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A FORMALISED INDUCTIVE APPROACH TO ESTABLISH THE INVARIANCE OF ANTI-DIAGONAL RATIOS WITH EXPONENTIATION FOR A TRI-DIAGONAL MATRIX OF FIXED DIMENSION

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Abstract We offer a formalised proof argument to establish the invariance, with respect to matrix power, of the $n-1$ anti-diagonal ratios within a fixed n-dimensional tri-diagonal matrix.

1 Introduction

1.1 Result and Background

This paper follows on from a recent one by the authors [2] in which an invariance property was proved for a tri-diagonal matrix of arbitrary dimension. The result may be stated thus:

Theorem 1.1. *Suppose* $\mathbf{M} = \mathbf{M}(a_1, \ldots, a_n, u_1, \ldots, u_{n-1}, l_1, \ldots, l_{n-1}) = \mathbf{M}(\mathbf{a}_n, \mathbf{u}_{n-1}, \mathbf{l}_{n-1})$ *is an* $n \times n$ *tri-diagonal matrix*

$$\mathbf{M} = \begin{pmatrix} a_1 & u_1 & & & & \\ l_1 & a_2 & u_2 & & & \\ & l_2 & a_3 & u_3 & & \\ & & \ddots & \ddots & \ddots & \\ & & & l_{n-2} & a_{n-1} & u_{n-1} \\ & & & & l_{n-1} & a_n \end{pmatrix},$$

with anti-diagonal ratios $u_1/l_1, u_2/l_2, \ldots, u_{n-1}/l_{n-1}$. *Then, unless otherwise indeterminate, the immediate off-diagonal terms of* $\mathbf{M}^k$ *form anti-diagonal ratios that remain invariant as the power* $k > 1$ *to which* $\mathbf{M}$ *is raised increases.*

We can illustrate the result for a couple of low values of n (excluding $n = 2$ since then $\mathbf{M}$ reduces to $\mathbf{M}(\mathbf{a}_2, \mathbf{u}_1, \mathbf{l}_1) = \left(\begin{smallmatrix} a_1 & u_1 \\ l_1 & a_2 \end{smallmatrix}\right)$ whose single anti-diagonals ratio u_1/l_1 has been established as a matrix power invariant elsewhere in the fully general case [1, 4], and for the instance $a_2 = a_1$ [3], using a variety of methods).

1.2 Examples

Consider the 3-square tri-diagonal matrix

$$\mathbf{M}(\mathbf{a}_3, \mathbf{u}_2, \mathbf{l}_2) = \begin{pmatrix} a_1 & u_1 & 0 \\ l_1 & a_2 & u_2 \\ 0 & l_2 & a_3 \end{pmatrix}, \tag{1.1}$$

whose successive powers

$$\mathbf{M}^2(\mathbf{a}_3, \mathbf{u}_2, \mathbf{l}_2) = \begin{pmatrix} a_1^2 + l_1 u_1 & (a_1 + a_2) u_1 & u_1 u_2 \\ (a_1 + a_2) l_1 & a_2^2 + l_1 u_1 + l_2 u_2 & (a_2 + a_3) u_2 \\ l_1 l_2 & (a_2 + a_3) l_2 & a_3^2 + l_2 u_2 \end{pmatrix}, \tag{1.2}$$

and so on, each have two anti-diagonal ratios u_1/l_1 and u_2/l_2. Likewise, successive powers of the 4-square tri-diagonal matrix

$$\mathbf{M}(\mathbf{a}_4, \mathbf{u}_3, \mathbf{l}_3) = \begin{pmatrix} a_1 & u_1 & 0 & 0 \\ l_1 & a_2 & u_2 & 0 \\ 0 & l_2 & a_3 & u_3 \\ 0 & 0 & l_3 & a_4 \end{pmatrix} \tag{1.3}$$

are

$$\mathbf{M}^2(\mathbf{a}_4, \mathbf{u}_3, \mathbf{l}_3) = \begin{pmatrix} a_1^2 + l_1u_1 & (a_1+a_2)u_1 & u_1u_2 & 0 \\ (a_1+a_2)l_1 & a_2^2 + l_1u_1 + l_2u_2 & (a_2+a_3)u_2 & u_2u_3 \\ l_1l_2 & (a_2+a_3)l_2 & a_3^2 + l_2u_2 + l_3u_3 & (a_3+a_4)u_3 \\ 0 & l_2l_3 & (a_3+a_4)l_3 & a_4^2 + l_3u_3 \end{pmatrix}, \tag{1.4}$$

and so on, each with three anti-diagonal ratios u_1/l_1, u_2/l_2 and u_3/l_3. Higher powers of these two matrices have been checked algebraically using computer software, as have powers ≥ 2 for other fully general n-square tri-diagonal matrices (containing $3n-2$ variables) of specific dimension $n = 5$ and greater; the interplay between matrix parameters is considerable, as expected, but Theorem 1.1 bears out in all of the many cases (that is, values of n) examined.

The presentation here details a new proof approach to the result using an inductive line of argument. *It is emphasised that the dimension of* $\mathbf{M}$ *(the value of* n*) is taken as fixed, and further that invariance of those anti-diagonal ratios throughout the exponentiated matrices* $\mathbf{M}, \mathbf{M}^2, \ldots, \mathbf{M}^n$ *must be pre-established (for it is needed within the proof).*

2 The Proof Approach

We here set down a line of argument to establish that the $n-1$ anti-diagonal ratios of $\mathbf{M}^k$ are, for all $k \geq 1$, the invariants $u_1/l_1, u_2/l_2, \ldots, u_{n-1}/l_{n-1}$ (each assumed to be well defined) for any n-square tri-diagonal matrix $\mathbf{M}$ (where n is fixed).

Proof. For $n \geq 2$, let $\mathcal{M}_n[\mathbb{F}]$ be the set of $n \times n$ matrices with entries from a field $\mathbb{F}$ (the set forming a vector space over $\mathbb{F}$ of dimension n^2), and define, for any $i = 1, \ldots, n-1$, $G_i : \mathcal{M}_n[\mathbb{F}] \to \mathbb{F}$ to be the linear map

$$G_i(\mathbf{S}) = \mathbf{S}_{i,i+1} - (u_i/l_i)\mathbf{S}_{i+1,i} \tag{P.1}$$

acting on any matrix $\mathbf{S} \in \mathcal{M}_n[\mathbb{F}]$, where $\mathbf{S}_{p,q}$ is the row p, column q, element of $\mathbf{S}$. We seek to show that

$$G_i(\mathbf{M}^k) = 0 \tag{P.2}$$

for every power $k \geq 1$ ($i = 1, \ldots, n-1$), and $n \times n$ tri-diagonal $\mathbf{M}$; we will induct on k.

We assume the result holds for n consecutive values of $k = d, d-1, d-2, \ldots, d-(n-1)$, where $k = d$ is arbitrary and defines the others in the sequence—in other words, $0 = G_i(\mathbf{M}^d) = G_i(\mathbf{M}^{d-1}) = G_i(\mathbf{M}^{d-2}) = \cdots = G_i(\mathbf{M}^{d-(n-1)})$ ($i = 1, \ldots, n-1$, $d \geq n$), noting that for $n = 2$ this would be an assumption for some $k = d, d-1$ ($d \geq 2$) having shown it is true for particular initial powers $k = 1, 2$, while the power values $k = 1, 2, 3$ are those needed to be checked for invariance when $n = 3$, in which case the assumption is of validity for some $k = d, d-1, d-2$ ($d \geq 3$), and so on.[1] Denoting the n-square identity matrix by $\mathbf{I}_n$ our inductive step proceeds as follows, based on the Cayley-Hamilton result that for constants $s_0, s_1, \ldots, s_{n-1} \in \mathbb{F}$,

$$\mathbf{M}^n = s_0\mathbf{I}_n + s_1\mathbf{M} + \cdots + s_{n-2}\mathbf{M}^{n-2} + s_{n-1}\mathbf{M}^{n-1} \tag{P.3}$$

[1]For any fixed matrix size n the number of initial powers that require checking is n (starting at power 1), which can be done computationally (note that it is possible to write down (for $i = 1, \ldots, n-1$) $G_i(\mathbf{M}^1) = G_i(\mathbf{M}) = \mathbf{M}_{i,i+1} - (u_i/l_i)\mathbf{M}_{i+1,i} = u_i - (u_i/l_i) \cdot l_i = 0$, and, additionally, $G_i(\mathbf{M}^2) = (\mathbf{M}^2)_{i,i+1} - (u_i/l_i)(\mathbf{M}^2)_{i+1,i} = (a_i + a_{i+1})u_i - (u_i/l_i) \cdot (a_i + a_{i+1})l_i = 0$ for *any* chosen value of n, but beyond this such relations are difficult to determine since as matrix power increases so does the algebraic complexity of the resulting matrix entries, and rapidly.

(that is, $\mathbf{M}$ satisfies its own order n characteristic equation;[2] in the instance $n = 2$ it reduces to the familiar identity $\mathbf{M}^2 = s_0\mathbf{I}_2 + s_1\mathbf{M}$ for a 2×2 matrix $\mathbf{M}$, where s_0, s_1 are the familiar constants $s_0 = -\mathrm{Det}\{\mathbf{M}\} = -(a_1a_2 - l_1u_1)$ and $s_1 = \mathrm{Tr}\{\mathbf{M}\} = a_1 + a_2$). This reads

$$\mathbf{M}^{d+1} = s_0\mathbf{M}^{d-(n-1)} + s_1\mathbf{M}^{d-(n-2)} + \cdots + s_{n-2}\mathbf{M}^{d-1} + s_{n-1}\mathbf{M}^d \tag{P.4}$$

on multiplying throughout by $\mathbf{M}^{d-(n-1)}$, whereupon (by linearity of G), for $i = 1, \ldots, n-1$,

$$\begin{aligned} G_i(\mathbf{M}^{d+1}) &= s_0G_i(\mathbf{M}^{d-(n-1)}) + s_1G_i(\mathbf{M}^{d-(n-2)}) + \cdots + s_{n-2}G_i(\mathbf{M}^{d-1}) + s_{n-1}G_i(\mathbf{M}^d) \\ &= s_0 \cdot 0 + s_1 \cdot 0 + \cdots + s_{n-2} \cdot 0 + s_{n-1} \cdot 0 \end{aligned} \tag{P.5}$$

(by assumption) $= 0$, and the inductive step is upheld. □

For clarity, the argument establishes that if invariance holds for any consecutive run of n powers of n-square $\mathbf{M}$, then it holds for the next in the sequence—thus, if it holds for powers $1, \ldots, n$ then it does so for power $n+1$ and (being true for the n powers $2, \ldots, n+1$) in turn for $n+2$, and so on; the $n = 2$ version of the proof is Proof II in [4], of which this is its natural extension. We have not presented a constructive proof of Theorem 1.1 (of the type seen in [2]), but instead demonstrated that the proof can be reduced to checking a finite number of power cases for any fixed $n \geq 2$—the only limitation here is that for all but small values of n this realistically has to be done by computer (which was impossible before the advent of algebraic software).

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[2]To be more precise, every n-square matrix over a commutative ring (such as the real or complex field) satisfies its own characteristic equation which is monic of degree n.