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ON SOME ASPECTS OF HORADAM SEQUENCE PERIODICITY VIA GENERATING FUNCTIONS

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Abstract We develop some recent work on interesting periodicity properties of a second order linear recurrence sequence known as the Horadam sequence. Sufficient conditions for cyclicity are stated and proved, using a new generating function approach, which address both non-degenerate and degenerate characteristic root cases—through these we consolidate our knowledge of, and recover, some previously observed periodic sequence behaviours which include the phenomenon of so called 'masked' cyclicity.

1 Introduction

1.1 The Horadam Sequence

Consider the second order linear recursion ($n \geq 2$)

$$w_n = pw_{n-1} - qw_{n-2}; \qquad w_0 = a, w_1 = b, \tag{1.1}$$

which, characterised by the four parameters $a, b, p, q \in \mathbb{C}$, defines the so called Horadam recurrence sequence written $\{w_n\}_{n=0}^{\infty} = \{w_n\}_0^{\infty} = \{w_n(a, b; p, q)\}_0^{\infty}$. Roots of the characteristic polynomial $P(\lambda; p, q) = \lambda^2 - p\lambda + q$ for (1.1) give rise to separate degenerate ($p^2 = 4q$) and non-degenerate ($p^2 \neq 4q$) root closed forms for w_n which are standard undergraduate exercises to construct. For $p^2 \neq 4q$ ($p, q \neq 0$), there are two distinct characteristic roots

$$\alpha(p, q) = (p + \sqrt{p^2 - 4q})/2, \qquad \beta(p, q) = (p - \sqrt{p^2 - 4q})/2, \tag{1.2}$$

with $\alpha + \beta = p$, $\alpha\beta = q$ and, for $n \geq 0$, a closed form

$$w_n(a, b; p, q) = w_n(\alpha(p, q), \beta(p, q), a, b) = \frac{(b - a\beta)\alpha^n - (b - a\alpha)\beta^n}{\alpha - \beta}. \tag{1.3}$$

For $p^2 = 4q$, on the other hand, the characteristic roots co-incide as simply

$$\alpha(p) = \beta(p) = p/2, \tag{1.4}$$

and, for $n \geq 0$,

$$w_n(a, b; p, p^2/4) = w_n(\alpha(p), a, b) = bn\alpha^{n-1} - a(n - 1)\alpha^n. \tag{1.5}$$

1.2 Ordinary Generating Functions in Context

Ordinary generating functions for sequences of Horadam terms, and powers thereof, have been formulated and studied for over half a century, beginning with A. F. Horadam who seems to have been the first to consider them [3] as he introduced the mathematical community to his sequence and some preliminary results in two seminal papers of 1965. Stănică addressed exponentiated sequence terms in 2003, but gave generating functions only in series form—for the non-degenerate characteristic roots case considered by Horadam—which contained elements from a particular sequence $\{w_n(2, p; p, -q)\}_0^{\infty}$ [10, Theorem 1, p. 322] (some selected (initial value dependent) closed forms duly followed). In 2004 Mansour expressed a general form for (non-degenerate case) power sequence generating functions by way of matrix determinant ratios [7, Theorem 1.1, p. 208] (extending to square powers of an arbitrary degree recurrence sequence in [8]), and listed some special case instances; closed forms relating to the first four powers of Horadam type terms were given at the end of the paper (see p. 211 therein). The most notable aspect of Mező's 2009 article is—amongst other results of a similar nature—a general form for the exponential generating function of Horadam power terms [9, Theorem 14, p. 9]. Elsewhere results have appeared scattered in the literature, such as the generating function for the square power sequence $\{[w_n(a,b;p,q)]^2\}_0^\infty$ of Haukkanen [2, Corollary 3.4, p. 361] who recovered the form given originally in [3]. Generating functions have, not surprisingly, been derived for some well-known sequences which are specialisations of the general Horadam sequence, as they have for those comprising products of Horadam terms and such like (on which latter the reader is referred to the survey [6]).

The paper is organised as follows. On concluding this opening section with more context for the paper, Section 2 details our analysis and results which include proofs, examples and remarks. A summarising section completes the paper, and offers future potential work for any interested reader.

1.3 This Paper and Previous Work on Cyclicity

This paper is a development arising from previous work by the authors in which periodicity properties of Horadam sequences have been examined using a matrix orientated approach to the topic. It is motivated by the existence of so called 'masked' periodicity first noted in [4], of which more explanation was given in a separate paper [5] and whose underpinning ideas appear naturally here as part of the theory presented. We state and prove sufficient conditions for cyclicity which cover instances of *both* characteristic root types, the results differing with each case. Accordingly, they strengthen our insight into the occurrence of masked periodicity described (for the non-degenerate roots case) in [5], and we are also able to recover degenerate roots case periodicity criteria as discussed in Section 2.2 of [4]. It is worth making the point that since any periodic Horadam sequence $\{w_n\}_0^\infty$ yields, for integer $k > 1$, further cyclic ones $\{(w_n)^k\}_0^\infty$, the lines of argument employed here can in principle also be applied to establish periodicity conditions for these power sequences via appropriate generating functions either taken from the literature or else produced by means of algebraic computing—this, however, lies beyond our remit here.

In [5] sequence masking was, as we have noted, discussed in detail for the non-degenerate roots case where those masking behaviours possible were shown to be dictated by the manner in which primitive roots of unity can be used to characterise a Horadam sequence. What is meant by the term 'masking' is that a fully general (*arbitrary* initial values) Horadam sequence can mask, or hide, one or two special case (*specific* initial values) sequence(s) of smaller period. We will see that Theorems 2.1 and 2.2 together combine to offer results supporting the phenomenon of masking through the insight into periodicity that they offer, although the mathematics is couched in a somewhat different manner in this paper which is in itself of interest and may prove useful in examining sequences arising from linear recurrences of higher degree (see the Summary).

Although we do not address masking explicitly here, it is worth mapping the ideas of the two forerunner works [4, 5] to what we will present below. In short, given initial (and arbitrary) values a, b, and respective primitive nth and mth roots of unity ζ_1 and ζ_2, the Horadam recurrence $w_n = (\zeta_1 + \zeta_2)w_{n-1} - (\zeta_1\zeta_2)w_{n-2}$ ($n \geq 2$) of (1.1) produces a $k(m,n)$-cyclic sequence as we will see (Theorem 2.1). One crucial fact to note, from [5], is that a matrix $\mathbf{A} = \mathbf{A}(\zeta_1,\zeta_2) = \left(\begin{smallmatrix}\zeta_1+\zeta_2 & -\zeta_1\zeta_2\\ 1 & 0\end{smallmatrix}\right)$ associated intimately with the sequence, and which provided a convenient tool for the analysis of periodicity in [4], is such that $\mathbf{A}^k(\zeta_1,\zeta_2)$ can be shown to possess eigenvalues $(\zeta_{1,2})^k$ in relation to its eigenvectors $(\zeta_{1,2},1)^T$ (T denoting transposition). By then fixing the initial value b to be a multiple of a whose constant of proportionality is one of the primitive roots (Theorem 2.2), the pair form a vector $(b,a)^T$ which lies along one of these eigenvectors (scaled by a) and—through the Horadam recursion—a new sequence can be revealed (that is, 'unmasked') of period $< k$, as demonstrated in [5] (and, to a lesser extent, in [4]) by the theory and examples provided; precisely how many sequences are available for unmasking, and what period(s) they take, depends on the relationship between m and n in any given instance.

2 Analysis and Results

2.1 Non-Degenerate Roots Case

We denote by

$$N(x) = N(x; a, b, p, q) = \sum_{n=0}^{\infty} w_n(a, b; p, q)x^n = \frac{a + (b - ap)x}{1 - px + qx^2} \tag{2.1}$$

the ordinary generating function of the Horadam sequence $\{w_n(a, b; p, q)\}_0^\infty$ which, on evidence, appeared first in the work of Horadam [3, Eq. (22), p. 440]; its formulation is routine (see the Appendix, where we appeal to a formula of his) and it lies at the center of the proof of our first result.

Theorem 2.1. *Suppose that distinct ζ_1 and ζ_2 are respective primitive nth and mth roots of unity ($m > n \geq 2$) and, in addition, roots of the Horadam characteristic polynomial $P(\lambda; p, q)$. Then the Horadam sequence $\{w_n(a, b; \zeta_1 + \zeta_2, \zeta_1\zeta_2)\}_0^\infty$ is a cyclic one of period k for some $k = k(m, n)$.*

Proof. We show that for all possible scenarios of the primitive characteristic root orders m, n, there exists an integer $k = k(m, n)$ such that the said Horadam sequence is a k-periodic one—we do this by establishing that the power series expansion of $N(x)$ has self-repeating k-blocks of Horadam term coefficients, and thus acts as a generating function for a period k sequence.

Given ζ_1 and ζ_2 are primitive roots of unity of respective order n and m ($m > n \geq 2$), and further that they are characteristic roots (so that $\zeta_1 + \zeta_2 = p$, $\zeta_1\zeta_2 = q$), then clearly $\hat{\zeta}_1 = 1/\zeta_1$ and $\hat{\zeta}_2 = 1/\zeta_2$ are (respectively) primitive nth and mth roots of unity too, and

$$0 = P(\zeta_{1,2}; p, q) = (\zeta_{1,2})^2 - p\zeta_{1,2} + q = \frac{1 - p\hat{\zeta}_{1,2} + q(\hat{\zeta}_{1,2})^2}{(\hat{\zeta}_{1,2})^2}, \tag{P.1}$$

from which we see that $x = \hat{\zeta}_{1,2}$ are distinct roots of the quadratic $1 - px + qx^2$; in other words, up to some multiplicative constant $\alpha^* \in \mathbb{C}$,

$$(x - \hat{\zeta}_1)(x - \hat{\zeta}_2) = \alpha^*(1 - px + qx^2), \tag{P.2}$$

which gives relations $\hat{\zeta}_1 + \hat{\zeta}_2 = p/q$, $\hat{\zeta}_1\hat{\zeta}_2 = 1/q$. Let us choose an integer $k = k(m, n)$ such that $x - \hat{\zeta}_1$ and $x - \hat{\zeta}_2$ are both factors of $1 - x^k$, or equivalently that $\hat{\zeta}_{1,2}$ are kth roots of unity (which is true by inference). Expedient selection scenarios for $k(m, n)$ are but three in number, readily identified as the following: Case I: m and n are coprime $\Rightarrow k = mn$; Case II: n is a divisor of $m \Rightarrow k = m$; Case III: m and n possess a lowest common multiple $l > m \Rightarrow k = l$ (this case assumes that neither $n|m$ nor that m, n are coprime, so that they have a lowest common multiple l for which $m, n < l < mn$). In all three cases $k = \text{lcm}(m, n)$, but separation is convenient in helping to see more clearly the way masking arises and how many sequences are hidden in each instance—examples supplied in [5] illustrated that two sequences (of period m and n) are masked in Cases I and III, while Case II reveals a single period n masked sequence.

Continuing, we consider now the function

$$F(x) = (1 - x^k)N(x) = \frac{(1 - x^k)[a + (b - ap)x]}{1 - px + qx^2} = \frac{(1 - x^k)[a + (b - ap)x]}{q(x - \hat{\zeta}_1)(x - \hat{\zeta}_2)}. \tag{P.3}$$

Since $(x - \hat{\zeta}_{1,2})|(1 - x^k)$ then $F(x) = F(x; a, b, p, q)$ is a finite polynomial of degree $(k+1) - 2 = k - 1$ and we write, for some $f_0, f_1, \ldots, f_{k-1}$,

$$F(x) = f_0 + f_1x + f_2x^2 + \cdots + f_{k-1}x^{k-1} = \sum_{s=0}^{k-1} f_s(a, b, p, q)x^s, \tag{P.4}$$

from which, by (P.3),

$$\begin{aligned} N(x) &= F(x)(1-x^k)^{-1} \\ &= F(x)(1+x^k+x^{2k}+\cdots) \\ &= F(x)+x^kF(x)+x^{2k}F(x)+\cdots \\ &= (f_0+f_1x+f_2x^2+\cdots+f_{k-1}x^{k-1}) \\ &\quad + (f_0x^k+f_1x^{k+1}+f_2x^{k+2}+\cdots+f_{k-1}x^{2k-1}) \\ &\quad\quad + (f_0x^{2k}+f_1x^{2k+1}+f_2x^{2k+2}+\cdots+f_{k-1}x^{3k-1})+\cdots, \end{aligned} \tag{P.5}$$

being a generating function for the sequence $\{f_0, f_1, f_2, \ldots, f_{k-1}, \ldots\} = \{w_0, w_1, w_2, \ldots, w_{k-1}, \ldots\}$ that self-repeats with period k; this completes the proof. □

Remark 2.1. Theorem 2.1 actually holds for $n = m$ (when the characteristic roots are distinct but have the same order); the sequence $\{w_n(a, b; \zeta_1 + \zeta_2, \zeta_1\zeta_2)\}_0^\infty$ has period m but is of no concern in relation to masking which cannot occur (see [5, Remark 3, p. 117]).

The degenerate roots case is in fact covered as a corollary to the following result that is different in nature (and in which ζ_1 and ζ_2 are interchangeable).

Theorem 2.2. *Suppose that ζ_1 and ζ_2 are roots (which may or may not co-incide) of the characteristic polynomial $P(\lambda; p, q)$. If ζ_1 is also a primitive kth root of unity and, in addition, if the Horadam sequence initial values are related proportionally as $b = a\zeta_1$, then the sequence $\{w_n(a, a\zeta_1; \zeta_1 + \zeta_2, \zeta_1\zeta_2)\}_0^\infty$ is a cyclic one of period k.*

Proof. Setting $\hat{\zeta}_1 = 1/\zeta_1$ and $\hat{\zeta}_2 = 1/\zeta_2$, then we again consider the function $F(x)$ of (P.3). Since $\hat{\zeta}_1$ is also a primitive kth root of unity then $(x - \hat{\zeta}_1)|(1 - x^k)$ and it suffices merely to show that $(x - \hat{\zeta}_2)|[a + (b - ap)x]$ to establish k-periodicity in the same manner as seen in the proof of Theorem 2.1. This, given $b = a\zeta_1$, is routine, for $[a + (b - ap)x]/(x - \hat{\zeta}_2) = a[1 + (\zeta_1 - p)x]/(x - \hat{\zeta}_2) = a[1 + (-\zeta_2)x]/(x - \hat{\zeta}_2) = -a\zeta_2(x - \frac{1}{\zeta_2})/(x - \hat{\zeta}_2) = -a\zeta_2 \in \mathbb{R}$. □

By way of example, consider first a case where $\zeta_1 = i$ (a primitive fourth root of unity) and $\zeta_2 = 3$, which results in the period 4 sequence $\{w_n(a, ai; 3 + i, 3i)\}_0^\infty = \{a, ai, -a, -ai, \ldots\}$. For $\zeta_1 = -1$ (the primitive square root of unity) and $\zeta_2 = \sqrt{5}i$ the sequence $\{w_n(a, -a; -1 + \sqrt{5}i, -\sqrt{5}i)\}_0^\infty = \{a, -a, \ldots\}$ is period 2. More useful to look at, perhaps, is the period 4 sequence $\{w_n(a, -ai; \zeta_2 - i, -\zeta_2 i)\}_0^\infty = \{a, -ai, -a, ai, \ldots\}$ where ζ_2 has been left *unassigned* (with $\zeta_1 = -i$ the other primitive fourth root of unity) but does not appear within the sequence at all. Execution of the Horadam recurrence (1.1) reveals the anticipated self-cancellation of ζ_2 throughout the process of sequence term generation—this is an observation found to be not specific to this example and, holding generally, it is readily explained: re-casting (1.3) as $w_n(\zeta_1, \zeta_2, a, b) = [(b - a\zeta_2)(\zeta_1)^n - (b - a\zeta_1)(\zeta_2)^n]/(\zeta_1 - \zeta_2)$ for $n \geq 0$, then upon setting $b = a\zeta_1$ this closed form contracts to $w_n(\zeta_1, a) = a(\zeta_1)^n$ and describes the sequence $\{a(\zeta_1)^n\}_0^\infty$ which is k-periodic if ζ_1 is a primitive kth root of unity.

We complete the section, and so the paper, by addressing the degenerate roots case and making some wider concluding remarks.

2.2 Degenerate Roots Case

The case of degenerate characteristic roots Horadam sequence periodicity is dealt with as a simple instance of Theorem 2.2 that we state and then discuss briefly in context.

Corollary 2.1. *Suppose (given $q(p) = p^2/4$) that $\zeta_1 = \zeta_2 = \zeta$ are repeated (degenerate) roots of $P(\lambda; p, q(p))$, with ζ also a primitive kth root of unity. Then the sequence $\{w_n(a, a\zeta; 2\zeta, \zeta^2)\}_0^\infty$ is a cyclic one of period k.*

Here, the function

$$D(x) = D(x; a, b, p) = \sum_{n=0}^{\infty} w_n(a, b; p, p^2/4)x^n = \frac{4[a + (b - ap)x]}{(2 - px)^2} = N(x; a, b, p, q(p)) \tag{2.2}$$

is the Horadam sequence $\{w_n(a,b;p,p^2/4)\}_0^\infty$ ordinary generating function associated with degenerate characteristic roots $\zeta_1 = \zeta_2 = \zeta = p/2$ (1.4), and is immediate from (2.1) with $q = q(p) = p^2/4$. Setting $b = a\zeta$ then $\{w_n(a,b;p,p^2/4)\}_0^\infty = \{w_n(a,a\zeta;2\zeta,\zeta^2)\}_0^\infty$ is k-cyclic, and is simply the sequence $\{a,a\zeta,a\zeta^2,\ldots,a\zeta^{k-1},\ldots\}$ delivered by the collapsed form of $D(x) = D(x;a,b(a,\zeta),p(\zeta)) = a(1-\zeta x)^{-1}$; the sequence term closed form $w_n(\zeta,a,b) = bn\zeta^{n-1} - a(n-1)\zeta^n$ (1.5) reduces in this instance (that is, with $b = a\zeta$) to $w_n(\zeta,a) = a\zeta^n$ trivially, and confirms the result.

Remark 2.2. The interested reader is referred to [4, Section 2.2] for examples which illustrate Corollary 2.1 in practice. Note that the conditions in [4] are shown to be necessary ones too. We do not attempt to establish that any of those in this paper are necessary, as an appeal to generating functions such as we have made here does not appear to lend itself to any meaningful proof strategies in this respect.

Note that while we do not address cyclicity of the exponentiated sequence $\{(w_n)^k\}_0^\infty$ for general $k > 1$, the (non-degenerate case) generating function for $\{(w_n)^2\}_0^\infty$ is derived in the Appendix—along with that for $\{w_n\}_0^\infty$—using a formulation of Horadam himself; this is done for the purposes of completeness and reader interest (degenerate case generating functions follow readily as special case ones, of course). However they are constructed, one gets a sense of how algebraically convoluted these generating functions become with increasing power k, although extrapolation of the approach taken here to study periodicity for values of k beyond 1 remains, as already mentioned, a valid potential means to deal with the sequences they represent even if its implementation looks to be non-trivial.

3 Summary

The originality of this paper lies in the fact that periodicity analysis has been undertaken for the first time through the fundamental (non-degenerate case) ordinary generating function of the Horadam sequence which connects with Horadam's very earliest work. We note that relatively little attention was devoted to the degenerate roots case at the start of this era, and Horadam's article [3]—in which a closed form for $D(x)$ (2.2) is absent and to which there is but the briefest of reference made—reflects this. Quite recently, the notion of masked cyclicity has been explored from the viewpoint of complex 'generators' (these are primitive roots of unity written in exponential form), and the ideas have been generalised to higher order recurrence systems [1] using an established methodology; periodicity conditions are illustrated in particular for a degree three linear recurrence model.

One final thought rests with the possibility that, working from a known ordinary generating function, the proof strategies employed here might be applied in future to yield counterpart cyclicity results for sequences governed by a fully general order three type extension of (1.1), or one of even higher degree still—this is left as an open problem.

Appendix

Background to Formulations of Generating Functions. Denote by $N_k(x)$ the *non-degenerate* roots case ordinary generating function

$$N_k(x) = N_k(x;a,b,p,q) = \sum_{n=0}^{\infty}[w_n(a,b;p,q)]^k x^n \tag{A.1}$$

for the sequence $\{(w_n)^k\}_0^\infty$ $(k \geq 1)$, noting that $N_1(x) = N(x)$ of (2.1) in Section 2.1; the corresponding *degenerate* case function is $D_k(x) = \sum_{n=0}^{\infty}[w_n(a,b;p,p^4/4)]^k x^n = N_k(x;a,b,p,p^4/4)$. We derive $N_1(x)$ and (in full detail) $N_2(x)$ as examples of Horadam's 1965 formula [3, Eq. (42), p. 442]

$$N_k(x) = \sum_{s=0}^{k}\binom{k}{s}A^{k-s}B^s(1-\alpha^{k-s}\beta^s x)^{-1}, \qquad k \geq 1, \tag{A.2}$$

where α, β are the characteristic roots (1.2) and the constants A, B are $A = (b - a\beta)/(\alpha - \beta)$, $B = (a\alpha - b)/(\alpha - \beta)$ (used in Horadam's closed form $w_n = A\alpha^n + B\beta^n$ matching (1.3); Horadam gave (A.2) as an alternative version of the sum $\sum_{n=0}^{\infty}(A\alpha^n + B\beta^n)^k x^n$ he wrote down for $N_k(x)$). This is an instructive procedure in itself, and we can appreciate how quickly the algebraic complexity of $N_k(x)$ increases with k; formulations for neither $N_1(x)$ nor $N_2(x)$ were detailed in [3], the latter function being merely stated with reference to (A.2) and the former offered as an implied reader exercise in confirmation.

Case $k = 1$**.** Noting the relations $A + B = a$ and $A\beta + B\alpha = ap - b$, (A.2) gives

$$\begin{aligned} N_1(x) &= \sum_{s=0}^{1} \binom{1}{s} A^{1-s} B^s (1 - \alpha^{1-s}\beta^s x)^{-1} \\ &= \frac{A}{(1-\alpha x)} + \frac{B}{(1-\beta x)} \\ &= \frac{A(1-\beta x) + B(1-\alpha x)}{(1-\alpha x)(1-\beta x)} \\ &= \frac{A + B - (A\beta + B\alpha)x}{1 - (\alpha+\beta)x + (\alpha\beta)x^2} \\ &= \frac{a + (b - ap)x}{1 - px + qx^2}, \end{aligned} \tag{A.3}$$

which is (2.1); $D_1(x) = D(x)$, as given in (2.2).

Case $k = 2$**.** This case is surprisingly more involved, with (A.2) yielding

$$\begin{aligned} N_2(x) &= \sum_{s=0}^{2} \binom{2}{s} A^{2-s} B^s (1 - \alpha^{2-s}\beta^s x)^{-1} \\ &= \frac{A^2}{(1-\alpha^2 x)} + \frac{2AB}{(1-\alpha\beta x)} + \frac{B^2}{(1-\beta^2 x)} \\ &= \frac{A^2(1-\alpha\beta x)(1-\beta^2 x) + 2AB(1-\alpha^2 x)(1-\beta^2 x) + B^2(1-\alpha^2 x)(1-\alpha\beta x)}{(1-\alpha^2 x)(1-\alpha\beta x)(1-\beta^2 x)} \\ &= N_2^u(x)/N_2^l(x), \end{aligned} \tag{A.4}$$

say. The denominator is dealt with easily, for we see that, employing the relation $\alpha^2 + \beta^2 = (\alpha+\beta)^2 - 2\alpha\beta = p^2 - 2q$,

$$\begin{aligned} N_2^l(x) &= (1-\alpha^2 x)(1-\alpha\beta x)(1-\beta^2 x) \\ &= [1 - (\alpha^2+\beta^2)x + (\alpha\beta)^2 x^2](1-\alpha\beta x) \\ &= [1 - (p^2 - 2q)x + q^2 x^2](1 - qx) \\ &= 1 - (p^2 - q)x + q(p^2 - q)x^2 - q^3 x^3. \end{aligned} \tag{A.5}$$

The numerator requires more effort, for as a polynomial in x it is, after some re-writing,

$$\begin{aligned} N_2^u(x) &= (A+B)^2 - [p(A^2\beta + B^2\alpha) + 2AB(p^2 - 2q)]x + q(A\beta + B\alpha)^2 x^2 \\ &= a^2 - [p(A^2\beta + B^2\alpha) + 2AB(p^2 - 2q)]x + q(b - ap)^2 x^2, \end{aligned} \tag{A.6}$$

where the coefficient of x needs some attention. First, we note that $A^2\beta + B^2\alpha = [(b - a\beta)^2\beta + (a\alpha - b)^2\alpha]/(\alpha-\beta)^2 = [b^2(\alpha+\beta) - 2ab(\alpha^2+\beta^2) + a^2(\alpha^3+\beta^3)]/(\alpha-\beta)^2 = [b^2 p - 2ab(p^2 - 2q) + a^2(p^3 - 3pq)]/(\alpha-\beta)^2$ (having used $\alpha^3 + \beta^3 = (\alpha+\beta)^3 - 3\alpha\beta(\alpha+\beta) = p^3 - 3pq$).

With $AB = (b - a\beta)(a\alpha - b)/(\alpha - \beta)^2 = \cdots = (abp - a^2q - b^2)/(\alpha - \beta)^2$, then

$$\begin{aligned}[x]\{N_2^u(x)\} &= -\frac{p[b^2p - 2ab(p^2 - 2q) + a^2(p^3 - 3pq)] + 2(abp - a^2q - b^2)(p^2 - 2q)}{(\alpha - \beta)^2} \\ &\vdots \\ &= -\frac{(4q - p^2)[b^2 - a^2(p^2 - q)]}{(\alpha - \beta)^2} \\ &= b^2 - a^2(p^2 - q), \end{aligned} \tag{A.7}$$

since $\alpha - \beta = \sqrt{p^2 - 4q}$. Thus, with $N_2^l(x)$ (A.5), and $N_2^u(x)$ (A.6) (completed by (A.7)),

$$N_2(x) = \frac{N_2^u(x)}{N_2^l(x)} = \frac{a^2 + [b^2 - a^2(p^2 - q)]x + q(b - ap)^2x^2}{1 - (p^2 - q)x + q(p^2 - q)x^2 - q^3x^3}, \tag{A.8}$$

whose numerator is in agreement with [3, Eq. (65), p.446] and whose full form tallies with others elsewhere. Nowadays, of course, such a result is readily available from standard algebraic software which we have used to check it. Calculation of $N_k(x)$ for $k = 3$ (which is prohibitively difficult by hand) and higher values may, if required, be found computationally.

Finally, note that

$$D_2(x) = 16\frac{4a^2 + (4b^2 - 3a^2p^2)x + p^2(b - ap)^2x^2}{(4 - p^2x)^3} \tag{A.9}$$

is immediate from (A.8).

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