Palestine Journal of Mathematics

Vol. 9(2)(2020) , 698–710

Sasakian manifolds admitting a non-symmetric non-metric connection

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Communicated by Ayman Badawi

MSC 2010 Classifications: 53B15, 53C25

Keywords and phrases: Non-symmetric non-metric connection, Sasakian structure, Nijenhuis tensor, Curvature tensors, Symmetric spaces, Ricci soliton

Abstract Sasakian manifolds with respect to a non-symmetric non-metric connection are studied. Some properties of the curvature, the conformal curvature, the conharmonic curvature of Sasakian manifolds admitting a non-symmetric non-metric connection are studied. Semisymmetric and Ricci semisymmetric Sasakian manifolds with respect to a non-symmetric non-metric connection studies as well.

1 Introduction

The infinitesimal perspective of connections in Riemannian geometry began to some extant with Christoffel. Later Levi-Civita observed that a connection also allowed for a notion of parallel transport. Levi-Civita focussed on connection as a kind of differential operator. In 20th century, Carton established connection as a certain kind of differential form. In 1950, Koszul gave an algebraic framework for a connection as a differential operator. The main invariants of an affine connection are its torsion and curvature [11, 12]. A torsion tensor of a connection is a mapping $T : \chi(M) \times \chi(M) \to \chi(M)$ defined as

$$T(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y], \tag{1.1}$$

for arbitrary vector fields $X, Y \in \chi(M)$. A connection ∇ is said to be torsion-free or symmetric if its torsion tensor $T(X,Y)$ vanishes, otherwise it is non-symmetric. A connection ∇ on a manifold M is called a metric connection if there is a Riemannian metric g in M such that $\nabla g = 0$, otherwise it is non-metric. A Levi-Civita connection is symmetric as well as metric. Semisymmetric and quater-symmetric connections are also one kind of non-symmertic connections. Systematic study of semisymmetric connection in a Riemannian manifold was initiated by Yano [22]. In 1992, Agashe and Chafle [1] introduced the notion of semisymmetric non-metric connection. In 2007, S. K. Chaubey [6] introduced another non-symmetric non-metric connection. Later he studied the same connection on different contact manifolds [7, 8, 9]. The properties of such connection have been studied by many geometers in different extant.

On the other hand, Sasakian manifolds [2, 5, 20] were introduced in 1960 by Shigeo Sasaki. It further studied by several geometers [10, 13, 16, 18]. The non-symmetric non-metric connection on Sasakian manifolds have also been studied by a few others [9]. However semisymmetric and Ricci semisymmetric Sasakian manifolds with respect to the non-symmetric non-metric connection have not been studied so far. The paper is organised as under. Section-1 is introductory. Section-2 contains some basic results of Sasakian manifolds. The definition of $\eta-$Einstein manifold, Ricci soliton and Nijenhuis tensor are also given in Section-2. Required results of the non-symmetric non-metric connection are given in the Section-3. The initial bridge of Sasakian manifold and the non-symmetric non-metric connection established in Section-4. Semisymmetric and Ricci semisymmetric Sasakina manifolds with respect to the non-symmetric non-metric connection have been consecutively studied in Section 5 and 6. It is also proved that Ricci soliton of data (g, ξ, λ) is shrinking on Ricci semisymmetric and semisymmetric Sasakian manifolds admitting the non-symmetric non-metric connection. Lastly in Section-7, we establish an example of Sasakian manifold admitting the non-symmetric non-metric connection. The example verifies several results of the article.

2 Preliminaries

Let M be a $(2n+1)$-dimensional almost contact metric manifold [2] equipped with almost contact metric structure (ϕ, ξ, η, g), where ϕ is $(1,1)$ tensor field, ξ is a vector field, η is 1-form and g is compatible Riemannian metric such that

$$\phi^2 = -I + \eta \otimes \xi, \qquad \eta(\xi) = 1, \qquad \phi\xi = 0, \qquad \eta o \phi = 0, \tag{2.1}$$

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \tag{2.2}$$

$$g(\phi X, Y) = -g(X, \phi Y), \qquad g(X, \xi) = \eta(X), \tag{2.3}$$

for all $X, Y \in TM$. Here TM is a tangent space of the manifold M.

An almost contact metric manifold M [2] is called Sasakian manifold if

$$(\nabla_X \phi)Y = g\left(X, Y\right)\xi - \eta\left(Y\right)X, \tag{2.4}$$

where ∇ is Levi-Civita connection of Riemannian metric g on M. From equation (2.4) and equations (2.1), (2.2) and (2.3), we have

$$\nabla_X \xi = -\phi X, \tag{2.5}$$

$$(\nabla_X \eta)Y = g\left(X, \phi Y\right). \tag{2.6}$$

Also the following relations hold in a Sasakian manifold:

$$R\left(X, Y\right)\xi = \eta\left(Y\right)X - \eta\left(X\right)Y, \tag{2.7}$$

$$R\left(\xi, Y\right)X = g\left(X, Y\right)\xi - \eta\left(X\right)Y, \tag{2.8}$$

$$R\left(\xi, X\right)\xi = \eta\left(X\right)\xi - X, \tag{2.9}$$

$$S\left(X, \xi\right) = 2n\eta\left(X\right), \tag{2.10}$$

$$Q\xi = 2n\xi, \tag{2.11}$$

where R is the curvature tensor, S is the Ricci-curvature and Q is the Ricci-operator of Sasakian manifold. S and Q are related to each other by

$$S\left(X, Y\right) = g\left(QX, Y\right). \tag{2.12}$$

Definition 2.1. An almost contact metric manifold M is said to be η-Einstein if its Ricci-tensor S is of the form

$$S\left(X, Y\right) = ag\left(X, Y\right) + b\eta\left(X\right)\eta(Y), \tag{2.13}$$

equivalently an almost contact metric manifold M is said to be η -Einstein if its Ricci-operator is of the form

$$Q\left(X\right) = aX + b\eta\left(X\right)\xi, \tag{2.14}$$

where a and b are smooth functions on M. It is also known that if a $(2n+1)-$dimensional Sasakian manifold is $\eta-$Einstain then $a + b = 2n$ [9].

Definition 2.2. A Ricci soliton (g, V, λ) on a Riemannian manifold is defined by

$$\pounds_V g + 2S + 2\lambda g = 0, \tag{2.15}$$

where $\pounds_V g$ is a Lie-derivative of Riemannian metric g with respect to vector field V and λ is a real constant. It is said to be shrinking, steady, or expanding according as $\lambda < 0$, $\lambda = 0$ and $\lambda > 0$ [4].

The Nijenhuis tensor $N(X,Y)$ of ϕ in (M_{2n+1}, g) is a vector valued bilinear function such that

$$N(X,Y) = (\nabla_{\phi X}\phi)(Y) - (\nabla_{\phi Y}\phi)(X) - \phi((\nabla_X\phi)(Y)) + \phi((\nabla_Y\phi)(X)). \tag{2.16}$$

If we define

$$'N(X,Y,Z) = g(N(X,Y),Z). \tag{2.17}$$

then

$$\begin{aligned} 'N(X,Y,Z) &= g((\nabla_{\phi X}\phi)(Y),Z) - g((\nabla_{\phi Y}\phi)(X),Z) \\ &\quad -g(\phi((\nabla_X\phi)(Y)),Z) + g(\phi((\nabla_Y\phi)(X)),Z). \end{aligned} \tag{2.18}$$

3 A non-symmetric non-metric connection

On the other hand, a linear connection $\widetilde{\nabla}$ [6], [8] defined as

$$\widetilde{\nabla}_X Y = \nabla_X Y + g(\phi X, Y)\xi \tag{3.1}$$

satisfying

$$\widetilde{T}(X,Y) = 2g(\phi X, Y)\xi \tag{3.2}$$

and

$$\left(\widetilde{\nabla}_X g\right)(Y,Z) = -\eta(Z)g(\phi X, Y) - \eta(Y)g(\phi X, Z) \tag{3.3}$$

for arbitrary vector field X, Y and Z, is called a non-symmetric non-metric connection. It is also known [6]

$$\left(\widetilde{\nabla}_X \phi\right)(Y) = (\nabla_X\phi)(Y) + g(\phi X, \phi Y)\xi, \tag{3.4}$$

$$\left(\widetilde{\nabla}_X \eta\right)(Y) = (\nabla_X\eta)(Y) - g(\phi X, Y), \tag{3.5}$$

$$\left(\widetilde{\nabla}_X g\right)(\phi Y, Z) = -\eta(Z)g(\phi X, \phi Y). \tag{3.6}$$

On changing Y by ξ in the equation (3.1), we have

$$\widetilde{\nabla}_X \xi = \nabla_X \xi. \tag{3.7}$$

We also have

Proposition 3.1. *The vector field ξ is invariant with respect to Levi-Civita connection ∇ and non-symmetric non-metric connection $\widetilde{\nabla}$ [16].*

Proposition 3.2. *Co-variant differentiation of Riemannian metric g with respect to contra-variant vector field ξ vanish identically in a contact metric manifold admitting the non-symmetric non-metric connection $\widetilde{\nabla}$ [16].*

The curvature tensor $\widetilde{R}$ of $\widetilde{\nabla}$ defined as follows

$$\widetilde{R}(X,Y)Z = \widetilde{\nabla}_X\widetilde{\nabla}_Y Z - \widetilde{\nabla}_Y\widetilde{\nabla}_X Z - \widetilde{\nabla}_{[X,Y]}Z, \tag{3.8}$$

where $X, Y, Z \in TM$. Using equations (3.1), (3.4), (3.5), (3.6) and (3.7), we get

$$\begin{aligned} \widetilde{R}(X,Y)Z &= R(X,Y)Z + g((\nabla_X\phi)Y,Z)\xi - g((\nabla_Y\phi)X,Z)\xi \\ &\quad +g(\phi Y,Z)\nabla_X\xi - g(\phi X,Z)\nabla_Y\xi, \end{aligned} \tag{3.9}$$

where

$$R(X,Y)Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z \tag{3.10}$$

is the Riemannian curvature tensor of Levi-Civita connection [2].

4 Sasakian manifolds with the non-symmetric non-metric connection

If we define

$$\widetilde{d}\eta(X,Y) = \left(\widetilde{\nabla}_X\eta\right)Y - \left(\widetilde{\nabla}_Y\eta\right)X. \tag{4.1}$$

Using equations (2.6) and (3.5), we get

$$\widetilde{d}\eta(X,Y) = -4g(\phi X, Y). \tag{4.2}$$

Let us define the Nijenhuis tensor $\widetilde{N}(X,Y)$ of ϕ in an (M_{2n+1}, g) admitting the non-symmetric non-metric connection

$$\begin{aligned}\widetilde{N}(X,Y) &= \left(\widetilde{\nabla}_{\phi X}\phi\right)(Y) - \left(\widetilde{\nabla}_{\phi Y}\phi\right)(X) \\ &\quad -\phi\left(\left(\widetilde{\nabla}_X\phi\right)(Y)\right) + \phi\left(\left(\widetilde{\nabla}_Y\phi\right)(X)\right).\end{aligned} \tag{4.3}$$

In view of equations (2.1) and (3.4), we get

$$\begin{aligned}\widetilde{N}(X,Y) &= \left(\nabla_{\phi X}\phi\right)(Y) - \left(\nabla_{\phi Y}\phi\right)(X) \\ &\quad -\phi\left(\left(\nabla_X\phi\right)(Y)\right) + \phi\left(\left(\nabla_Y\phi\right)(X)\right) + 2g(\phi X, Y)\xi.\end{aligned} \tag{4.4}$$

Using equation (2.4), we get

$$\widetilde{N}(X,Y) = 3g(\phi X, Y)\xi. \tag{4.5}$$

From equations (4.2) and (4.5), we get

$$4\widetilde{N}(X,Y) + 3\widetilde{d}\eta(X,Y)\xi = 0. \tag{4.6}$$

Hence we can say

Theorem 4.1. *Sasakian manifolds admitting the non-symmetric non-metric connection $\widetilde{\nabla}$ satisfy the equation (4.6).*

In view of equations (3.2) and (4.6), we can also write the following corollary:

Corollary 4.2. *Sasakian manifolds admitting the non-symmetric non-metric connection $\widetilde{\nabla}$ satisfy the equation* $2\widetilde{N}(X,Y) - 3\widetilde{T}(X,Y) = 0$.

Lemma 4.3. *On a Sasakian manifold equipped with the non-symmetric non-metric connection $\widetilde{\nabla}$, Ricci tensor, Ricci operator and scalar curvature are invariant for the non-symmetric non-metric connection $\widetilde{\nabla}$ and Livi-Civita connection ∇.*

Proof. Using equations (2.1), (2.3), (2.4) and (2.5) in (3.9), we have

$$\begin{aligned}\widetilde{R}(X,Y)Z &= R(X,Y)Z + g(Y,Z)\eta(X)\xi - g(X,Z)\eta(Y)\xi \\ &\quad + g(\phi X, Z)(\phi Y) - g(\phi Y, Z)(\phi X).\end{aligned} \tag{4.7}$$

Contracting equation (4.7) with respect to X, we have

$$\widetilde{S}(Y,Z) = S(Y,Z). \tag{4.8}$$

By virtue of equation (2.12), equation (4.8) gives

$$\widetilde{Q}(Y) = Q(Y) \tag{4.9}$$

Again contracting equation (4.8), we get

$$\widetilde{r} = r, \tag{4.10}$$

where $\widetilde{S}(Y,Z)$; $S(Y,Z)$, $\widetilde{Q}$; Q and $\widetilde{r}$; r are the Ricci tensors, Ricci operators and scalar curvatures of the non-symmetric non-metric connection $\widetilde{\nabla}$ and Livi-Civita connection ∇. □

On replacing $X = \xi$ in the equation (4.7) and using equations (2.1) and (2.3), we have

$$\widetilde{R}(\xi, Y)Z = R(\xi, Y)Z + g(Y, Z)\xi - \eta(Z)\eta(Y)\xi. \tag{4.11}$$

In view of equations (2.8) and (4.11), we can easily get

$$\widetilde{R}(\xi, Y)Z = 2g(Y, Z)\xi - \eta(Z)Y - \eta(Z)\eta(Y)\xi. \tag{4.12}$$

Again on replacing $Z = \xi$ in the equation (4.7) and using equations (2.1), (2.3) and (2.7), we have

$$\begin{aligned} \widetilde{R}(X, Y)\xi &= R(X, Y)\xi \\ &= \eta(Y)X - \eta(X)Y. \end{aligned} \tag{4.13}$$

In view of equations (4.7), (2.1) and $g(R(X, Y, Z), W) = -g(R(X, Y, W), Z)$, we have

$$\eta(\widetilde{R}(X, Y)Z) = 2[g(Y, Z)\eta(X) - g(X, Z)\eta(Y)]. \tag{4.14}$$

On contracting equation (4.13) with respect to X, we have

$$\widetilde{S}(Y, \xi) = 2n\eta(Y). \tag{4.15}$$

Theorem 4.4. *If Riemannian curvature tensor of $\widetilde{\nabla}$ in a Sasakian manifold admitting the non-symmetric non-metric connection $\widetilde{\nabla}$ vanishes, then the manifold is Ricci-flat.*

Proof. On taking $\widetilde{R}(X, Y)Z = 0$ in the equation (4.7), we have

$$\begin{aligned} R(X, Y)Z &= g(X, Z)\eta(Y)\xi - g(Y, Z)\eta(X)\xi \\ &\quad + g(\phi Y, Z)(\phi X) - g(\phi X, Z)(\phi Y). \end{aligned} \tag{4.16}$$

In view of $'R(X, Y, Z, W) = g(R(X, Y)Z, W)$ and equation (4.16), we have

$$\begin{aligned} 'R(X, Y, Z, W) &= g(X, Z)\eta(Y)\eta(W) - g(Y, Z)\eta(X)\eta(W) \\ &\quad + g(\phi Y, Z)g(\phi X, W) + g(X, \phi Z)g(\phi Y, W). \end{aligned} \tag{4.17}$$

Contracting equation (4.17) with respect to vector fields X, we get

$$S(Y, Z) = 0, \tag{4.18}$$

which leads to the proof. □

Theorem 4.5. *In a Sasakian manifold admitting the non-symmetric non-metric connection $\widetilde{\nabla}$, the necessary and sufficient condition for the conformal curvature tensor of $\widetilde{\nabla}$ coincides with that of ∇ is that the conharmonic curvature tensor of $\widetilde{\nabla}$ is equal to that of ∇.*

Proof. The conformal curvature tensor of $\widetilde{\nabla}$ is defined as

$$\begin{aligned} \widetilde{C}(X, Y)Z &= \widetilde{R}(X, Y)Z - \frac{1}{(2n-1)}[\widetilde{S}(Y, Z)X - \widetilde{S}(X, Z)Y \\ &\quad + g(Y, Z)\widetilde{Q}X - g(X, Z)\widetilde{Q}Y] \\ &\quad + \frac{\widetilde{r}}{2n(2n-1)}\left[g(Y, Z)X - g(X, Z)Y\right]. \end{aligned} \tag{4.19}$$

Using equations (4.7), (4.8), (4.9) and (4.10) in the equation (4.19), we have

$$\begin{aligned} \widetilde{C}(X, Y)Z - C(X, Y)Z &= g(Y, Z)\eta(X)\xi - g(X, Z)\eta(Y)\xi \\ &\quad + g(\phi Y, Z)X - g(\phi X, Z)Y, \end{aligned} \tag{4.20}$$

where

$$\begin{aligned} C(X,Y)Z &= R(X,Y)Z - \frac{1}{(2n-1)}[S(Y,Z)X - S(X,Z)Y \\ &\quad + g(Y,Z)QX - g(X,Z)QY] \\ &\quad + \frac{r}{2n(2n-1)}[g(Y,Z)X - g(X,Z)Y]. \end{aligned} \tag{4.21}$$

Again we define conharmonic curvature tensor of $\widetilde{\nabla}$ as

$$\begin{aligned} \widetilde{L}(X,Y)Z &= \widetilde{R}(X,Y)Z - \frac{1}{(2n-1)}[\widetilde{S}(Y,Z)X - \widetilde{S}(X,Z)Y \\ &\quad + g(Y,Z)\widetilde{Q}X - g(X,Z)\widetilde{Q}Y]. \end{aligned} \tag{4.22}$$

Using equations (4.7), (4.8), (4.9) and (4.10) in the equation (4.22), we have

$$\begin{aligned} \widetilde{L}(X,Y)Z - L(X,Y)Z &= g(Y,Z)\eta(X)\xi - g(X,Z)\eta(Y)\xi \\ &\quad + g(\phi Y,Z)X - g(\phi X,Z)Y, \end{aligned} \tag{4.23}$$

where

$$\begin{aligned} L(X,Y)Z &= R(X,Y)Z - \frac{1}{(2n-1)}[S(Y,Z)X - S(X,Z)Y \\ &\quad + g(Y,Z)QX - g(X,Z)QY]. \end{aligned} \tag{4.24}$$

Proof of the theorem is obvious in view of equations (4.20) and (4.23). □

Theorem 4.6. *In a Sasakian manifold admitting the non-symmetric non-metric connection $\widetilde{\nabla}$, the necessary and sufficient condition for the concircular curvature tensor coincides with curvature tensor is scalar curvature of $\widetilde{\nabla}$ to be zero.*

Proof. The concircular curvature tensor [19] of a Riemannian manifold is defined as

$$V(X,Y)Z = R(X,Y)Z - \frac{r}{2n(2n+1)}[g(Y,Z)X - g(X,Z)Y]. \tag{4.25}$$

Proof of the theorem is obvious in view of equations (4.10), (4.25) and $\widetilde{r} = 0$. □

5 Semisymmetric Sasakian manifolds admitting the non-symmetric non-metric connection $\widetilde{\nabla}$

A $(2n+1)-$ dimensional contact metric manifold M with the non-symmetric non-metric connection is said to be semisymmetric if $(\widetilde{R}(X,Y).\widetilde{R})(Z,U)V = 0$. Suppose M is a semisymmetric contract metric manifold admitting the non-symmetric non-metric connection $\widetilde{\nabla}$. Which implies

$$\begin{aligned} &\widetilde{R}(X,Y)\widetilde{R}(Z,U)V - \widetilde{R}(\widetilde{R}(X,Y)Z,U)V \\ &-\widetilde{R}(Z,\widetilde{R}(X,Y)U)V - \widetilde{R}(Z,U)\widetilde{R}(X,Y)V = 0. \end{aligned} \tag{5.1}$$

On changing $X = \xi$, we get

$$\begin{aligned} &\widetilde{R}(\xi,Y)\widetilde{R}(Z,U)V - \widetilde{R}(\widetilde{R}(\xi,Y)Z,U)V \\ &-\widetilde{R}(Z,\widetilde{R}(\xi,Y)U)V - \widetilde{R}(Z,U)\widetilde{R}(\xi,Y)V = 0. \end{aligned} \tag{5.2}$$

In view of equation (4.11), we obtain

$$
\begin{aligned}
&2g(Y,\widetilde{R}(Z,U)V)\xi-\eta(\widetilde{R}(Z,U)V)Y-\eta(Y)\eta(\widetilde{R}(Z,U)V)\xi\\
&-2g(Y,Z)\widetilde{R}(\xi,U)V+\eta(Z)\widetilde{R}(Y,U)V+\eta(Y)\eta(Z)\widetilde{R}(\xi,U)V\\
&-2g(Y,U)\widetilde{R}(Z,\xi)V+\eta(U)\widetilde{R}(Z,Y)V+\eta(Y)\eta(U)\widetilde{R}(Z,\xi)V\\
&-2g(Y,V)\widetilde{R}(Z,U)\xi+\eta(V)\widetilde{R}(Z,U)Y+\eta(Y)\eta(V)\widetilde{R}(Z,U)\xi=0,
\end{aligned}\tag{5.3}
$$

which implies

$$
\begin{aligned}
2'\widetilde{R}(Z,U,V,Y) \;=\;& \eta(\widetilde{R}(Z,U)V)\eta(Y)+\eta(Y)\eta(\widetilde{R}(Z,U)V)\\
&+2g(Y,Z)\eta(\widetilde{R}(\xi,U),V)-\eta(Z)\eta(\widetilde{R}(Y,U)V)\\
&-\eta(Y)\eta(Z)\eta(\widetilde{R}(\xi,U)V)+2g(Y,U)\eta(\widetilde{R}(Z,\xi)V)\\
&-\eta(U)\eta(\widetilde{R}(Z,Y)V)-\eta(Y)\eta(U)\eta(\widetilde{R}(Z,\xi)V)\\
&+2g(Y,V)\eta(\widetilde{R}(Z,U)\xi)-\eta(V)\eta(\widetilde{R}(Z,U)Y)\\
&-\eta(Y)\eta(V)\eta(\widetilde{R}(Z,U)\xi).
\end{aligned}\tag{5.4}
$$

Again by using equations (2.1), (2.3), (4.12), (4.13) and (4.14) in the above equation, we get

$$
\begin{aligned}
'\widetilde{R}(Z,U,V,Y) \;=\;& 2[g(Y,Z)g(U,V)-g(Y,U)g(Z,V)]\\
&+[\eta(Z)\eta(V)g(Y,U)-\eta(U)\eta(V)g(Y,Z)].
\end{aligned}\tag{5.5}
$$

Hence we can say

$$
\begin{aligned}
\widetilde{R}(Z,U)V \;=\;& 2[g(U,V)Z-g(Z,V)U]\\
&+[\eta(Z)\eta(V)U-\eta(U)\eta(V)Z].
\end{aligned}\tag{5.6}
$$

Contracting above equation with respect to Z, we get

$$\widetilde{S}(U,V)=4ng(U,V)-2n\eta(U)\eta(V).\tag{5.7}$$

In vies of above equation, we also have the following results

$$\widetilde{Q}(U)=4nU-2n\eta(U)\xi\tag{5.8}$$

and

$$\widetilde{r}=2n(4n+1).\tag{5.9}$$

Equations (4.8), (4.9), (4.10) and (5.7), (5.8), (5.9) respectively lead to the following equations

$$S(U,V)=4ng(U,V)-2n\eta(U)\eta(V),\tag{5.10}$$

$$Q(U)=4nU-2n\eta(U)\xi\tag{5.11}$$

and

$$r=2n(4n+1).\tag{5.12}$$

In view of definition 2.1 and equation (5.10), we can state the following theorem:

Theorem 5.1. *A semisymmetric Sasakian manifold admitting the non-symmetric non-metric connection $\widetilde{\nabla}$ is an $\eta-$Einstain manifold.*

The conformal curvature tensor of the non-metric non-symmetric connection $\widetilde{\nabla}$ is defined by equation (4.19). On substituting the value of $\widetilde{S}(X,Y)$, $\widetilde{Q}(X)$ and $\widetilde{r}$ from equations (5.7), (5.8) and (5.9) in the equation (4.19), we get

$$
\begin{aligned}
\widetilde{C}(X,Y)Z \;=\;& \frac{1}{(2n-1)}[g(X,Z)Y-g(Y,Z)X\\
&+\eta(Y)\eta(Z)X-\eta(X)\eta(Z)Y]\\
&+\frac{2n}{(2n-1)}\left[\eta(X)g(Y,Z)\xi-\eta(Y)g(X,Z)\xi\right].
\end{aligned}\tag{5.13}
$$

On taking $Z = \xi$ in the above equation, we get $\widetilde{C}(X, Y, \xi) = 0$. Hence we can state the following theorem:

Theorem 5.2. *A semisymmetric Sasakian manifold admitting the non-symmetric non-metric connection $\widetilde{\nabla}$ is an $\xi-$conformally flat with respect to the non-symmetric non-metric connection $\widetilde{\nabla}$.*

Again in view of equations (4.7) and (5.6), we get

$$\begin{aligned} R(X,Y)Z &= 2[g(Y,Z)g(U,V) - g(Y,U)g(Z,V)] \\ &\quad +[\eta(Z)\eta(V)g(Y,U) - \eta(U)\eta(V)g(Y,Z)] \\ &\quad -g(Y,Z)\eta(X)\xi + g(X,Z)\eta(Y)\xi \\ &\quad -g(\phi X,Z)(\phi Y) + g(\phi Y,Z)(\phi X). \end{aligned} \tag{5.14}$$

Now by using equations (5.10), (5.11), (5.12) and (5.14) in the equation (4.21), we get

$$\begin{aligned} C(X,Y)Z &= \frac{1}{(2n-1)}[g(X,Z)Y - g(Y,Z)X \\ &\quad +\eta(Y)\eta(Z)X - \eta(X)\eta(Z)Y \\ &\quad +\eta(X)g(Y,Z)\xi - \eta(Y)g(X,Z)\xi] \\ &\quad -g(\phi X,Z)(\phi Y) + g(\phi Y,Z)(\phi X). \end{aligned} \tag{5.15}$$

On taking $Z = \xi$ in the above above equation, we get $C(X, Y, \xi) = 0$. Hence we can also state the following corollary:

Corollary 5.3. *A semisymmetric Sasakian manifold admitting the non-symmetric non-metric connection $\widetilde{\nabla}$ is $\xi-$conformally flat as well.*

6 Ricci semisymmetric Sasakian manifolds admitting the non-symmetric non-metric connection $\widetilde{\nabla}$

A $(2n+1)-$ dimensional contact metric manifold M with the non-symmetric non-metric connection is said to be Ricci semisymmetric if $\widetilde{R}(X,Y).\widetilde{S}$ vanish identically. Hence in a $(2n+1)-$ dimensional Ricci semisymmetric contact metric manifold M with the non-symmetric non-metric connection, we have

$$\widetilde{S}(\widetilde{R}(X,Y)Z,U) + \widetilde{S}(Z,\widetilde{R}(X,Y)U) = 0. \tag{6.1}$$

Taking $X = \xi$ and using equation (4.12) in the above equation, we have

$$\begin{aligned} &2g(Y,Z)\widetilde{S}(\xi,U) - \eta(Z)\widetilde{S}(Y,U) - \eta(Y)\eta(Z)\widetilde{S}(\xi,U) \\ &+2g(Y,U)\widetilde{S}(Z,\xi) - \eta(U)\widetilde{S}(Z,Y) - \eta(Y)\eta(U)\widetilde{S}(\xi,Z) = 0. \end{aligned} \tag{6.2}$$

In view of equation (4.15), the above equation yields

$$\begin{aligned} &4n\eta(U)g(Y,Z) - \eta(Z)\widetilde{S}(Y,U) - 2n\eta(Y)\eta(Z)\eta(U) \\ &+4n\eta(Z)g(Y,U) - \eta(U)\widetilde{S}(Z,Y) - 2n\eta(Y)\eta(Z)\eta(U) = 0. \end{aligned} \tag{6.3}$$

Again change $U = \xi$ and using equation (4.15) in the above equation, we have

$$\widetilde{S}(Y,Z) = 4ng(Y,Z) - 2n\eta(Y)\eta(Z). \tag{6.4}$$

As the Ricci curvature tensor of a Sasakian manifold admitting the non-symmetric non-metric connection is invariant with respect to the non-symmetric non-metric connection $\widetilde{\nabla}$ and Livi-civita connection ∇, we can say

$$S(Y,Z) = 4ng(Y,Z) - 2n\eta(Y)\eta(Z). \tag{6.5}$$

Hence we conclude the following theorem:

Theorem 6.1. *A Ricci semisymmetric Sasakian manifold admitting the non-symmetric non-metric connection $\widetilde{\nabla}$ is an $\eta-$Einstain manifold.*

On the other hand if we use equation (4.7) in the following equation

$$(\widetilde{R}(X,Y).\widetilde{S})(Z,U) = -\widetilde{S}(\widetilde{R}(X,Y)Z,U) - \widetilde{S}(Z,\widetilde{R}(X,Y)U), \tag{6.6}$$

we get

$$\begin{aligned}(\widetilde{R}(X,Y).\widetilde{S})(Z,U) \quad = & \\ & +g(\phi Y,Z)S(\phi X,U) - g(\phi X,Z)S(\phi Y,U) \\ & +\eta(Y)g(X,Z)S(\xi,U) - \eta(X)g(Y,Z)S(\xi,U) \\ & +g(\phi Y,U)S(Z,\phi X) - g(\phi X,U)S(Z,\phi Y) \\ & +\eta(Y)g(X,U)S(Z,\xi) - \eta(X)g(Y,U)S(Z,\xi).\end{aligned} \tag{6.7}$$

The above equation can be re-write as

$$\begin{aligned}&(\widetilde{R}(X,Y).\widetilde{S})(Z,U) - (R(X,Y).S)(Z,U) \\ &= g(\phi Y,Z)S(\phi X,U) - g(\phi X,Z)S(\phi Y,U) \\ &+\eta(Y)g(X,Z)S(\xi,U) - \eta(X)g(Y,Z)S(\xi,U) \\ &+g(\phi Y,U)S(Z,\phi X) - g(\phi X,U)S(Z,\phi Y) \\ &+\eta(Y)g(X,U)S(Z,\xi) - \eta(X)g(Y,U)S(Z,\xi),\end{aligned} \tag{6.8}$$

where

$$(R(X,Y).S)(Z,U) = -S(R(X,Y)Z,U) - S(Z,R(X,Y)U). \tag{6.9}$$

If we assume

$$(R(X,Y).S)(Z,U) = (\widetilde{R}(X,Y).\widetilde{S})(Z,U), \tag{6.10}$$

equation (6.8) can be write as under

$$\begin{aligned}&g(\phi Y,Z)S(\phi X,U) - g(\phi X,Z)S(\phi Y,U) \\ &+\eta(Y)g(X,Z)S(\xi,U) - \eta(X)g(Y,Z)S(\xi,U) \\ &+g(\phi Y,U)S(Z,\phi X) - g(\phi X,U)S(Z,\phi Y) \\ &+\eta(Y)g(X,U)S(Z,\xi) - \eta(X)g(Y,U)S(Z,\xi) = 0.\end{aligned} \tag{6.11}$$

If we change $U = \xi$ in the equation (6.11) and using equations (2.1), (2.3) and (6.5), we get

$$2n[\eta(X)g(Y,Z) - \eta(Y)g(X,Z)] = 0. \tag{6.12}$$

Which is not possible. Hence we have the following corollary:

Corollary 6.2. *In a Ricci semisymmetric Sasakian manifold admitting the non-symmetric non-metric connection*

$$(R(X,Y).S)(Z,U) \neq (\widetilde{R}(X,Y).\widetilde{S})(Z,U). \tag{6.13}$$

Ricci soliton of data (g,V,λ) is defined by the equation (2.15), where g,V,λ are Riemannian metric, a vector field, and a real constant. Naturally two situations appear regarding the vector field V: $V \in Span(\xi)$ and $V \perp Span(\xi)$. Here we discuss first case that is $V \in Span(\xi)$. Ricci soliton of data (g,ξ,λ) on a Sasakian manifold admitting the non-symmetric non-metric connection $\widetilde{\nabla}$ can be defined as under:

$$(\widetilde{\pounds}_\xi g)(X,Y) + 2\widetilde{S}(X,Y) + 2\lambda g(X,Y) = 0, \tag{6.14}$$

for all $X, Y \in TM$. Here $\widetilde{\mathcal{L}}_\xi g$, the Lie-derivative of Riemannian metric g with respect to ξ admitting the non-symmetric non-metric connection $\widetilde{\nabla}$, is defined as

$$\begin{aligned}(\widetilde{\mathcal{L}}_\xi g)(X,Y) &= \xi g(X,Y) + g(\widetilde{\mathcal{L}}_\xi X, Y) + g(X, \widetilde{\mathcal{L}}_\xi Y) \\ &= \xi g(X,Y) + g(\widetilde{\nabla}_\xi X - \widetilde{\nabla}_X \xi, Y) + g(X, \widetilde{\nabla}_\xi Y - \widetilde{\nabla}_Y \xi) \\ &= \xi g(X,Y) + g(\widetilde{\nabla}_\xi X, Y) + g(X, \widetilde{\nabla}_\xi Y) \\ &\quad - g(\widetilde{\nabla}_X \xi, Y) - g(X, \widetilde{\nabla}_Y \xi). \end{aligned} \tag{6.15}$$

The above equation can be re-write as

$$(\widetilde{\mathcal{L}}_\xi g)(X,Y) = (\widetilde{\nabla}_\xi g)(X,Y) - g(\widetilde{\nabla}_X \xi, Y) - g(X, \widetilde{\nabla}_Y \xi). \tag{6.16}$$

Now in view of equations (2.3), (2.5), (3.3), (3.7) and (6.16), we have

$$(\widetilde{\mathcal{L}}_\xi g)(X,Y) = 0. \tag{6.17}$$

Hence on a Ricci semisymmetric and semisymmetric Sasakian manifold admitting the non-symmetric non-metric connection $\widetilde{\nabla}$, the equation (6.14) in view of equations (5.10), (6.5), (6.18), can we written as

$$(4n+\lambda)g(X,Y) - 2n\eta(X)\eta(Y) = 0. \tag{6.18}$$

On taking $X = Y = \xi$ in the above equation (6.18), we get $\lambda = -2n < 0$. Hence we can state the following theorem:

Theorem 6.3. *On a Ricci semisymmetric and semisymmetric Sasakian manifold admitting the non-symmetric non-metric connection $\widetilde{\nabla}$, the Ricci soliton of data (g, ξ, λ) is shrinking.*

7 Example of Sasakian Manifold

Example 7.1. Consider a three dimensional manifold $M^3 = \{(x,y,z) \in R^3 : z \neq 0\}$ with the standard coordinate system (x,y,z) of R^3. Let $e_1 = \left(y\frac{\partial}{\partial z} - \frac{\partial}{\partial x}\right), e_2 = -2\frac{\partial}{\partial y}, e_3 = \frac{\partial}{\partial z} = \xi$ are linear independent vector fields at each point of M^3 and form basis of tangent space at each point.

Let g be a Riemannian metric of M^3 defined by

$$\begin{aligned} g(e_1,e_2) = g(e_2,e_3) = g(e_3,e_1) = 0, \\ g(e_1,e_1) = g(e_2,e_2) = g(e_3,e_3) = 1, \end{aligned} \tag{7.1}$$

and ϕ is an $(1,1)-$tensor field defined by

$$\phi(e_1) = e_2, \phi(e_2) = -e_1, \phi(e_3) = 0. \tag{7.2}$$

Using linearity of ϕ and g, we have

$$\eta(e_3) = 1, \quad \phi^2 X = -X + \eta(X)e_3 \tag{7.3}$$

for any $X \in TM$. Here η is a 1-form on M^3 defined by $\eta(X) = g(X, e_3)$ for any $X \in TM$. Hence for $\xi = e_3$, the structure (ϕ, ξ, η, g) defines an almost contact metric structure on M^3. The Lie bracket for the example can be calculated by using the definition $[X,Y]f = X(Yf) - Y(Xf)$. All possible Lie brackets for the example are as follows:

$$\begin{array}{lll} [e_1,e_1] = 0, & [e_1,e_2] = 2e_3, & [e_1,e_3] = 0, \\ [e_2,e_1] = -2e_3, & [e_2,e_2] = 0, & [e_2,e_3] = 0, \\ [e_3,e_1] = 0, & [e_3,e_2] = 0, & [e_3,e_3] = 0\,. \end{array} \tag{7.4}$$

Let ∇ is a Levi-civita connection with respect to Riemannian metric g. Using the Koszul formula

$$\begin{aligned} 2g(\nabla_X Y, Z) &= Xg(Y,Z) + Yg(Z,X) - Zg(X,Y) \\ &\quad + g([X,Y],Z) - g([Y,Z],X) + g([Z,X],Y) \end{aligned} \tag{7.5}$$

and the Riemannian metric g, we can easily calculate

$$\begin{aligned} &\nabla_{e_1} e_1 = 0, && \nabla_{e_1} e_2 = e_3, && \nabla_{e_1} e_3 = -e_2, \\ &\nabla_{e_2} e_1 = -e_3, && \nabla_{e_2} e_2 = 0, && \nabla_{e_2} e_3 = e_1, \\ &\nabla_{e_3} e_1 = -e_2, && \nabla_{e_3} e_2 = e_1, && \nabla_{e_3} e_3 = 0\,. \end{aligned} \tag{7.6}$$

Now for $X = X^1e_1 + X^2e_2 + X^3e_3$ and $\xi = e_3$, we have

$$\begin{aligned} \nabla_X \xi &= \nabla_{X^1e_1+X^2e_2+X^3e_3} e_3 \\ &= X^1\nabla_{e_1} e_3 + X^2\nabla_{e_2} e_3 + X^3\nabla_{e_3} e_3 \\ &= X^2e_1 - X^1e_2, \end{aligned} \tag{7.7}$$

$$\begin{aligned} \phi X &= \phi(X^1e_1 + X^2e_2 + X^3e_3e_3) \\ &= X^1\phi e_1 + X^2\phi e_2 + X^3\phi e_3 \\ &= X^1e_2 - X^2e_1, \end{aligned} \tag{7.8}$$

where X^1, X^2, X^3 are scalars. In view of equations (7.7) and (7.8), we can say that the structure (ϕ, ξ, η, g) is Sasakian structure on M. *Consequently $M^3(\phi, \xi, \eta, g)$ is a Sasakian manifold.*

In reference of equations (2.1), (2.3), (3.1) and (7.6), we have the following

$$\begin{aligned} &\widetilde{\nabla}_{e_1} e_1 = 0, && \widetilde{\nabla}_{e_1} e_2 = 2e_3, && \widetilde{\nabla}_{e_1} e_3 = -e_2, \\ &\widetilde{\nabla}_{e_2} e_1 = -2e_3, && \widetilde{\nabla}_{e_2} e_2 = 0, && \widetilde{\nabla}_{e_2} e_3 = e_1, \\ &\widetilde{\nabla}_{e_3} e_1 = -e_2, && \widetilde{\nabla}_{e_3} e_2 = e_1, && \widetilde{\nabla}_{e_3} e_3 = 0\,. \end{aligned} \tag{7.9}$$

On changing $X = e_1$, $Y = e_2$ and $Z = e_3$ in the equations (3.2) and (3.3), we have

$$\widetilde{T}(e_1, e_2) = 2g(\phi e_1, e_2)e_3 = 2g(e_2, e_2)e_3 = 2e_3 \neq 0$$

and

$$\begin{aligned} (\widetilde{\nabla}_{e_1} g)(e_2, e_3) &= -\eta(e_3)g(\phi e_1, e_2) - \eta(e_2)g(\phi e_1, e_3) \\ &= -1g(e_2, e_2) = -1 \neq 0. \end{aligned}$$

Hence the connection defined in (3.1) is a non-symmetric non-metric connection. Again for $X = X^1e_1 + X^2e_2 + X^3e_3$ and $\xi = e_3$, we have

$$\begin{aligned} \widetilde{\nabla}_X \xi &= \widetilde{\nabla}_{X^1e_1+X^2e_2+X^3e_3} e_3 \\ &= X^1\widetilde{\nabla}_{e_1} e_3 + X^2\widetilde{\nabla}_{e_2} e_3 + X^3\widetilde{\nabla}_{e_3} e_3 \\ &= X^2e_1 - X^1e_2, \end{aligned} \tag{7.10}$$

In view of equations (7.7)and (7.10), we can say that the example verify proposition 3.1.

The Riemannian curvature tensor $R(e_i, e_j)X; i, j = 1, 2, 3,$ of connection ∇ can be calculated by using equations (3.10), (7.4) and (7.6). The all possible values for $X = X^1e_1 + X^2e_2 + X^3e_3$ are as follows:

$$R(e_1,e_2)X = 3(-X^2e_1 + X^1e_2), \quad R(e_1,e_3)X = X^3e_1 - X^1e_3,$$
$$R(e_2,e_1)X = 3(X^2e_1 - X^1e_2), \quad R(e_2,e_3)X = X^3e_2 - X^2e_3, \tag{7.11}$$
$$R(e_3,e_1)X = -X^3e_1 + X^1e_3, \quad R(e_3,e_2)X = -X^3e_2 + X^2e_3,$$

along with $R(e_i,e_i)X = 0;\ \forall\ i = 1,2,3.$ *From straight forward calculations, it can be easily proved the equations (2.7), (2.8) and (2.9) hold.*

In the same manner, we calculate the Riemannian curvature tensor $\widetilde{R}(e_i,e_j)X; i,j = 1,2,3$ of the non-symmetric non-metric connection $\widetilde{\nabla}$ by using equations (3.10), (7.4) and (7.9).

$$\widetilde{R}(e_1,e_2)X = 4(-X^2e_1 + X^1e_2), \quad \widetilde{R}(e_1,e_3)X = X^3e_1 - 2X^1e_3,$$
$$\widetilde{R}(e_2,e_1)X = 4(X^2e_1 - X^1e_2), \quad \widetilde{R}(e_2,e_3)X = X^3e_2 - 2X^2e_3, \tag{7.12}$$
$$\widetilde{R}(e_3,e_1)X = -X^3e_1 + 2X^1e_3, \quad \widetilde{R}(e_3,e_2)X = -X^3e_2 + 2X^2e_3,$$

along with $\widetilde{R}(e_i,e_i)X = 0;\ \forall\ i = 1,2,3.$

In consequence of equations (7.11) and (7.12), we can verify the equations (4.7), (4.11), (4.12), (4.13) and (4.14).

The Ricci tensors $S(e_j,X); j = 1,2,3,$ of connection ∇ for the given Sasakian manifold, can be calculated by using the results of equation (7.11) in the equation $S(e_j,X) = \sum_{i=1}^{3} g(R(e_i,e_j)X,e_i)$. It is as under:

$$S(e_1,X) = -2X^1, \quad S(e_2,X) = -2X^2, \quad S(e_3,X) = 2X^3. \tag{7.13}$$

The Ricci tensors $\widetilde{S}(e_j,X); j = 1,2,3,$ of the non-symmetric non-metric connection $\widetilde{\nabla}$ for the same given Sasakian manifold, can also be calculated by using the results of equation (7.12) in the equation $\widetilde{S}(e_j,X) = \sum_{i=1}^{3} g(\widetilde{R}(e_i,e_j)X,e_i)$. It is as follows:

$$\widetilde{S}(e_1,X) = -2X^1, \quad \widetilde{S}(e_2,X) = -2X^2, \quad \widetilde{S}(e_3,X) = 2X^3. \tag{7.14}$$

In view of equations (7.13) and (7.14), the scalar curvature tensor with respect to connection ∇ as well as with respect to the non-symmetric non-metric connection $\widetilde{\nabla}$ of the given Sasakian manifold can be calculated as under:

$$r = \sum_{i=1}^{3} S(e_i,e_i) = -2 - 2 + 2 = -2,$$
$$\widetilde{r} = \sum_{i=1}^{3} \widetilde{S}(e_i,e_i) = -2 - 2 + 2 = -2.$$

Hence we can say that the taken example of three dimensional Sasakian manifold verify Lemma 4.3 as well.

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Received: March 13, 2018.
Accepted: April 3, 2019.