of $\Lambda(\zeta,\varsigma)=\phi_{\zeta\zeta}{}''(\zeta,\varsigma)\phi_{\varsigma\varsigma}{}''(\zeta,\varsigma)-\left[\phi_{\zeta\varsigma}{}''(\zeta,\varsigma)\right]^2$, for each $t\in[0,2]$.
By simple computation, we can easily see that

$$\phi_\zeta{}'(\zeta,\varsigma)=2c_1(t)\zeta+c_2(t),\ \ \phi_\varsigma{}'(\zeta,\varsigma)=2c_1(t)\varsigma+c_2(t)$$

and

$$\phi_{\zeta\varsigma}{}''(\zeta,\varsigma)=\phi_{\varsigma\zeta}{}''(\zeta,\varsigma)=0,$$
$$\phi_{\zeta\zeta}{}''(\zeta,\varsigma)=\phi_{\varsigma\varsigma}{}''(\zeta,\varsigma)=2c_1(t),\ (\zeta,\varsigma)\in\Omega.$$

Thus, (ζ_0,ς_0), where $\zeta_0=\varsigma_0=\frac{-c_2(t)}{2c_1(t)}$, is likely a critical point of the function $\phi(\zeta,\varsigma)$. We can easily show that $(\zeta_0,\varsigma_0)\in\Omega$; that is, $\frac{-c_2(t)}{2c_1(t)}\leq 1$ for $t\leq t_0$, where $t_0=\frac{6(1+\beta)(1+2\beta)}{5\alpha|\tau|(1+3\beta)+6(1+\beta)(1+2\beta)}<$

1. Therefore, the function $\phi(\zeta,\varsigma)$ cannot have a critical point for $t \in (t_0,2]$. Hence, we must investigate the maximum of the function $\phi(\zeta,\varsigma)$ for $t \in [0,t_0]$.
Since

$$\Lambda(\zeta_0,\varsigma_0) = 4c_1^2(t) > 0 \text{ and } \phi_{\zeta\zeta}''(\zeta_0,\varsigma_0) = 2c_1(t) < 0,$$

(ζ_0,ς_0) is a maximum point of the function $\phi(\zeta,\varsigma)$, for each $t \in [0,t_0]$. Therefore,

$$\max\{\phi(\zeta,\varsigma) : (\zeta,\varsigma) \in \Omega\} = \phi(\zeta_0,\varsigma_0) = c_3(t) - \frac{c_2^2(t)}{2c_1(t)}.$$

Let the function $H : [0,t_0] \to \mathbb{R}$ defined as follows

$$H(t) = c_3(t) - \frac{c_2^2(t)}{2c_1(t)}. \tag{2.18}$$

Substituting the value $c_1(t)$, $c_2(t)$ and $c_3(t)$ in (2.18), we write

$$H(t) = h_1(\alpha,\beta,\tau)t^3 + h_2(\alpha,\beta,\tau)t^2 + h_3(\alpha,\beta,\tau),$$

where

$$h_1(\alpha,\beta,\tau) = \frac{5\alpha^2|\tau|^2\left[5\alpha\,|\tau|\,(1+3\beta)+6\,(1+\beta)\,(1+2\beta)\right]}{144(1+\beta)^2(1+2\beta)^2} + \frac{\alpha\,|\tau|}{8\,(1+3\beta)},$$

$$h_2(\alpha,\beta,\tau) = \frac{5\alpha^2|\tau|^2\left[5\alpha\,|\tau|\,(1+3\beta)+6\,(1+\beta)\,(1+2\beta)\right]}{72(1+\beta)^2(1+2\beta)^2}$$

and

$$h_3(\alpha,\beta,\tau) = \frac{\alpha\,|\tau|}{2\,(1+3\beta)}.$$

By simple computation, we have

$$H'(t) = \left[3h_1(\alpha,\beta,\tau)t + 2h_2(\alpha,\beta,\tau)\right]t.$$

Since $h_1(\alpha,\beta,\tau) > 0$ and $h_2(\alpha,\beta,\tau) > 0$ for each $\alpha \in (0,1)\,, \beta \in [0,1]\,, |\tau| > 0$, then $H'(t) > 0$. So, the function $H(t)$ is an strictly increasing function on $[0,t_0]$.
Therefore,

$$\min\{H(t) : t \in (0,t_0]\} = H(0) = h_3(\alpha,\beta,\tau). \tag{2.19}$$

Thus, from (2.17) and (2.19), we obtain

$$|a_4| \le \frac{\alpha\,|\tau|}{2\,(1+3\beta)}. \tag{2.20}$$

Thus, from (2.10), (2.15) and (2.20) the proof of Theorem 2.1 is completed. □

In the special cases from Theorem 2.1, we arrive at the following results.

Corollary 2.2. *Let the function $f(z)$ given by (1.1) be in the class $\aleph_\Sigma(\alpha,\beta,1) = H_\Sigma(\alpha,\beta)$, $\alpha \in (0,1)\,, \beta \in [0,1]$. Then*

$$|a_2| \le \frac{\alpha}{1+\beta}$$

and

$$|a_3| \le \begin{cases} \frac{2\alpha}{3(1+2\beta)}, & if \;\; \alpha \in (0,\alpha_0)\,, \\ \frac{\alpha^2}{(1+\beta)^2}, & if \;\; \alpha \in [\alpha_0,1)\,, \end{cases}$$

where $\alpha_0 = \frac{2(1+\beta)^2}{3(1+2\beta)}$.
Also,

$$|a_4| \le \frac{\alpha}{2\,(1+3\beta)}.$$

Corollary 2.3. *Let the function $f(z)$ given by (1.1) be in the class $\aleph_\Sigma(\alpha,0,\tau)$, $\alpha\in(0,1)$, $\tau\in\mathbb{C}^*=\mathbb{C}-\{0\}$. Then*

$$|a_2|\le\alpha|\tau|$$

and

$$|a_3|\le\begin{cases}\frac{2\alpha|\tau|}{3}, & if\ \ |\tau|\in(0,\tau_0),\\ \alpha^2|\tau|^2, & if\ \ |\tau|\in[\tau_0,+\infty),\end{cases}$$

where $\tau_0=\frac{2}{3\alpha}$.
Also,

$$|a_4|\le\frac{\alpha|\tau|}{2}.$$

Corollary 2.4. *Let the function $f(z)$ given by (1.1) be in the class $\aleph_\Sigma(\alpha,0,1)=\aleph_\Sigma(\alpha,0)=N_\Sigma(\alpha)$, $\alpha\in(0,1)$. Then,*

$$|a_2|\le\alpha,$$

$$|a_3|\le\begin{cases}\frac{2\alpha}{3}, & if\ \ \alpha\in\left(0,\frac{2}{3}\right),\\ \alpha^2, & if\ \ \alpha\in\left[\frac{2}{3},1\right)\end{cases}$$

and

$$|a_4|\le\frac{\alpha}{2}.$$

Corollary 2.5. *Let the function $f(z)$ given by (1.1) be in the class $\aleph_\Sigma(\alpha,1,\tau)$, $\alpha\in(0,1)$, $\tau\in\mathbb{C}^*=\mathbb{C}-\{0\}$. Then,*

$$|a_2|\le\frac{\alpha|\tau|}{2}$$

and

$$|a_3|\le\begin{cases}\frac{2\alpha|\tau|}{9}, & if\ \ |\tau|\in(0,\tau_0),\\ \frac{\alpha^2|\tau|^2}{4}, & if\ \ |\tau|\in[\tau_0,+\infty),\end{cases}$$

where $\tau_0=\frac{8}{9\alpha}$.
Also,

$$|a_4|\le\frac{\alpha|\tau|}{8}.$$

Corollary 2.6. *Let the function $f(z)$ given by (1.1) be in the class $\aleph_\Sigma(\alpha,1,1)=\aleph_\Sigma(\alpha)$, $\alpha\in(0,1)$. Then,*

$$|a_2|\le\frac{\alpha}{2},$$

$$|a_3|\le\begin{cases}\frac{2\alpha}{9}, & if\ \ \alpha\in\left(0,\frac{8}{9}\right),\\ \frac{\alpha^2}{4}, & if\ \ \alpha\in\left[\frac{1}{9},1\right)\end{cases}$$

and

$$|a_4|\le\frac{\alpha}{8}.$$

3 Fekete-Szegö problem for the function class $\aleph_\Sigma(\alpha,\beta,\tau)$

In this section, we will prove the following theorem on the Fekete-Szegö problem of the function class $\aleph_\Sigma(\alpha,\beta,\tau)$.

Theorem 3.1. *Let the function $f(z)$ given by (1.1) be in the class $\aleph_\Sigma(\alpha,\beta,\tau)$, $\alpha\in(0,1)$, $\beta\in[0,1]$, $\tau\in\mathbb{C}^*=\mathbb{C}-\{0\}$ and $\mu\in\mathbb{C}$. Then,*

$$\left|a_3-\mu a_2^2\right|\le\begin{cases}\frac{2\alpha|\tau|}{3(1+2\beta)}, & if\ \ |1-\mu|\in[0,\mu_0),\\ |1-\mu|\,\frac{\alpha^2|\tau|^2}{(1+\beta)^2}, & if\ \ |1-\mu|\in[\mu_0,+\infty),\end{cases}$$

where $\mu_0=\frac{2(1+\beta)^2}{3\alpha|\tau|(1+2\beta)}$.

Proof. Let $f \in \aleph_{\Sigma}(\alpha,\beta,\tau)$,$\alpha \in (0,1)$,$\beta \in [0,1]$, $\tau \in \mathbb{C}^{*}$=$\mathbb{C}-\{0\}$and $\mu \in \mathbb{C}$.
From (2.5) and (2.6), we find that

$$a_3-\mu a_2^2=(1-\mu)\,\frac{\alpha^2\tau^2}{4(1+\beta)^2}{p_1}^2+\frac{\alpha\tau}{6(1+2\beta)}\left(p_2-q_2\right). \tag{3.1}$$

Substituting the expression (2.8) in (3.1) and using triangle inequality, taking $|x|=\theta$, $|y|=\vartheta$, we can easily obtain that

$$\left|a_3-\mu a_2^2\right|\leq d_1(t)+d_2(t)\left(\theta+\vartheta\right)=\psi\left(\theta,\vartheta\right), \tag{3.2}$$

where

$$d_1(t)=|1-\mu|\,\frac{\alpha^2|\tau|^2}{4(1+\beta)^2}t^2\geq 0 \text{ and } d_2(t)=\frac{\alpha\,|\tau|\left(4-t^2\right)}{12\left(1+2\beta\right)}\geq 0.$$

It is clear that the maximum of the function $\psi\left(\theta,\vartheta\right)$ occurs at $(\theta,\vartheta)=(1,1)$. Therefore,

$$\max\left\{\psi\left(\theta,\vartheta\right):\theta,\vartheta\in[0,1]\right\}=\psi\left(1,1\right)=d_1(t)+2d_2(t).$$

Let us define the function $H:[0,2]\to\mathbb{R}$ as follows

$$H(t)=d_1(t)+2d_2(t) \tag{3.3}$$

for fixed $\tau\in\mathbb{C}^*=\mathbb{C}-\{0\}$.
Substituting the value $d_1(t)$ and $d_2(t)$ in (3.3), we obtain

$$H(t)=C\left(\alpha,\beta,\mu;\tau\right)t^2+D\left(\alpha,\beta;\tau\right),$$

where

$$C\left(\alpha,\beta,\mu;\tau\right)=\frac{\alpha^2|\tau|^2}{4(1+\beta)^2}\left[|1-\mu|-\frac{2(1+\beta)^2}{3\alpha\,|\tau|\left(1+2\beta\right)}\right] \text{ and } D\left(\alpha,\beta;\tau\right)=\frac{2\alpha\,|\tau|}{3\left(1+2\beta\right)}.$$

Now, we must investigate the maximum of the function $H(t)$ in the interval $[0,2]$.
By simple computation, we have

$$H'(t)=2C\left(\alpha,\beta,\mu;\tau\right)t.$$

It is clear that $H'(t)<0$ if $C\left(\alpha,\beta,\mu;\tau\right)<0$; that is, if $|1-\mu|\in\left(0,\frac{2(1+\beta)^2}{3\alpha|\tau|(1+2\beta)}\right)$.
Thus, the function $H(t)$ is a strictly decreasing function if$|1-\mu|\in(0,\mu_0)$, where $\mu_0=\frac{2(1+\beta)^2}{3\alpha|\tau|(1+2\beta)}$.
Therefore,

$$\max\left\{H(t):t\in[0,2]\right\}=H(0)=2d_2(0)=\frac{2\alpha\,|\tau|}{3\left(1+2\beta\right)}. \tag{3.4}$$

Also, $H'(t)\geq 0$; that is, the function $G(t)$ is an increasing function for $|1-\mu|\geq\mu_0$. Therefore,

$$\max\left\{H(t):t\in[0,2]\right\}=H(2)=d_1(2)=\frac{\alpha^2|\tau|^2}{(1+\beta)^2}\,|1-\mu|\,. \tag{3.5}$$

Thus, from (3.2) and (3.5), we easily see that

$$\left|a_3-\mu a_2^2\right|\leq\begin{cases}\frac{2\alpha|\tau|}{3(1+2\beta)}, & if\quad |1-\mu|\in[0,\mu_0)\,,\\ \frac{\alpha^2|\tau|^2}{(1+\beta)^2}\,|1-\mu|\,, & if\quad |1-\mu|\in[\mu_0,+\infty)\,,\end{cases}$$

where $\mu_0=\frac{2(1+\beta)^2}{3\alpha|\tau|(1+2\beta)}$.
Thus, the proof of Theorem 3.1 is completed. □

In the special cases from Theorem 3.1, we arrive at the following results.

Corollary 3.2. *Let the function $f(z)$ given by (1.1) be in the class $\aleph_\Sigma(\alpha,0,\tau)$, $\alpha\in(0,1)$, $\tau\in\mathbb{C}^*=\mathbb{C}-\{0\}$ and $\mu\in\mathbb{C}$. Then,*

$$\left|a_3-\mu a_2^2\right|\leq\begin{cases}\frac{2\alpha|\tau|}{3}, & if \quad |1-\mu|\in[0,\mu_0),\\ |1-\mu|\,\alpha^2|\tau|^2, & if \quad |1-\mu|\in[\mu_0,+\infty),\end{cases}$$

where $\mu_0=\frac{2}{3\alpha|\tau|}$.

Corollary 3.3. *Let the function $f(z)$ given by (1.1) be in the class $\aleph_\Sigma(\alpha,1,\tau)$, $\alpha\in(0,1)$, $\tau\in\mathbb{C}^*=\mathbb{C}-\{0\}$ and $\mu\in\mathbb{C}$. Then,*

$$\left|a_3-\mu a_2^2\right|\leq\begin{cases}\frac{2\alpha|\tau|}{9}, & if \quad |1-\mu|\in[0,\mu_0),\\ |1-\mu|\,\frac{\alpha^2|\tau|^2}{4}, & if \quad |1-\mu|\in[\mu_0,+\infty),\end{cases}$$

where $\mu_0=\frac{8}{9\alpha|\tau|}$.

Taking $\mu=0$ and $\mu=1$ in Theorem 3.1, we can easily arrive at the following result.

Corollary 3.4. *Let the function $f(z)$ given by (1.1) be in the class $\aleph_\Sigma(\alpha,\beta,\tau)$, $\alpha\in(0,1)$, $\beta\in[0,1]$, $\tau\in\mathbb{C}^*=\mathbb{C}-\{0\}$. Then,*

$$|a_3|\leq\begin{cases}\frac{2\alpha|\tau|}{3(1+2\beta)}, & if \quad |\tau|\in(0,\tau_0),\\ \frac{\alpha^2|\tau|^2}{(1+\beta)^2}, & if \quad |\tau|\in[\tau_0,+\infty),\end{cases}$$

where $\tau_0=\frac{2(1+\beta)^2}{3\alpha(1+2\beta)}$ *and*

$$\left|a_3-a_2^2\right|\leq\frac{2\alpha|\tau|}{3(1+2\beta)}.$$

Remark 3.5. The first result of Corollary 3.4 confirms the second inequality of Theorem 2.1.

Remark 3.6. Numerous consequences of the results obtained in the Corollary 3.2, 3.3 and 3.4 can indeed be deduced by specializing the various parameters involved.

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