Palestine Journal of Mathematics

Vol. 9(2)(2020) , 620–625

ON WEAKLY SEMIPRIME SUBSEMIMODULES

Farkhonde Farzalipour and Peyman Ghiasvand

Communicated by Ayman Badawi

MSC 2010 Classifications: Primary 16Y60; Secondary 13C05, 13C99.

Keywords and phrases: Semiring, semimodule, weakly prime, weakly semiprime.

Abstract In this paper we study weakly prime and weakly semiprime subsemimodules of a semimodule over a commutative semiring with nonzero identity. Also, we give a number of results concerning weakly semiprime subsemimodules of a multiplication semimodule.

1 Introduction

The concept of semirings and semimodules has been studied by several authors, for example see [1], [2], [3], [4], [5], [6], [9], [10], [11]. Weakly prime submodules of a module over a commutative ring with a nonzero identity have been introduced and studied by S. Ebrahmi Atani and F. Farzalipour [7]. Also, weakly semiprime subsemimodules of a semimodule over a commutative semiring have been studied in [11]. In this paper we study the weakly prime and weakly semiprime subsemimodules of a semimodule over commutative semiring with nonzero identity. Before we state some results let us introduce some notation and terminology. By a commutative semiring we mean an algebraic system $R = (R, +, \cdot)$ such that $R = (R, +)$ and $R = (R, \cdot)$ are commutative semigroup, connected by $a(b + c) = ab + bc$ for all $a, b, c \in R$, and there exists $0 \in R$ such that $r + 0 = 0$ and $r \cdot 0 = 0 \cdot r = 0$ for all $r \in R$. Throughout this paper let R be a commutative semiring. A semiring R is said to be semidomain whenever $a, b \in R$ with $ab = 0$, implies that $a = 0$ or $b = 0$. A subtractive ideal (=k-ideal) I is an ideal such that if $x, x + y \in I$, then $y \in I$. A proper ideal I of semiring R is called maximal (k-maximal) if J is an ideal of R (resp. k-ideal) in R such that $I \subsetneqq J$, then $J = R$. A nonzero element a of R is said to be semiunit in R if there exist $r, s \in R$ such that $1 + ra = sa$. R is called a local semiring if and only if R has a unique k-maximal ideal. A (left) semimodule M over a semiring R is a commutative additive semigroup which has a zero element, together with a mapping from $R \times M$ into M such that $(r + s)m = rm + sm$, $r(m + n) = rm + rn$, $r(sm) = (rs)m$ and $0m = r0_M = 0_M r = 0_M$ for all $m, n \in M$ and $r, s \in R$. Let M be a semimodule over a semiring R and let N be a subset of M, we say that N is a subsemimodule of M when N is itself an R-semimodule with respect to the operations for M (so $0_M \in N$). It is easy to see that if $r \in R$, then $rM = \{rm : m \in M\}$ is a subsemimodule of M. A subtractive subsemimodule (=k-subsemimodule) N is subsemimodule such that if $x, x + y \in N$, then $y \in N$. A proper subsemimodule N of R-semimodule M is called prime, if $rm \in N$ where $r \in R$ and $m \in M$, then $m \in N$ or $rM \subseteq N$. A semimodule M is called prime if the zero subsemimodule of M is prime subsemimodule. The semiring R is a semimodule over itself. In this case, the subsemimodules of R are called ideals of R. If R is a semiring (not necessarily a semidomain) and M an R-semimodule, then we define the subset $T(M)$ as $T(M) = \{m \in M : rm = 0 \text{ for some } 0 \neq r \in R\}$.

It is clear that if R a semidomain, then $T(M)$ is a subsemimodule of M (see [4]). Let R is a semidomain and M an R-semimodule, then M is called torsion if $T(M) = M$ and M is called torsion free if $T(M) = 0$.

2 Weakly Prime Subsemimodules

Let R be a semiring and M an R-semimodule. A proper subsemimodule N of M is called weakly prime, if $0 \neq rm \in N$ where $r \in R$ and $m \in M$, then $m \in N$ or $r \in (N : M)$ (see [1]).

It is clear that every prime subsemimodule is a weakly prime subsemimodule. However, since 0 is always weakly prime(by definition), a weakly prime subsemimodule need not be prime. Let R be a semiring which is not semidomain and let M be faithful R-semimodule. If 0 is a prime subsemimodule, then $(0 : M) = 0$ is a prime ideal of semiring R, which is not the case, but we have the following results:

Proposition 2.1. *Let M be an R-semimodule with $T(M) = 0$. Then every weakly prime subsemimodule of M is prime.*

Proof. Let N be a weakly prime subsemimodule of M. Suppose that $rm \in N$ where $r \in R$, $m \in M$. If $0 \neq rm \in N$, N weakly prime gives $m \in N$ or $rM \subseteq N$. If $rm = 0$, then $r = 0$ or $m = 0$ since $T(M) = 0$. So N is prime. □

Proposition 2.2. *Let M be a semimodule over a local semiring R with k-maximal ideal P such that $PM = 0$. Then every proper k-subsemimodule of M is weakly prime.*

Proof. Let N be a proper k-subsemimodule of M, and $0 \neq rm \in N$ where $r \in R$ and $m \in M$. If r is semiunit, then $1 + ar = sr$ for some $a, s \in R$. So $m + (rm)a = s(rm) \in N$, thus $m \in N$ since N is a k-subsemimodule. Let r is not semiunit, so $rm \in PM = 0$ by [5, Theorem 2], a contradiction. Hence N is weakly prime. □

We know that if N is a prime subsemimodule of an R-semimodule M, then $(N : M)$ is a prime ideal of R (see [4, Lemma 4]. This is not always true for case of weakly prime subsemimodules. For example, let M be $\mathbb{Z}_0^+$-semimodule $\mathbb{Z}_8$. Let $N = \{0\}$. Certainly, N is a weakly prime subsemimodule of M, but $(N : M) = (0 : M) = 8\mathbb{Z}_0^+$ is not a weakly prime ideal of $\mathbb{Z}_0^+$, because $0 \neq 4 \cdot 4 \in 8\mathbb{Z}_0^+$ and $4 \notin 8\mathbb{Z}_0^+$.

Now we consider the case in which from a weakly prime subsemimodule we reach to a weakly prime ideal.

Proposition 2.3. *Let M be a P-prime R-semimodule. If N is a weakly prime k-subsemimodule of M, then $(N : M)$ is a weakly prime ideal of R.*

Proof. Since M is a prime semimodule its zero subsemimodule is prime. So $P = (0 : M)$ is a prime ideal of R. Let $0 \neq ab \in (N : M)$ and $a \notin (N : M)$ where $a, b \in R$. Hence there exists $m \in M$ such that $am \notin N$. Now $(ab)M \subseteq N$. If $(ab)M = 0$, then $ab \in (0 : M) = P$, and so $a \in P$ or $b \in P$. But $a \notin (N : M)$ and $P = (0 : M) \subseteq (N : M)$, hence $b \in (N : M)$. If $(ab)M \neq 0$, then there exist $0 \neq n \in M$ such that $(ab)n \neq 0$. If $an \notin N$, then $b(an) \in N$ implies that $b \in (N : M)$. If $an \in N$, then $a(m + n) \notin N$, because if $a(m + n) \in N$, then $am \in N$ since N is k-subsemimodule, a contradiction. Hence $ab(m + n) = b(a(m + n)) \in N$ and $a(m + n) \notin N$ so $b \in (N : M)$. In any case $0 \neq ab \in (N : M)$ and $a \notin (N : M)$ implies that $b \in (N : M)$. □

3 Weakly semiprime subsemimodules

Definition 3.1. Let R be a semiring and M an R-semimodule. A proper subsemimodule N of M is called weakly semiprime, if $0 \neq r^k m \in N$ for some $r \in R$, $m \in M$ and $k \in \mathbb{Z}^+$, then $rm \in N$.

Since the semiring R is an R-semimodule by itself, according to our definition, a proper ideal I of R is a weakly semiprime ideal, if whenever $0 \neq a^k b \in I$ for some $a, b \in R$ and $k \in \mathbb{Z}^+$, then $ab \in I$. It is clear that every semiprime is weakly semiprime, but the converse is not true in general. In fact the zero subsemimodule of an R-semimodule M is always weakly semiprime, but is not necessarily semiprime. For example in the semiring $\mathbb{Z}_4$, the ideal $\{0\}$ is weakly semiprime, but not semiprime, because $2^2 \cdot 3 \in I$ but $2 \cdot 3 \notin I$. Also, it is clear that if N is a weakly prime, then N is weakly semiprime.

If N is a weakly semiprime subsemimodule of R-semimodule M, then it is possible that $(N : M)$ is not a weakly semiprime ideal of R. For example, let M be $\mathbb{Z}_0^+$-semimodule $\mathbb{Z}_4$. Let

$N = \{0\}$. Certainly, N is a weakly semiprime subsemimodule of M, but $(N : M) = (0 : M) = 4\mathbb{Z}_0^+$ is not a weakly semiprime ideal of $\mathbb{Z}_0^+$, because $0 \neq 2^2 \in 4\mathbb{Z}_0^+$ but $2 \notin 4\mathbb{Z}_0^+$.

Now we consider several cases in which from a weakly semiprime subsemimodule, we reach a weakly semiprime ideal.

Proposition 3.2. *Let M be a faithful cyclic R-semimodule and N be a weakly semiprime subsemimodule of M. Then $(N : M)$ is a weakly semiprime ideal of R.*

Proof. Assume that $M = Rx$ for some $x \in M$. Let $0 \neq a^k b \in (N : M)$ where $a, b \in R$ and $k \in \mathbb{Z}^+$. So $a^k bM \subseteq N$ and since M is faithful, $0 \neq a^k bM$. Hence $0 \neq a^k bx \in N$, so $a(bx) \in N$ since N is weakly semiprime. Therefore $(ab)M \subseteq N$, so $(N : M)$ is a weakly semiprime ideal of R. □

Remark 3.3. Let M be an R-semimodule. Then M is a P-prime semimodule if and only if $(0 : M) = (0 : m)$ for every nonzero element $m \in M$.

Proposition 3.4. *Let M be a P-prime R-semimodule and N a weakly semiprime subsemimodule of M. Then $(N : M)$ is a weakly semiprime ideal of R.*

Proof. Let $0 \neq a^k b \in (N : M)$ where $a, b \in R$ and $k \in \mathbb{Z}^+$. Let x be an arbitrary element of M, so $a^k bx \in N$. If $a^k bx = 0$, then $a^k b \in (0 : m) = (0 : M) = P$. This implies that $ab \in P \subseteq (N : M)$. If $a^k bx \neq 0$, then from $a^k bx \in N$ we conclude that $abx \in N$ since N is weakly semiprime. In any case $abx \in N$ for every $x \in M$ and so $ab \in (N : M)$, as required. □

The next Theorem gives an alternative definition for weakly semiprime subsemimodules when a semimodule is prime.

Theorem 3.5. *Let M be an R-semimodule and N a proper subsemimodule of M. If for every ideal of semiring R, subsemimodule K of M and $t \in \mathbb{Z}^+$, $0 \neq I^t K \subseteq N$ implies that $IK \subseteq N$, then N is a weakly semiprime subsemimodule of M. The converse is true if M is a P-prime semimodule.*

Proof. Let $0 \neq r^k m \in N$ where $r \in R$, $m \in M$ and $k \in \mathbb{Z}^+$. We take $I = Rr$ and $K = Rm$. Now $0 \neq I^k K \subseteq N$ and so by the hypothesis $IK \subseteq N$ which implies that $rm \in N$. Therefore N is a weakly semiprime subsemimodule of M. Conversely, let I be an ideal of R, K a subsemimodule of M and $t \in \mathbb{Z}^+$. Assume that $0 \neq I^t K \subseteq N$. Consider the set $S = \{ra | r \in I, a \in K\}$. Now $r^t a \in I^t K$. If $r^t a \neq 0$ then clearly $ra \in N$. Let $r^t a = 0$ where $a \neq 0$, then $r^t \in (0 : a) = (0 : M) = P$ by Remark 3.3. So $r \in P = (0 : a)$ and so $ra = 0$. In any case $ra \in N$ and $S \subseteq N$. But S generates IK and therefore $IK \subseteq N$. The proof is complete. □

Remark 3.6. Since the semiring R is an R-semimodule, so if I is a weakly semiprime ideal of a semidomain R, then by Theorem 3.5, for every ideals J, K of R and positive integer t, $0 \neq J^t K \subseteq I$.

Proposition 3.7. *Let R be a semidomain and M be a torsion free R-semimodule. Then every weakly semiprime subsemimodule of M is semiprime.*

Proof. Let N be a weakly semiprime subsemimodule of M. Suppose that $r^k m \in N$ where $r \in R$, $m \in M$ and $k \in \mathbb{Z}^+$. If $r^k m \neq 0$ then $rm \in N$. Let $r^k m = 0$ with $m \neq 0$. Then $r^k = 0$ as $T(M) = 0$. Since R is semidomain we have $r = 0$. In any case we get $rm \in N$, hence N is a weakly semiprime subsemimodule of M. □

Proposition 3.8. *Let M be a semimodule over local semiring R with k-maximal ideal P such that $PM = 0$. Then every proper k-subsemimodule of M is weakly semiprime.*

Proof. Let N be a weakly semiprime subsemimodule of M and $r^k m \in N$ where $r \in R$, $m \in M$ and $k \in \mathbb{Z}^+$. If r is semiunit, then $1 + ra = sr$ for some $a, s \in R$, so $m + (ra)m = (sr)m \in N$. Hence $m \in N$ since N is k-subsemimodule. Let r is not semiunit, then $r \in P$, so $rm \in PM = 0 \subseteq N$. □

We study weakly semiprime subsemimodules in quotient semimodules.

A subsemimodule N of an R-semimodule M is called partitioning subsemimodule (=Q-subsemimodule) if there exists a subset Q of M such that
(1) $M = \bigcup\{q + N : q \in Q\}$.
(2) If $q_1, q_2 \in Q$, then $(q_1 + N) \cap (q_2 + N) \neq \emptyset \Leftrightarrow q_1 = q_2$ [2].

Let N be a partitioning subsemimodule of an R-semimodule M. Then $M/N_{(Q)} = \{q + N : q \in Q\}$ forms an R-semimodule under the following addition $\bigoplus$ and scaler multiplication $\bigodot$, $(q_1 + N) \bigoplus (q_2 + N) = q_3 + N$ where q_3 is a unique element of Q such that $q_1 + q_2 + N \subseteq q_3 + N$ and $r \odot (q_1 + N) = q_4 + N$ where $q_4 \in Q$ is unique such that $rq_1 + N \subseteq q_4 + N$. This R-semimodule $M/N_{(Q)}$ is called a quotient semimodule of M by N and denoted $(M/N_{(Q)}, \bigoplus, \bigodot)$ [2].

Theorem 3.9. *Let N be a Q-subsemimodule of an R-semimodule M and P a k-subsemimodule of M with $N \subseteq P$. Then*
(i) If P is a weakly semiprime subsemimodule of M, then $P/N_{(Q\cap P)}$ is a weakly semiprime subsemimodule of $M/N_{(Q)}$.
(ii) If N, $P/N_{(Q\cap P)}$ are weakly semiprime subsemimodules of M and $M/N_{(Q)}$ respectively, then P is a weakly semiprime subsemimodule of M.

Proof. (i) Let P be a weakly semiprime subsemimodule of M. Let q_0 be the unique element of Q such that $q_0 + N$ is the zero element of $M/N_{(Q)}$. Let $q_0 + N \neq r^k \odot (q_1 + N) \in P/N_{(Q\cap P)}$ where $r \in R$, $q_1 \in Q$ and $k \in \mathbb{Z}^+$. By [2, Lemma 3.4] there exists a unique $q_2 \in Q \cap P$ such that $r^k \odot (q_1 + N) = q_2 + N$ such that $r^k q_1 + N \subseteq q_2 + N$. Since $N \subseteq P$ and P is k-subsemimodule, so $r^k q_1 \in P$. If $r^k q_1 = 0$, then $r^k q_1 \in (q_0 + N) \cap (q_2 + N)$ (because $0 \in q_0 + N$ by [2, Lemma 2.3]), thus $q_0 = q_2$ and hence $q_0 + N = q_2 + N$, a contradiction. Thus $0 \neq r^k q_1 \in P$, as P is weakly semiprime, so $rq_1 \in P$. Hence $r \odot (q_1 + N) = q_2 + N$ where q_2 is a unique element of Q such that $rq_1 + N \subseteq q_2 + N$. Since $N \subseteq P$ and P is a k-subsemimodule of M, so $q_2 \in P$. Hence $q_2 \in Q \cap P$ and so $r \odot (q_1 + N) = q_2 + N \in P/N_{(Q\cap P)}$. Thus $P/N_{(Q\cap P)}$ is weakly semiprime.

(ii) Suppose that N, $P/N_{(Q\cap P)}$ are weakly semiprime subsemimodules of M and $M/N_{(Q)}$ respectively. Let $0 \neq r^k m \in P$ where $r \in R$, $m \in M$ and $k \in \mathbb{Z}^+$. If $0 \neq r^k m \in N$, then $rm \in N \subseteq P$, as needed. Let $r^k m \in P - N$. By using [1, Lemma 3.6], there exists a unique $q_1 \in Q$ such that $m \in q_1 + N$ and $r^k m \in r^k \odot (q_1 + N) = q_2 + N$ where q_2 is a unique element of Q such that $r^k q_1 + N \subseteq q_2 + N$. Since $r^k m \in P$ and $r^k m \in q_2 + N$, so $q_2 \in P$ as P is k-subsemimodule and $N \subseteq P$. Hence $q_0 + N \neq r^k \odot (q_1 + N) = q_2 + N \in P/N_{(Q\cap P)}$. As $P/N_{(Q\cap P)}$ is weakly semiprime, so $r \odot (q_1 + N) \in P/N(Q\cap P)$. Therefore, $r \odot (q_1 + N) = q_3 + N$ where $q_3 \in Q \cap P$ such that $rq_1 + N \subseteq q_3 + N$. So $rq_1 \in P$ since P is k-subsemimodule and $N \subseteq P$. As $m \in q_1 + N$, so $rm \in rq_1 + N$. Therefore $rm \in P$ since $N \subseteq P$, as required. □

An R-semimodule M is called a multiplication semimodule, if for every subsemimodule N of M, $N = IM$ for some ideal I of semiring R (see [11]).

Let M be a multiplication R-semimodule and N, K are subsemimodules of M. Then there exist ideals I, J of R such that $N = IM$ and $K = JM$. We define the product of N, K, NK, as $(IJ)M$, i.e. $NK = (IM)(JM) = (IJ)M$. By [11, Theorem 2], the product of two subsemimodule is independent of its presentations.

Now we study weakly semiprime subsemimodules of multiplication semimodules.

Theorem 3.10. *Let R be a semidomain, M a multiplication R-semimodule and $(N : M)$ a weakly semiprime ideal of R. Then N is a weakly semiprime subsemimodule of M.*

Proof. Let $0 \neq I^t K \subseteq N$ where I is an ideal of R, K a subsemimodule of M and t a positive integer. Since M is a multiplication R-semimodule we can write $K = JM$ for some ideal J of semiring R and so $0 \neq I^t JM \subseteq N$, that is $0 \neq I^t J \subseteq (N : M)$. Hence by Remark 3.6, $IJ \subseteq (N : M)$, so $I(JM) \subseteq N$. From this we have $IK \subseteq N$, therefore N is a weakly semiprime subsemimodule of M. □

Theorem 3.11. *Let M be a P-prime multiplication semimodule and N be a weakly semiprime subsemimodule of M. Then for every subsemimodule K of M and positive integer t, $0 \neq K^t \subseteq N$ implies that $K \subseteq N$.*

Proof. Let N be a weakly semiprime subsemimodule of M and $0 \neq K^t \subseteq N$ where K is a subsemimodule of M and $t \in \mathbb{Z}^+$. Hence $K = IM$ for some ideal I of R. So $0 \neq K^t = I^t M \subseteq N$. Since M is P-prime and N weakly semiprime, then $K = IM \subseteq N$ by Theorem 3.5. □

Corollary 3.12. *Let M be a P-prime multiplication R-semimodule and N be a weakly semiprime subsemimodule of M. Then for every $m \in M$ and $t \in \mathbb{Z}^+$, $0 \neq m^t \subseteq N$ implies that $m \in N$.*

Proof. Let $0 \neq m^t \subseteq N$ where $m \in M$ and $t \in \mathbb{Z}^+$. Since M is multiplication, so there exists an ideal I of R such that $Rm = IM$ and so $0 \neq Rm^t = I^t M \subseteq N$. Since N is weakly semiprime and M is P-prime so by Theorem 3.5, we have $IM \subseteq N$. Hence $m \in Rm = IM \subseteq N$, as needed. □

Theorem 3.13. *Let M be a multiplication R-semimodule which has no nonzero nilpotent subsemimodue and N be a proper subsemimodule of M. If for every subsemimodule U of M and positive integer t, $0 \neq U^t \subseteq N$ implies that $U \subseteq N$, then N is a weakly semiprime subsemimodule of M.*

Proof. Let $0 \neq I^t K \subseteq N$ where I is an ideal of R, K a subsemimodule of M and $t \in \mathbb{Z}^+$. So $K = JM$ for some ideal J of R. Therefore $0 \neq I^t K = I^t JM \subseteq N$. Since M has no nonzero nilpotent subsemimodule, so $0 \neq (IK)^t$ and hence $0 \neq (IK)^t = (IJ)^t M \subseteq N$. Hence $IK \subseteq N$ by hypothesis, so the proof is complete. □

Corollary 3.14. *Let M be a multiplication R-semimodule which has no nonzero nilpotent subsemimodue and N be a proper subsemimodule of M. If for every $m \in M$ and $t \in \mathbb{Z}^+$, $0 \neq m^t \subseteq N$ implies that $m \in N$, then N is a weakly semiprime subsemimodule.*

Proof. Let $0 \neq K^t \subseteq N$ for some subsemimodule K of M and $t \in \mathbb{Z}^+$ but $K \nsubseteq N$. Let $x \in K - N$. So $Rx = JM$ for some ideal J of R. Clearly, $x \neq 0$ and so $0 \neq Rx = JM$. Since M has no nonzero nilpotent subsemimodule, so $0 \neq R(x)^t$ and hence $0 \neq R(x)^t = (Rx)^t = (JM)^t = J^t M \subseteq N$. Hence $Rx \subseteq N$ by Theorem 3.13, so $x \in N$ which is a contradiction. Hence $K \subseteq N$, thus N is a weakly semiprime subsemimodule of M. □

References

[1] J. N. Chaudhari and D. R. Bonde, Weakly prime subsemimodules of semimodules over semirings, *Int. J. of Algebra* **4(5)**, 167–174 (2010).

[2] J. N. Chaudhari and D. R. Bonde, On partitioning and subtractive subsemimodules of semimodules over semirings, *Kyungpook Math. J.* **50**, 329–336 (2010).

[3] R. Ebrahimi Atani, On prime subsemimodules of semimodules, *Int. J. of Algebra* **4(26)**, 1299–1306 (2010).

[4] R. Ebrahimi Atani and S. Ebrahimi Atani, *On subsemimodules of semimodules*, *Buletinul Academiei De Stiinte a Repablicii Moldova Mathematica* **2(63)**, 20–30 (2010).

[5] R. Ebrahimi Atani and S. Ebrahimi Atani, *Ideal theory in commutative semirings*, *Buletinul Academiei De Stiinte a Repablicii Moldova Mathematica* **2(57)**, 14–23 (2010).

[6] S. Ebrahimi Atani, On k-weakly primary ideals over semirings, *Sarajevo J. of Math.* **3(15)**, 9–13 (2007).

[7] S. Ebrahimi Atani and F. Farzalipour, On weakly prime submodules, *Tamkang J. of Math.* **38**, 247–252 (2007).

[8] F. Farzalipour and P. Ghiasvand, Quasi Multiplication Modules, *Thai J. of Math.* **1(2)**, 361–366 (2009).

[9] J. S. Golan, *Semiring and their applications*, Kluwer Academic publisher Dordrecht (1999).

[10] Gupta. Vishnu and J. N. Chaudhari, Characterization of weakly prime subtractive ideals in semirings, *Bull. Inst. Math. Acad. Sinica (New Series)* **3**, 347–352 (2008).

[11] G. Yesilot, K. H. Oral and U. Tekir, On prime subsemimodule of semimodules, *Int. J. of Algebra* **4(1)**, 53–60 (2010).

Author information

Farkhonde Farzalipour and Peyman Ghiasvand, Department of Mathematics, Payame Noor University, Tehran 19395-3697, Iran.
E-mail: f_farzalipour@pnu.ac.ir

Received: December 24, 2018..
Accepted: April 11, 2019.