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Approximation of functions with first and second derivatives $f^{'}$, $f^{''}$ belonging to Lipschitz class of order $0 < \alpha \leq 1$ by generalized Legendre wavelet Expansion

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Abstract In this paper , five new generalized Legendre wavelet estimators of functions having first and second derivative belonging to $Lip_\alpha[0,1]$ have been established.

1 Introduction

Orthogonal functions have attracted the researchers of the modern analysis due to its properties like optical control, applicability in a dynamical system. A function or signal of $L^2[0,1)$ can be expressed in the form of wavelet expansion. There are several Fourier series which is not convergent.Thus the difference of the function and n^{th} partial sum of its expansion can be calculated in some specific cases. Keeping this approach of Fourier analysis in mind, such differences can be calculated in case of wavelet expansion in wavelet analysis. The whole concept of approximation theory is based on Weierstrass approximation theorem. According to this theorem, if a function f is continuous in [a,b] then there is a sequence $(B_n(x))_{n=0}^{\infty}$of Bernstein polynomials which converges uniformly to f on [a,b]. Legendre wavelet approximates a function of Lip_α class of order $0 < \alpha \leq 1$ by piecewise Legendre polynomials. Thus the Legendre wavelets defined on the interval [0,1), are generally obtained by a translation operator on Legendre wavelet, defined on the subintervals$[0, \frac{1}{2^n})$ of [0,1). In best of our knowledge, till date nothing seems to have been done so for the degree of approximation of the function $f \in L^2[0,1)$ whose first and second derivatives$f^{'}$ and$f^{''}$ belonging to $Lip_\alpha[0,1)$ of order $0 < \alpha \leq 1$ by Generalized Legendre wavelet methods. The purpose of this research paper is to make an advanced study in this direction, five new wavelet estimators of $f^{'}$ and $f^{''}$ have been estimated by $\Psi^{(\mu)}_{n,m}$ i.e. Generalized Legendre methods. Our estimators are better, sharper and new in wavelet analysis and its applications.

2 Definitions and Preliminaries

Definition 2.1. Generalized or Extended Legendre wavelet:
Let N denotes the set all natural numbers, $N_0 = N \bigcup \{0\}$ and $Z_\mu = \{0, 1, 2, 3, ..., \mu - 1\}$ for a positive integer μ. For a positive integer $\mu > 1$, define the contracting mapping on the interval I=[0,1] by

$$\psi_\xi(x) = \frac{x + \xi}{\mu}, x \in [0,1], \xi \in Z_\mu.$$

Then

$$\psi_\xi(I) \subset I, \forall \xi \in Z_\mu, \bigcup_{\xi \in Z_\mu} \psi_\xi(I) = I.$$

The Legendre polynomials $P_m(x)$ are defined by

$$P_m(x) = \frac{1}{2^m m!} \frac{d^m}{dx^m}[(x^2 - 1)^m], m \geq 0.$$

$\{P_m\}_{m=0}^{\infty}$ are orthogonal and satisfy :

$$P_0(t) = 1, P_1(t) = t$$

$$(m+1)P_{m+1}(t) = (2m+1)tP_m(t) - mP_{m-1}(t), m \in N.$$

Let

$$G_0 = span\left\{P_m(2t-1)|t \in [0,1], m \in Z_\mu\right\}.$$

Let $\xi \in Z_\mu$, define T_ξ on $L^2[0,1]$ by

$$T_\xi f(t) = \begin{cases} \sqrt{\mu} f(\psi_\xi^{-1}(t)) & t \in \psi_\xi(I) \\ 0 & \text{, otherwise} \end{cases}$$

$$\begin{aligned} ||T_\xi(f)||_2^2 &= \int_0^1 (T_\xi f(x))^2 dx \\ &= \int_0^1 (\sqrt{\mu} f(\psi_\xi^{-1}(x)))^2 dx \\ &= \mu \int_0^1 f^2(\mu x - \xi) dx \\ &= \int_{-\xi}^{\mu-\xi} f^2(v) dv, \quad \mu x - \xi = v \\ &= ||f||_2^2 \\ \therefore ||T_\xi(f)||_2^2 &= ||f||_2^2. \end{aligned}$$

Therefore, $T_\xi f$ is an isometry.
Suppose

$$G_{k+1} = \bigoplus_{\xi \in Z_\mu} T_\xi G_k, k \in N_0.$$

Then,

$$\begin{aligned} (i) G_k &\subset G_{k+1}, \forall\ k \in N_0. \\ (ii) dim G_k &= M\mu^k. \\ (iii) \overline{\bigcup_{k=0}^{\infty} G_k} &= L^2[0,1]. \end{aligned}$$

Let

$$F_0 = \left\{\sqrt{2m+1} P_m(2t-1)|t \in [0,1], m \in Z_M\right\}$$

. Then (i) F_0 is an orthonormal basis for G_0.
$(ii) supp\left\{T_\xi f\right\} \bigcap supp\left\{T_{\xi'} f\right\} = \phi, \xi \neq \xi' \forall f \in L^2[0,1]$,
where $supp(f)$ denotes the support of the function f.
For $m \neq m'$
$< \sqrt{2m+1} P_m(2x-1), \sqrt{2m'+1} P_{m'}(2x-1) >$

$$\begin{aligned} &= \sqrt{2m+1}\sqrt{2m'+1} \int_0^1 P_m(2x-1) dx P_{m'}(2x-1) dx \\ &= \sqrt{2m+1}\sqrt{2m'+1} \int_{-1}^1 P_m(y) dx P_{m'}(y) dy, 2x-1 = y \\ &= 0, \text{ by property of Legendre polynomial.} \end{aligned}$$

$$< \sqrt{2m+1}P_m(2x-1), \sqrt{2m+1}P_m(2x-1) >$$

$$\begin{aligned} &= (2m+1)\int_0^1 P_m^2(2x-1)dx \\ &= \frac{2m+1}{2}\int_{-1}^1 P_m^2(y)dy, 2x-1=y \\ &= 1. \end{aligned}$$

$$F_k = \left\{T_{\xi 0} o T_{\xi 1} o T_{\xi 2} o ... o T_{\xi k-1}(\sqrt{2m+1}P_m(2t-1))|m \in Z_M\right\}$$

$$\begin{aligned} F_1 &= \{T_{\xi 0}(\sqrt{2m+1}P_m(2t-1)|m \in Z_M)\} \\ &= \{\sqrt{\mu}\sqrt{2m+1}P_m(2\psi_{\xi}^{-1}(x)-1)|m \in Z_M\} \\ &= \{\sqrt{2m+1}\mu^{\frac{1}{2}}P_m(2\mu x-(2\xi+1))|m \in Z_M\} \end{aligned}$$

Let$F_{1,m} \in F_1$

$$\begin{aligned} ||F_{1,m}||_2^2 &= \int_0^1 (2m+1)\mu P_m^2(2\mu x-(2\xi+1))dx \\ &= \int_{\frac{\xi}{\mu}}^{\frac{\xi+1}{\mu}} \mu(2m+1)P_m^2(2\mu x-(2\xi+1)) \\ &= \mu(2m+1)\int_{-1}^1 P_m^2(u)\frac{du}{2\mu}, \ \ 2\mu x-(2\xi+1)=u \\ &= 1. \\ < F_{1,m}, F_{1,m'} > &= 0, m \neq m' \end{aligned}$$

Similarly ,

$$< F_{k,m}, F_{k,m'} > = \begin{cases} 1, m = m' \\ 0, m \neq m' \end{cases}$$

F_k is an orthonormal basis for the vector space G_k, where "o" denotes composition of functions.[6] Similarly, for $n = 1, 2, 3, ..., \mu^k, k \in N$,

$$\Psi^{(\mu)}_{n,m}(t) = \Psi^{(\mu)}(k,m,n,t) = \begin{cases} \sqrt{2m+1}\mu^{\frac{k}{2}}P_m(2\mu^k t - 2n + 1) & t \in [\frac{n-1}{\mu^k}, \frac{n}{\mu^k}) \\ 0 & \text{, otherwise.} \end{cases}$$

is called extended Legendre wavelet and $\left\{\Psi^{(\mu)}_{n,m}\right\}$ is an orthonormal basis of G_k.
For $\mu = 2$, $\left\{\Psi^{(\mu)}_{n,m}\right\}$ reduces to known Legendre wavelet

$$\Psi^{(L)}_{n,m}(t) = \begin{cases} \sqrt{m+\frac{1}{2}}\ 2^{\frac{k}{2}}P_m(2^k t - \hat{n}), & \frac{\hat{n}-1}{2^k} \leq t < \frac{\hat{n}+1}{2^k} \\ 0 & \text{, otherwise.} \end{cases}$$

Thus $\left\{\Psi^{(\mu)}_{n,m}\right\}$ is a generalization of $\left\{\Psi^{(L)}_{n,m}\right\}$.

Definition 2.2. Extended Legendre approximation:
A function $f \in L^2[0,1]$ is expressed in the form of extended Legendre wavelet as

$$f(t) = \sum_{n=1}^{\infty}\sum_{m=0}^{\infty} c_{n,m}\Psi^{(\mu)}_{n,m}(t) = \sum_{n=1}^{\infty}\sum_{m=0}^{\infty} < f, \Psi^{(\mu)}_{n,m} > \Psi^{(\mu)}_{n,m}(t) \tag{2.1}$$

If the infinite series(2.1) is truncated, then it can be expressed as:

$$f(t) = \sum_{n=1}^{\mu^k} \sum_{m=0}^{M-1} c_{n,m} \Psi_{n,m}^{(\mu)}(t) = C^T \Psi^{(\mu)}(t) \tag{2.2}$$

where C and $\Psi^{(\mu)}(t)$ are $\mu^k M$ column vectors given by

$$C = [c_{1,0}, ..., c_{1,(M-1)}, c_{2,0}, ..., c_{2,(M-1)}, ..., c_{\mu^k,0}, ..., c_{\mu^k,(M-1)}]^T,$$

and $\Psi^{(\mu)}(t) = [\Psi_{1,0}^{(\mu)}(t), ..., \Psi_{1,(M-1)}^{(\mu)}, \Psi_{2,0}^{(\mu)}(t), ..., \Psi_{2,(M-1)}^{(\mu)}(t), ..., \Psi_{\mu^k,0}^{(\mu)}(t), ..., \Psi_{\mu^k,(M-1)}^{(\mu)}(t)]^T$ respectively. We define

$$||f||_2 = \left(\int_0^1 |f(x)|^2 dx \right)^{1/2}$$

and $E_{\mu^k,M}(f)$ of f by $(S_{\mu^k,M} f)(x) = \sum_{n=1}^{\mu^k} \sum_{m=0}^{M-1} c_{n,m} \Psi_{n,m}^{(\mu)}(x)$
is defined by

$E_{\mu^k,M}(f) = \min_{S_{\mu^k,M}(f)} \|f - S_{\mu^k,M}(f)\|_2$, where $E_{\mu^k,M}(f)$ is the extended Legendre approximation.

If $E_{\mu^k,M}(f) \to 0$ as $n \to \infty$ then $E_{\mu^k,M}(f)$ is the best approximation of f.(Zygmund[1],pp.115).

Definition 2.3. Lipschitz class :
A function $f \in Lip_\alpha[0,1]$ if

$$|f(v+u) - f(v)| = O(|u|^\alpha); 0 < \alpha \leq 1, v, u, u+v \in [0,1][2].$$

i.e.

$$|f(v+u) - f(v)| = M_f |u|^\alpha, M_f \geq 0, v, u, u+v \in [0,1]$$

Example:

A function $f : [0,1] \to R$ is given by

$$f(x) = sinx$$

Consider,

$$|f(x+t) - f(x)| = |sin(x+t) - sinx| = \left| 2cos\left(\frac{2x+t}{2}\right) sin\left(\frac{t}{2}\right) \right| \leq |t| \forall x, t, x+t \in [0,1]$$
$$= O(|t|)$$

Then

$$f \in Lip_1[0,1]$$

3 Theorems

We prove the following theorems:

Theorem 3.1. Let $f \in L^2[0,1)$ such that $f' \in Lip_\alpha[0,1]$ and its extended Legendre wavelet expansion be

$$f(x) = \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} c_{n,m} \Psi_{n,m}^{(\mu)}(x).$$

Then extended Legendre wavelet approximations satisfy:

(I)For $f(x) = \sum_{n=1}^{\infty} c_{n,0} \Psi_{n,0}^{(\mu)}(x)$, $E_{\mu^k,0}^{(1)}(f) = \min_{S_{\mu^k,0}(f)} \|f - \sum_{n=1}^{\mu^k} c_{n,0} \Psi_{n,0}^{(\mu)}\|_2 = O\left(\frac{1}{\mu^k}\left(1 + \frac{1}{\mu^{k\alpha}}\right)\right)$.

(II)For $f(x) = \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} c_{n,m} \Psi_{n,m}^{(\mu)}(x)$,

$E^{(1)}_{\mu^k,M}(f) = \min_{S_{\mu^k,M}(f)} \|f - \sum_{n=1}^{\mu^k} \sum_{m=0}^{M-1} c_{n,m}\Psi^{(\mu)}_{n,m}\|_2$
$= O\left(\frac{1}{\mu^k(2M-1)^{\frac{1}{2}}}\left(1+\frac{1}{\mu^{k\alpha}}\right)\right), M \geq 2.$

Theorem 3.2. Let $f \in L^2[0,1)$ such that $f^{''} \in Lip_{\alpha}[0,1]$ and its expansion be

$$f(x) = \sum_{n=1}^{\infty}\sum_{m=0}^{\infty} c_{n,m}\Psi^{(\mu)}_{n,m}(x).$$

Then extended Legendre wavelet approximations satisfy:
(I)For $f(x) = \sum_{n=1}^{\infty} c_{n,0}\Psi^{(\mu)}_{n,0}(x), E^{(2)}_{\mu^k,0}(f) = \min_{S_{\mu^k,0}(f)} \|f - \sum_{n=1}^{\mu^k} c_{n,0}\Psi^{(\mu)}_{n,0}\|_2 = O\left(\frac{1}{\mu^k}\left(1+\frac{1}{\mu^{k(\alpha+1)}}\right)\right).$
(II)For $f(x) = \sum_{n=1}^{\infty}\sum_{m=0}^{1} c_{n,m}\Psi^{(\mu)}_{n,m}(x),$
$E^{(2)}_{\mu^k,1}(f) = \min_{S_{\mu^k,1}(f)} \|f - \sum_{n=1}^{\mu^k}\sum_{m=0}^{1} c_{n,m}\Psi^{(\mu)}_{n,m}\|_2 = O\left(\frac{1}{\mu^{2k}}\left(1+\frac{1}{\mu^{k\alpha}}\right)\right).$
(III)For $f(x) = \sum_{n=1}^{\infty}\sum_{m=0}^{\infty} c_{n,m}\Psi^{(\mu)}_{n,m}(x),$
$E^{(2)}_{\mu^k,M}(f) = \min_{S_{\mu^k,M}(f)} \|f - \sum_{n=1}^{\mu^k} \sum_{m=0}^{M-1} c_{n,m}\Psi^{(\mu)}_{n,m}\|_2$
$= O\left(\frac{1}{(2M-3)^{\frac{3}{2}}}\frac{1}{\mu^{2k}}\left(1+\frac{1}{\mu^{k\alpha}}\right)\right), M \geq 3.$

Proof 3.1. (I)
For

$$f(x) = \sum_{n=1}^{\infty} c_{n,0}\Psi^{(\mu)}_{n,0}(x)$$

The error $e^{(1)}_n(x)$ between $f(x)$ having $f^{'} \in Lip_{\alpha}[0,1]$ and its expression over any subinterval is defined as

$$\begin{aligned} e^{(1)}_n(x) &= c_{n,0}\Psi^{(\mu)}_{n,0}(x) - f(x), x \in \left[\frac{n-1}{\mu^k}, \frac{n}{\mu^k}\right) \\ ||e^{(1)}_n||_2^2 &= \int_0^1 |e^{(1)}_n(x)|^2 dx \\ &= \int_{\frac{n-1}{\mu^k}}^{\frac{n}{\mu^k}} (f(x))^2 dx - c^2_{n,0} \qquad (3.1) \end{aligned}$$

Consider,

$$\int_{\frac{n-1}{\mu^k}}^{\frac{n}{\mu^k}} (f(x))^2 dx = \int_0^{\frac{1}{\mu^k}} \left[f\left(\frac{n-1}{\mu^k} + h\right)\right]^2 dh, x = \frac{n-1}{\mu^k} + h$$

$$= \int_0^{\frac{1}{\mu^k}} \left[f\left(\frac{n-1}{\mu^k}\right) + hf^{'}\left(\frac{n-1}{\mu^k} + \theta h\right)\right]^2 dh, 0 < \theta < 1$$

$$= \frac{1}{\mu^k}\left(f\left(\frac{n-1}{\mu^k}\right)\right)^2 + \int_0^{\frac{1}{\mu^k}} h^2 \left(f^{'}\left(\frac{n-1}{\mu^k} + \theta h\right)\right)^2 dh$$

$$+ \; 2f\left(\frac{n-1}{\mu^k}\right)\int_0^{\frac{1}{\mu^k}} hf^{'}\left(\frac{n-1}{\mu^k} + \theta h\right) dh.$$

Next,

$$c_{n,0} = < f, \Psi_{n,0}^{(\mu)} >$$

$$= \int_{\frac{n-1}{\mu^k}}^{\frac{n}{\mu^k}} f(x)\mu^{\frac{k}{2}}\, dx$$

$$= \mu^{\frac{k}{2}} \int_0^{\frac{1}{\mu^k}} f\left(\frac{n-1}{\mu^k} + h\right) dh, x = \frac{n-1}{\mu^k} + h$$

$$c_{n,0} = \mu^{\frac{k}{2}} \int_0^{\frac{1}{\mu^k}} \left[f\left(\frac{n-1}{\mu^k}\right) + hf^{'}\left(\frac{n-1}{\mu^k} + \theta h\right)\right] dh, 0 < \theta < 1$$

$$= \mu^{\frac{k}{2}} \left[\frac{1}{\mu^k} f\left(\frac{n-1}{\mu^k}\right) + \int_0^{\frac{1}{\mu^k}} hf^{'}\left(\frac{n-1}{\mu^k} + \theta h\right) dh\right]$$

Now,

$$c_{n,0}^2 = \frac{1}{\mu^k}\left(f\left(\frac{n-1}{\mu^k}\right)\right)^2 + \mu^k \left(\int_0^{\frac{1}{\mu^k}} h\left(f^{'}\left(\frac{n-1}{\mu^k}\right) + \theta h\right) dh\right)^2$$

$$+ \; 2f\left(\frac{n-1}{\mu^k}\right)\int_0^{\frac{1}{\mu^k}} hf^{'}\left(\frac{n-1}{\mu^k} + \theta h\right) dh.$$

Then,

$$||e_n^{(1)}||_2^2 = \int_0^{\frac{1}{\mu^k}} h^2 \left(f^{'}\left(\frac{n-1}{\mu^k} + \theta h\right)\right)^2 dh - \mu^k \left(\int_0^{\frac{2}{\mu^k}} hf^{'}\left(\frac{n-1}{\mu^k} + \theta h\right) dh\right)^2$$

$$= I_1 - I_2, say. \tag{3.2}$$

Consider,

$$I_1 = \int_0^{\frac{1}{\mu^k}} h^2 \left(f^{'}\left(\frac{n-1}{\mu^k} + \theta h\right)\right)^2 dh$$

$$
\begin{aligned}
I_1 &= \int_0^{\frac{1}{\mu^k}} h^2 \left[f^{'}\left(\frac{n-1}{\mu^k}+\theta h\right) - f^{'}\left(\frac{n-1}{\mu^k}\right) + f^{'}\left(\frac{n-1}{\mu^k}\right)\right]^2 dh \\
&= \int_0^{\frac{1}{\mu^k}} h^2 \left[f^{'}\left(\frac{n-1}{\mu^k}+\theta h\right) - f^{'}\left(\frac{n-1}{\mu^k}\right)\right]^2 dh + \int_0^{\frac{1}{\mu^k}} h^2 \left[f^{'}\left(\frac{n-1}{\mu^k}\right)\right]^2 \\
&+ 2\int_0^{\frac{1}{\mu^k}} h^2 \left[f^{'}\left(\frac{n-1}{\mu^k}+\theta h\right) - f^{'}\left(\frac{n-1}{\mu^k}\right)\right]\left[f^{'}\left(\frac{n-1}{\mu^k}\right)\right] dh \\
|I_1| &\leq \int_0^{\frac{1}{\mu^k}} M_1^2 h^{2\alpha+2} dh + M_2^2 \int_0^{\frac{1}{\mu^k}} h^2 dh + 2M_1M_2 \int_0^{\frac{1}{\mu^k}} h^{\alpha+2} dh, f \in Lip_{\alpha}[0,1] \\
&= \frac{M_1^2}{(2\alpha+3)\mu^{k(2\alpha+3)}} + \frac{M_2^2}{3.\mu^{3k}} + \frac{2M_1M_2}{(\alpha+3)\mu^{k(\alpha+3)}} \qquad (3.3)
\end{aligned}
$$

Also,

$$
\begin{aligned}
I_2 &= \mu^k \left(\int_0^{\frac{1}{\mu^k}} h f^{'}\left(\frac{n-1}{\mu^k}+\theta h\right) dh \right)^2 \\
&= \mu^k \left(\int_0^{\frac{1}{\mu^k}} h \left[f^{'}\left(\frac{n-1}{\mu^k}+\theta h\right) - f^{'}\left(\frac{n-1}{\mu^k}\right) + f^{'}\left(\frac{n-1}{\mu^k} dh\right)\right] dh \right)^2 \\
|I_2| &\leq \mu^k \left(\int_0^{\frac{1}{\mu^k}} h \left| f^{'}\left(\frac{n-1}{\mu^k}+\theta h\right) - f^{'}\left(\frac{n-1}{\mu^k}\right)\right| dh + \int_0^{\frac{1}{\mu^k}} h \left| f^{'}\left(\frac{n-1}{\mu^k}\right)\right| dh \right)^2 \\
&\leq \mu^k \left(\int_0^{\frac{1}{\mu^k}} h M_1^{'} h^{\alpha} dh + M_2^{'} \int_0^{\frac{2}{\mu^k}} h dh \right)^2 \\
&= \mu^k \left(\frac{M_1^{'}}{(\alpha+2)\mu^{k(\alpha+2)}} + \frac{M_2^{'}}{2\mu^{2k}} \right)^2 \\
&= \frac{M_1^{'2}}{(\alpha+2)^2\mu^{k(2\alpha+3)}} + \frac{M_2^{'2}}{4\mu^{3k}} + \frac{M_1^{'}M_2^{'}}{(\alpha+2)\mu^{k(\alpha+3)}} \qquad (3.4)
\end{aligned}
$$

By eqns (3.2), (3.3) and (3.4) , we have

$$
\begin{aligned}
||e_n^{(1)}||_2^2 &\leq \frac{M_1^2}{(2\alpha+3)\mu^{k(2\alpha+3)}} + \frac{M_2^2}{3\mu^{3k}} + \frac{2M_1M_2}{(\alpha+3)\mu^{k(\alpha+3)}} \\
&+ \frac{M_1^{'2}}{(\alpha+2)^2\mu^{k(2\alpha+3)}} + \frac{M_2^{'2}}{4\mu^{3k}} + \frac{M_1^{'}M_2^{'}}{(\alpha+2)\mu^{k(\alpha+3)}} \\
&\leq \frac{N_1^2}{\mu^{k(2\alpha+3)}}\left(\frac{1}{(2\alpha+3)} + \frac{1}{(\alpha+2)^2}\right) + \frac{7N_2^2}{12\mu^{3k}} + \frac{2N_1N_2}{\mu^{k(\alpha+3)}}\left(\frac{1}{(\alpha+2)} + \frac{1}{2(\alpha+3)}\right) \\
&\leq \frac{1}{\mu^k}\left(\frac{N_1}{\mu^{k(\alpha+1)}} + \frac{N_2}{\mu^k}\right)^2, N_1 = max[M_1, M_1^{'}], N_2 = max[M_2, M_2^{'}]
\end{aligned}
$$

$$\text{Now,}\quad (E^{(1)}_{\mu^k,0}(f))^2 = \int_0^1 \left(\sum_{n=1}^{\mu^k} e^{(1)}_n(x)\right)^2 dx$$

$$= \int_0^1 \sum_{n=1}^{\mu^k} (e^{(1)}_n(x))^2 dx + 2\sum_{n=1}^{\mu^k}\sum_{n\neq n'}^{\mu^k}\int_0^1 e^{(1)}_n(x)e^{(1)}_{n'}(x)dx$$

$$= \sum_{n=1}^{\mu^k}\int_0^1 (e_n(x))^2 dx, \text{ due to disjoint supports of } e_n \text{ and } e'_n$$

$$= \sum_{n=1}^{\mu^k} ||e^{(1)}_n||_2^2$$

$$\leq \sum_{n=1}^{\mu^k} \frac{1}{\mu^k}\left(\frac{N_1}{\mu^{k(\alpha+1)}} + \frac{N_2}{\mu^k}\right)^2$$

$$= \left(\frac{N_1}{\mu^{k(\alpha+1)}} + \frac{N_2}{\mu^k}\right)^2.$$

$$\leq N^2\left(\frac{1}{\mu^k} + \frac{1}{\mu^{k(\alpha+1)}}\right)^2, \text{where} N = max[N_1, N_2]$$

Therefore,

$$E^{(1)}_{\mu^k,0}(f) \leq N\left(\frac{1}{\mu^k} + \frac{1}{\mu^{k(\alpha+1)}}\right)$$

$$= O\left(\frac{1}{\mu^k}\left(1 + \frac{1}{\mu^{k\alpha}}\right)\right).$$

(II)

For

$$f(x) = \sum_{n=1}^{\infty}\sum_{m=0}^{\infty} c_{n,m}\Psi^{(\mu)}_{n,m}(x).$$

$$c_{n,m} = \int_{\frac{n-1}{\mu^k}}^{\frac{n}{\mu^k}} f(x)\sqrt{2m+1}\mu^{\frac{k}{2}}P_m(2\mu^k x - \hat{n})dx$$

$$= \frac{\sqrt{2m+1}}{2\mu^{\frac{k}{2}}}\int_{-1}^{1} f\left(\frac{\hat{n}+t}{2\mu^k}\right)P_m(t)dt, 2\mu^k x - \hat{n} = t$$

$$= \frac{\sqrt{2m+1}}{2\mu^{\frac{k}{2}}}\int_{-1}^{1} f\left(\frac{\hat{n}+t}{2\mu^k}\right)d\left(\frac{P_{m+1}(t) - P_{m-1}(t)}{2m+1}\right)$$

$$= \frac{1}{2\mu^{\frac{k}{2}}(2m+1)^{\frac{1}{2}}}\left[f\left(\frac{\hat{n}+t}{2\mu^k}\right)(P_{m+1}(t) - P_{m-1}(t))\right]_{-1}^{1}$$

$$-\frac{1}{4\mu^{\frac{3k}{2}}(2m+1)^{\frac{1}{2}}}\int_{-1}^{1} f'\left(\frac{\hat{n}+t}{2\mu^k}\right)(P_{m+1}(t) - P_{m-1}(t))dt, \text{integrating by parts.}$$

$$= \frac{-1}{4\mu^{\frac{3k}{2}}(2m+1)^{\frac{1}{2}}}\int_{-1}^{1}\left\{f'\left(\frac{\hat{n}+t}{2\mu^k}\right) - f'\left(\frac{\hat{n}}{2\mu^k}\right) + f'\left(\frac{\hat{n}}{2\mu^k}\right)\right\}(P_{m+1}(t) - P_{m-1}(t)$$

$$\begin{aligned}
c_{n,m} &= \frac{-1}{4\mu^{\frac{3k}{2}}(2m+1)^{\frac{1}{2}}}\int_{-1}^{1}\left\{f^{'}\left(\frac{\hat{n}+t}{2\mu^k}\right)-f^{'}\left(\frac{\hat{n}}{2\mu^k}\right)\right\}(P_{m+1}(t)-P_{m-1}(t))dt \\
&\quad -\frac{1}{4\mu^{\frac{3k}{2}}(2m+1)^{\frac{1}{2}}}\int_{-1}^{1} f^{'}\left(\frac{\hat{n}}{2\mu^k}\right)(P_{m+1}(t)-P_{m-1}(t))dt \\
|c_{n,m}| &\leq \frac{1}{4\mu^{\frac{3k}{2}}(2m+1)^{\frac{1}{2}}}\int_{-1}^{1}\left|f^{'}\left(\frac{\hat{n}+t}{2\mu^k}\right)-f^{'}\left(\frac{\hat{n}}{2\mu^k}\right)\right||P_{m+1}(t)-P_{m-1}(t)|dt \\
&\quad +\frac{1}{4\mu^{\frac{3k}{2}}(2m+1)^{\frac{1}{2}}}\int_{-1}^{1}\left|f^{'}\left(\frac{\hat{n}}{2\mu^k}\right)\right||P_{m+1}(t)-P_{m-1}(t)|dt \\
&\leq \frac{1}{4\mu^{\frac{3k}{2}}(2m+1)^{\frac{1}{2}}}\int_{-1}^{1}\left\{\left(\frac{Q_1}{\mu^{k\alpha}}\right)+Q_2\right\}|P_{m+1}(t)-P_{m-1}(t)|dt \\
&= \frac{1}{4\mu^{\frac{3k}{2}}(2m+1)^{\frac{1}{2}}}\left\{\left(\frac{Q_1}{\mu^{k\alpha}}\right)+Q_2\right\}\int_{-1}^{1}|P_{m+1}(t)-P_{m-1}(t)|dt
\end{aligned}$$

Since,

$$\begin{aligned}
&\int_{-1}^{1}|P_{m+1}(t)-P_{m-1}(t)|dt \leq \left(\int_{-1}^{1}1^2dt\right)^{\frac{1}{2}}\left(\int_{-1}^{1}|P_{m+1}(t)-P_{m-1}(t)|^2dt\right)^{\frac{1}{2}} \\
&= \sqrt{2}\left(\int_{-1}^{1}(P^2_{m+1}(t)-P^2_{m-1}(t))dt\right)^{\frac{1}{2}} \\
&= \sqrt{2}\left(\frac{2}{2m-1}+\frac{2}{2m+3}\right)^{\frac{1}{2}} \\
&\leq \frac{2\sqrt{2}}{\sqrt{2m-1}},
\end{aligned}$$

therefore

$$\begin{aligned}
|c_{n,m}| &\leq \frac{1}{4\mu^{\frac{3k}{2}}(2m+1)^{\frac{1}{2}}}\left\{\left(\frac{Q_1}{\mu^{k\alpha}}\right)+Q_2\right\}\frac{2\sqrt{2}}{\sqrt{2m-1}} \\
&\leq \frac{1}{\sqrt{2}\mu^{\frac{3k}{2}}(2m-1)}\left(\frac{Q_1}{\mu^{k\alpha}}+Q_2\right). \qquad (3.5)
\end{aligned}$$

$$\begin{aligned}
f(x)-S_{\mu^k,M}(f) &= \left(\sum_{n=1}^{\mu^k}\left(\sum_{m=0}^{M-1}+\sum_{m=M}^{\infty}\right)c_{n,m}\Psi^{(\mu)}_{n,m}(x)-\sum_{n=1}^{\mu^k}\sum_{m=0}^{M-1}c_{n,m}\Psi^{(\mu)}_{n,m}(x)\right) \\
&= \sum_{n=1}^{\mu^k}\sum_{m=M}^{\infty}c_{n,m}\Psi^{(\mu)}_{n,m}(x) \\
(E^1_{\mu^k,M}(f))^2 &= \min_{S_{\mu^k,M}(f)}\|f-S_{\mu^k,M}(f)\|_2^2 \\
&= \int_0^1|f(x)-S_{\mu^k,M}(f)|^2dx \\
&= \int_0^1\left(\sum_{n=1}^{\mu^k}\sum_{m=M}^{\infty}c_{n,m}\Psi^{(\mu)}_{n,m}(x)\right)^2dx
\end{aligned}$$

$$
\begin{aligned}
(E^1_{\mu^k,M}(f))^2 &= \int_0^1 \left(\sum_{n=1}^{\mu^k} \sum_{m=M}^{\infty} c^2_{n,m} \Psi^{(\mu)2}_{n,m}(x) \right.\\
&+ \left. 2 \sum\sum\sum\sum_{1\le n\ne n'\le \mu^k, M\le m\ne m'<\infty} c_{n,m} c_{n'm'} \Psi^{(\mu)}_{n,m}(x) \Psi^{(\mu)}_{n'm'}(x) \right) dx\\
&= \sum_{n=1}^{\mu^k} \sum_{m=M}^{\infty} c^2_{n,m} \int_0^1 \Psi^{(\mu)2}_{n,m}(x) dx\\
&+ 2 \sum\sum\sum\sum_{1\le n\ne n'\le \mu^k, M\le m\ne m'<\infty} c_{n,m} c_{n'm'} \int_0^1 \Psi^{(\mu)}_{n,m}(x) \Psi^{(\mu)}_{n'm'}(x) dx\\
&= \sum_{n=1}^{\mu^k} \sum_{m=M}^{\infty} c^2_{n,m} \|\Psi^{(\mu)}_{n,m}\|_2^2, \text{other term vanish due to orthonormality of } \psi^{(\mu)}_{n,m}.\\
&= \sum_{n=1}^{\mu^k} \sum_{m=M}^{\infty} |c_{n,m}|^2
\end{aligned}
$$

Then,

$$
\begin{aligned}
(E^{(1)}_{\mu^k,M}(f))^2 &= \min_{S_{\mu^k,M}(f)} \|f - S_{\mu^k,M}(f)\|_2^2 = \sum_{n=1}^{\mu^k} \sum_{m=M}^{\infty} |c_{n,m}|^2\\
&\le \sum_{n=1}^{\mu^k} \sum_{m=M}^{\infty} \left(\frac{1}{\sqrt{2}\mu^{\frac{3k}{2}} (2m-1)^{\frac{1}{2}}} \left(\frac{Q_1}{\mu^{k\alpha}} + Q_2 \right) \right)^2, \text{by eqn(3.5)}\\
&= \sum_{n=1}^{\frac{\mu^k}{2}} \left(\frac{Q_1}{\mu^{k\alpha}} + Q_2 \right)^2 \frac{1}{2\mu^{3k}} \sum_{m=M}^{\infty} \frac{1}{(2m-1)}\\
&= \frac{\mu^k}{2} \left(\frac{Q_1}{\mu^{k\alpha}} + Q_2 \right)^2 \frac{1}{2\mu^{3k}} \int_M^{\infty} \frac{1}{(2m-1)^2} dm\\
&= \left(\frac{Q_1}{\mu^{k\alpha}} + Q_2 \right)^2 \frac{1}{4\mu^{2k}} \frac{1}{(2M-1)}\\
&\le \frac{Q^2}{4\mu^{2k}} \left(1 + \frac{1}{\mu^{k\alpha}} \right)^2 \frac{1}{(2M-1)}, Q = max[Q_1, Q_2].
\end{aligned}
$$

Thus,

$$
E^{(1)}_{\mu^k,M}(f) \le \frac{Q}{2\mu^k} \left(1 + \frac{1}{\mu^{k\alpha}} \right) \frac{1}{(2M-1)^{\frac{1}{2}}}.
$$

Hence,

$$
E^{(1)}_{\mu^k,M}(f) = O\left(\frac{1}{(2M-1)^{\frac{1}{2}}} \frac{1}{\mu^k} \left(1 + \frac{1}{\mu^{k\alpha}} \right) \right).
$$

Thus, the Theorem (3.1)is completely established.

Proof 3.2. (I)

The error $e^{(2)}_n(x)$ between $f(x)$having $f^{''} \in Lip_\alpha[0,1]$ and its expression over any subinterval is defined as

$$\begin{aligned} e_n^{(2)}(x) &= c_{n,0}\Psi^{(\mu)}_{n,0}(x) - f(x), x \in \left[\frac{n-1}{\mu^k}, \frac{n}{\mu^k}\right) \\ ||e_n^{(2)}||_2^2 &= \int\limits_{\frac{n-1}{\mu^k}}^{\frac{n}{\mu^k}} (f(x))^2 dx - c_{n,0}^2. \end{aligned} \tag{3.6}$$

Consider,

$$\begin{aligned} \int\limits_{\frac{n-1}{\mu^k}}^{\frac{n}{\mu^k}} (f(x))^2 dx &= \int\limits_0^{\frac{1}{\mu^k}} \left[f\left(\frac{n-1}{\mu^k} + h\right)\right]^2 dh, x = \frac{n-1}{\mu^k} + h \\ &= \int\limits_0^{\frac{1}{\mu^k}} \left[f\left(\frac{n-1}{\mu^k}\right) + hf^{'}\left(\frac{n-1}{\mu^k}\right) + \frac{h^2}{2}f^{''}\left(\frac{n-1}{\mu^k} + \theta h\right)\right]^2 dh, 0 < \theta < 1 \\ &= \frac{1}{\mu^k}\left(f\left(\frac{n-1}{\mu^k}\right)\right)^2 + \frac{1}{3\mu^{3k}}\left(f^{'}\left(\frac{n-1}{\mu^k}\right)\right)^2 + \frac{1}{4}\int\limits_0^{\frac{1}{\mu^k}} h^4\left(f^{''}\left(\frac{n-1}{\mu^k} + \theta h\right)\right)^2 dh \\ &+ \frac{1}{\mu^{2k}} f\left(\frac{n-1}{\mu^k}\right) f^{'}\left(\frac{n-1}{\mu^k}\right) + f\left(\frac{n-1}{\mu^k}\right)\int\limits_0^{\frac{1}{\mu^k}} h^2 f^{''}\left(\frac{n-1}{\mu^k} + \theta h\right) dh \\ &+ f^{'}\left(\frac{n-1}{\mu^k}\right)\int\limits_0^{\frac{1}{\mu^k}} h^3 f^{''}\left(\frac{n-1}{\mu^k} + \theta h\right) dh \end{aligned} \tag{3.7}$$

$$\begin{aligned} c_{n,0} &= < f, \Psi^{(\mu)}_{n,0} > \\ &= \mu^{\frac{k}{2}}\int\limits_0^{\frac{1}{\mu^k}} f\left(\frac{n-1}{\mu^k} + h\right) dh, x = \frac{n-1}{\mu^k} + h \\ &= \mu^{\frac{k}{2}}\int\limits_0^{\frac{1}{\mu^k}} \left[f\left(\frac{n-1}{\mu^k}\right) + hf^{'}\left(\frac{n-1}{\mu^k}\right) + \frac{h^2}{2}f^{''}\left(\frac{n-1}{\mu^k} + \theta h\right)\right] dh, 0 < \theta < 1 \\ &= \mu^{\frac{k}{2}}\left[\frac{1}{\mu^k} f\left(\frac{n-1}{\mu^k}\right) + \frac{1}{2\mu^{2k}} f^{'}\left(\frac{n-1}{\mu^k}\right) + \frac{1}{2}\int\limits_0^{\frac{1}{\mu^k}} h^2 f^{''}\left(\frac{n-1}{\mu^k} + \theta h\right) dh\right]. \end{aligned}$$

Now,

$$\begin{aligned} c_{n,0}^2 &= \frac{1}{\mu^k}\left(f\left(\frac{n-1}{\mu^k}\right)\right)^2 + \frac{1}{4\mu^{3k}}\left(f^{'}\left(\frac{n-1}{\mu^k}\right)\right)^2 + \frac{\mu^k}{4}\left(\int\limits_0^{\frac{2}{\mu^k}} h^2\left(f^{''}\left(\frac{n-1}{\mu^k} + \theta h\right) dh\right)\right)^2 \\ &+ \frac{1}{\mu^{2k}} f\left(\frac{n-1}{\mu^k}\right) f^{'}\left(\frac{n-1}{\mu^k}\right) + f\left(\frac{n-1}{\mu^k}\right)\int\limits_0^{\frac{1}{\mu^k}} h^2 f^{''}\left(\frac{n-1}{\mu^k} + \theta h\right) dh \\ &+ \frac{1}{2\mu^k} f^{'}\left(\frac{n-1}{\mu^k}\right)\int\limits_0^{\frac{1}{\mu^k}} h^2 f^{''}\left(\frac{n-1}{\mu^k} + \theta h\right) dh \end{aligned} \tag{3.8}$$

By eqns (3.6), (3.7)and (3.8), we have

$$\begin{aligned}
||e_n^{(2)}||_2^2 &= \frac{1}{12\mu^{3k}}\left(f^{'}\left(\frac{n-1}{\mu^k}\right)\right)^2+\frac{1}{4}\int_0^{\frac{1}{\mu^k}} h^4\left(f^{''}\left(\frac{n-1}{\mu^k}+\theta h\right)\right)^2 dh\\
&+ f^{'}\left(\frac{n-1}{\mu^k}\right)\int_0^{\frac{1}{\mu^k}} h^3 f^{''}\left(\frac{n-1}{\mu^k}+\theta h\right)dh-\frac{\mu^k}{4}\left(\int_0^{\frac{1}{\mu^k}} h^2\left(f^{''}\left(\frac{n-1}{\mu^k}+\theta h\right)dh\right)\right)^2\\
&- \frac{1}{2\mu^k}f^{'}\left(\frac{n-1}{\mu^k}\right)\int_0^{\frac{2}{\mu^k}} h^2 f^{''}\left(\frac{n-1}{\mu^k}+\theta h\right)dh\\
&= I_1+I_2+I_3-I_4-I_5, say \qquad (3.9)\\
I_1 &= \frac{1}{12\mu^{3k}}\left(f^{'}\left(\frac{n-1}{\mu^k}\right)\right)^2\\
|I_1| &\leq \frac{R_1^2}{12\mu^{3k}}. \qquad (3.10)\\
I_2 &= \frac{1}{4}\int_0^{\frac{1}{\mu^k}} h^4\left(f^{''}\left(\frac{n-1}{\mu^k}+\theta h\right)\right)^2 dh\\
&= \frac{1}{4}\int_0^{\frac{1}{\mu^k}} h^4\left\{f^{''}\left(\frac{n-1}{\mu^k}+\theta h\right)-f^{''}\left(\frac{n-1}{\mu^k}\right)+f^{''}\left(\frac{n-1}{\mu^k}\right)\right\}^2 dh\\
|I_2| &\leq \frac{1}{4}\int_0^{\frac{1}{\mu^k}} h^4\left|f^{''}\left(\frac{n-1}{\mu^k}+\theta h\right)-f^{''}\left(\frac{n-1}{\mu^k}\right)\right|^2 dh+\frac{1}{4}\int_0^{\frac{2}{\mu^k}} h^4\left|f^{''}\left(\frac{n-1}{\mu^k}\right)\right|^2 dh\\
&+ \frac{1}{2}\int_0^{\frac{1}{\mu^k}} h^4\left|f^{''}\left(\frac{n-1}{\mu^k}+\theta h\right)-f^{''}\left(\frac{n-1}{\mu^k}\right)\right|\left|f^{''}\left(\frac{n-1}{\mu^k}\right)\right| dh\\
&\leq \frac{R_2^2}{4(2\alpha+5)\mu^{k(2\alpha+5)}}+\frac{R_3^2}{20\mu^{5k}}+\frac{R_2R_32^{\alpha}}{2(\alpha+5)\mu^{k(\alpha+5)}}. \qquad (3.11)\\
I_3 &= f^{'}\left(\frac{n-1}{\mu^k}\right)\int_0^{\frac{1}{\mu^k}} h^3 f^{''}\left(\frac{n-1}{\mu^k}+\theta h\right)dh\\
|I_3| &\leq \frac{R_1R_2}{(\alpha+4)\mu^{k(\alpha+4)}}+\frac{R_1R_3}{4\mu^{4k}} \qquad (3.12)\\
I_4 &= \frac{\mu^k}{4}\left(\int_0^{\frac{1}{\mu^k}} h^2\left(f^{''}\left(\frac{n-1}{\mu^k}+\theta h\right)dh\right)\right)^2\\
|I_4| &\leq \frac{R_2^2}{4(\alpha+3)^2 2^{k(2\alpha+5)}}+\frac{R_3^2}{36\mu^{5k}}+\frac{2R_2R_3}{12(\alpha+3)\mu^{k(\alpha+5)}}. \qquad (3.13)\\
I_5 &= \frac{1}{2\mu^k}f^{'}\left(\frac{n-1}{\mu^k}\right)\int_0^{\frac{2}{\mu^k}} h^2 f^{''}\left(\frac{n-1}{\mu^k}+\theta h\right)dh\\
|I_5| &\leq \frac{R_1R_2}{2(\alpha+3)\mu^{k(\alpha+4)}}+\frac{R_1R_3}{6\mu^{4k}}. \qquad (3.14)
\end{aligned}$$

By eqns (3.9) to (3.14), we have

$$
\begin{aligned}
||e_n^{(2)}||_2^2 &\leq \frac{R_1^2}{12\mu^{3k}} + \frac{R_2^2 2^{2\alpha}}{(2\alpha+5)\mu^{k(2\alpha+5)}} + \frac{R_3^2}{20\mu^{5k}} + \frac{R_2R_3}{2(\alpha+5)\mu^{k(\alpha+5)}} + \frac{R_1R_2}{(\alpha+4)\mu^{k(\alpha+4)}} \\
&+\frac{R_1R_3}{4\mu^{4k}} + \frac{R_2^2}{4(\alpha+3)^2\mu^{k(2\alpha+5)}} + \frac{R_3^2}{36\mu^{5k}} + \frac{2R_2R_3}{12(\alpha+3)\mu^{k(\alpha+5)}} + \frac{R_1R_2}{2(\alpha+3)\mu^{k(\alpha+4)}} \\
&+\frac{R_1R_3}{6\mu^{4k}} \\
&= \frac{R_1^2}{12\mu^{3k}} + \frac{R_2^2}{4\mu^{k(2\alpha+5)}}\left\{\frac{1}{2\alpha+5} + \frac{1}{(\alpha+3)^2}\right\} + \frac{R_3^2}{4\mu^{5k}}\left(\frac{1}{5}+\frac{1}{6}\right) \\
&+ \frac{R_1R_2}{\mu^{k(\alpha+4)}}\left(\frac{1}{\alpha+4} + \frac{1}{2(\alpha+3)}\right) + \frac{R_1R_3}{2\mu^{4k}}\left(\frac{1}{2}+\frac{1}{3}\right) \\
&+ \frac{R_2R_3}{\mu^{k(\alpha+5)}}\left(\frac{1}{(\alpha+5)} + \frac{1}{(\alpha+3)}\right) \\
&\leq \frac{1}{\mu^{3k}}\left\{R_1^2 + \frac{R_2^2}{\mu^{2k(\alpha+1)}} + \frac{R_3^2}{\mu^{2k}} + \frac{2R_1R_2}{\mu^{k(\alpha+1)}} + \frac{2R_1R_3}{\mu^k} + \frac{2R_2R_3}{\mu^{k(\alpha+2)}}\right\} \\
||e_n^{(2)}||_2^2 &\leq \frac{1}{\mu^{3k}}\left(R_1 + \frac{R_2}{\mu^{k(\alpha+1)}} + \frac{R_3}{\mu^k}\right)^2. \qquad (3.15)
\end{aligned}
$$

Next,

$$
\begin{aligned}
(E_{\mu^k,0}^{(2)}(f))^2 &= \int_0^1 \left(\sum_{n=1}^{\mu^k} e_n^{(2)}(x)\right)^2 dx \\
&= \int_0^1 \sum_{n=1}^{\mu^k} (e_n^{(2)}(x))^2 dx + 2\sum_{n=1}^{\mu^k}\sum_{n\neq n'}^{\mu^k}\int_0^1 e_n^{(2)}(x)e_{n'}^{(2)}(x)dx \\
&= \sum_{n=1}^{\mu^k}\int_0^1 (e_n^{(2)}(x))^2 dx, \text{ due to disjoint supports of } e_n \text{ and } e_n' \\
&= \sum_{n=1}^{\mu^k} ||e_n^{(2)}||_2^2 \\
&\leq \sum_{n=1}^{\mu^k}\frac{1}{\mu^{3k}}\left(R_1 + \frac{R_2}{\mu^{k(\alpha+1)}} + \frac{R_3}{\mu^k}\right)^2, \text{ by eqn (3.15)} \\
&= \mu^k\frac{1}{\mu^{3k}}\left(R_1 + \frac{R_2}{\mu^{k(\alpha+1)}} + \frac{R_3}{\mu^k}\right)^2.
\end{aligned}
$$

Then,

$$
\begin{aligned}
E_{\mu^k,0}^{(2)}(f) &\leq \frac{2R}{\mu^k}\left(1 + \frac{1}{\mu^{k(\alpha+1)}}\right), R = max[R_1,R_2,R_3] \\
&= O\left(\frac{1}{\mu^k}\left(1+\frac{1}{\mu^{k(\alpha+1)}}\right)\right).
\end{aligned}
$$

Thus,

$$
E_{\mu^k,0}^{(2)}(f) = O\left(\frac{1}{\mu^k}\left(1+\frac{1}{\mu^{k(\alpha+1)}}\right)\right).
$$

(II)

$$f(x) \quad = \quad \sum_{n=1}^{\mu^k}\sum_{m=0}^{1} c_{n,m}\Psi_{n,m}^{(\mu)}(x).$$

The error $e_n^{(3)}(x)$ between $f(x)$ having $f^{''} \in Lip_\alpha[0,1]$ and its expression over any subinterval is defined as

$$\begin{aligned} e_n^{(3)}(x) &= c_{n,0}\Psi_{n,0}^{(\mu)}(x) + c_{n,1}\Psi_{n,1}^{(\mu)} - f(x) \\ ||e_n^{(3)}||_2^2 &= \int_{\frac{n-1}{\mu^k}}^{\frac{n}{\mu^k}} (f(x))^2 dx - c_{n,0}^2 - c_{n,1}^2 \end{aligned} \tag{3.16}$$

Now, consider

$$\begin{aligned} c_{n,1} &= \sqrt{3}\mu^{\frac{k}{2}} \int_{\frac{n-1}{\mu^k}}^{\frac{n}{\mu^k}} f(x)P_1(2\mu^k x - \hat{n})dx \\ &= \sqrt{3}\mu^{\frac{k}{2}} \int_0^{\frac{1}{\mu^k}} f\left(\frac{n-1}{\mu^k} + h\right)(2\mu^k h - 1)dh \\ &= \sqrt{3}\mu^{\frac{k}{2}} \int_0^{\frac{1}{\mu^k}} \left\{ f\left(\frac{n-1}{\mu^k}\right) + hf^{'}\left(\frac{n-1}{\mu^k}\right) + \frac{h^2}{2}f^{''}\left(\frac{n-1}{\mu^k} + \theta h\right)\right\}(2\mu^k h - 1)dh \end{aligned}$$

$$c_{n,1} = \sqrt{3}\mu^{\frac{k}{2}} \left\{ \frac{1}{6\mu^{2k}} f^{'}\left(\frac{n-1}{\mu^k}\right) + \frac{1}{2}\int_0^{\frac{1}{\mu^k}} (2\mu^k h - 1)h^2 f^{''}\left(\frac{n-1}{\mu^k} + \theta h\right) \right\}. \tag{3.17}$$

By eqns (3.7), (3.8),(3.16)and (3.17),we have

$$\begin{aligned} ||e_n^{(3)}||_2^2 &= \frac{1}{4}\int_0^{\frac{1}{\mu^k}} h^2 \left(f^{''}\left(\frac{n-1}{\mu^k} + \theta h\right)dh\right)^2 - \frac{\mu^k}{4}\left(\int_0^{\frac{1}{\mu^k}} h^2 f^{''}\left(\frac{\hat{n}-1}{\mu^k} + \theta h\right)dh\right)^2 \\ &- \frac{3}{4}\mu^k \left(\int_0^{\frac{1}{\mu^k}} (2\mu^k h - 1)h^2 f^{''}\left(\frac{n-1}{\mu^k} + \theta h\right)dh\right)^2 \\ &= I_1 - I_2 - I_3, say \end{aligned} \tag{3.18}$$

$$I_1 = \frac{1}{4}\int_0^{\frac{1}{\mu^k}} h^2 \left(f^{''}\left(\frac{\hat{n}-1}{\mu^k} + \theta h\right)dh\right)^2$$

$$|I_1| \leq \frac{T_1^2}{4(2\alpha+5)\mu^{k(2\alpha+5)}} + \frac{T_2^2}{20\mu^{5k}} + \frac{T_1T_2}{2(\alpha+5)\mu^{k(\alpha+5)}}. \tag{3.19}$$

$$I_2 = \frac{\mu^k}{4}\left(\int_0^{\frac{1}{\mu^k}} h^2 f^{''}\left(\frac{n-1}{\mu^k}+\theta h\right)dh\right)^2$$

$$|I_2| \leq \frac{T_1^2}{4(\alpha+3)^2\mu^{k(2\alpha+5)}} + \frac{T_2^2}{36\mu^{5k}} + \frac{T_1T_2}{2(\alpha+5)\mu^{k(\alpha+5)}}. \quad (3.20)$$

$$I_3 = \frac{3}{4}\mu^k\left(\int_0^{\frac{1}{\mu^k}} (2\mu^k h-1)h^2 f^{''}\left(\frac{n-1}{\mu^k}+\theta h\right)dh\right)^2$$

$$|I_3| \leq \frac{27T_1^2}{4(\alpha+3)^2.\mu^{k(2\alpha+5)}} + \frac{75T_2^2}{144\mu^{5k}} + \frac{15T_1T_2}{4(\alpha+3)\mu^{k(\alpha+5)}}. \quad (3.21)$$

By eqns(3.18) to (3.21), we have

$$\begin{aligned} ||e_n^{(3)}||_2^2 &\leq \frac{29}{\mu^{5k}}\left(\frac{T_1}{\mu^{k\alpha}}+T_2\right)^2 \\ &= \frac{29T^2}{\mu^{5k}}\left(1+\frac{1}{\mu^{k\alpha}}\right)^2, T = max[T_1,T_2] \quad (3.22) \end{aligned}$$

Then,

$$\begin{aligned} (E^{(2)}_{\mu^k,1}(f))^2 &= \sum_{n=1}^{\mu^k} ||e_n^{(3)}||_2^2 \\ &\leq \frac{29T^2}{\mu^{4k}}\left(1+\frac{1}{\mu^{k\alpha}}\right)^2, by (3.22) \\ E^{(2)}_{\mu^k,1}(f) &\leq \frac{\sqrt{29}T}{\mu^{2k}}\left(1+\frac{1}{\mu^{k\alpha}}\right) \\ &= O\left(\frac{1}{\mu^{2k}}\left(1+\frac{1}{\mu^{k\alpha}}\right)\right). \end{aligned}$$

Hence,

$$E^{(2)}_{\mu^k,1}(f) = O\left(\frac{1}{\mu^{2k}}\left(1+\frac{1}{\mu^{k\alpha}}\right)\right).$$

(III)
For

$$f(x) = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} c_{n,m}\Psi^{(\mu)}_{n,m}(x).$$

Following the proof of the second part of the theorem (3.1),

$$\begin{aligned} c_{n,m} &= \frac{1}{4\sqrt{2m+1}}\frac{1}{\mu^{\frac{3k}{2}}}\int_{-1}^{1} f^{'}\left(\frac{\hat{n}+t}{2\mu^k}\right)P_{m-1}(t)-P_{m+1}(t)dt \\ &= \frac{1}{4\sqrt{2m+1}}\frac{1}{\mu^{\frac{3k}{2}}}\int_{-1}^{1} f^{'}\left(\frac{\hat{n}+t}{2\mu^k}\right)\left\{\frac{d(P_m(t)-P_{m-1}(t))}{2m-1}\right\} \\ &- \frac{1}{\sqrt{(2m+1)}}\frac{1}{\mu^{\frac{3k}{2}}}\int_{-1}^{1} f^{'}\left(\frac{\hat{n}+t}{2\mu^k}\right)\frac{d(P_{m+2}(t)-P_m(t))}{2m+3} \end{aligned}$$

$$
\begin{aligned}
c_{n,m} &= \frac{1}{8\sqrt{2m+1}}\frac{1}{\mu^{\frac{5k}{2}}}\int_{-1}^{1} f^{''}\left(\frac{\hat{n}+t}{2\mu^k}\right)\left\{\frac{(P_m(t)-P_{m-1}(t))}{2m-1}\right\}\\
&- \frac{1}{8\sqrt{2m+1}}\frac{1}{\mu^{\frac{5k}{2}}}\int_{-1}^{1} f^{''}\left(\frac{\hat{n}+t}{2\mu^k}\right)\frac{(P_{m+2}(t)-P_m(t))}{2m+3}\\
&= \frac{1}{8\sqrt{2m+1}}\frac{1}{\mu^{\frac{5k}{2}}}\int_{-1}^{1} f^{''}\left(\frac{\hat{n}+t}{2\mu^k}\right)\frac{\tau(t)}{(2m-1)(2m+3)}dt\\
\text{where}\quad \tau(t) &= (2m+3)P_{m-2}(t)-2(2m+1)P_m(t)+(2m-1)P_{m+2}(t).
\end{aligned}
$$

Then,

$$
\begin{aligned}
|c_{n,m}| &= \frac{1}{8(2m-1)(2m+3)\sqrt{2m+1}}\frac{1}{\mu^{\frac{5k}{2}}}\left|\int_{-1}^{1} f^{''}\left(\frac{\hat{n}+t}{2\mu^k}\right)\tau(t)\right| dt\\
&\leq \frac{1}{8(2m-1)(2m+3)\sqrt{2m+1}}\frac{1}{\mu^{\frac{5k}{2}}}\int_{-1}^{1}\left|f^{''}\left(\frac{\hat{n}+t}{2\mu^k}\right)\right| |\tau(t)|\, dt
\end{aligned}
$$

Consider,

$$
\begin{aligned}
\int_{-1}^{1}\left|f^{''}\left(\frac{\hat{n}+t}{2\mu^k}\right)\right||\tau(t)|\,dt &= \int_{-1}^{1}\left|f^{''}\left(\frac{\hat{n}+t}{2\mu^k}\right)-f^{''}\left(\frac{\hat{n}}{2\mu^k}\right)+f^{''}\left(\frac{\hat{n}}{2\mu^k}\right)\right||\tau(t)|\,dt\\
&\leq \int_{-1}^{1}\left|f^{''}\left(\frac{\hat{n}+t}{2\mu^k}\right)-f^{''}\left(\frac{\hat{n}}{2\mu^k}\right)\right||\tau(t)|\,dt+\int_{-1}^{1}\left|f^{''}\left(\frac{\hat{n}}{2\mu^k}\right)\right||\tau(t)|\,dt\\
&\leq \int_{-1}^{1}\left(\frac{A_1}{\mu^{k\alpha}}+A_2\right)|\tau(t)|dt\\
&\leq \left(\frac{A_1}{\mu^{k\alpha}}+A_2\right)\left(\int_{-1}^{1}1^2dt\right)^{\frac{1}{2}}\left(\int_{-1}^{1}|\tau(t)|^2dt\right)^{\frac{1}{2}}\\
&\leq \sqrt{24}\left(\frac{A_1}{\mu^{k\alpha}}+A_2\right)\frac{2m+3}{\sqrt{2m-3}}\\
|c_{n,m}| &\leq \frac{\sqrt{\frac{3}{8}}}{\mu^{\frac{5k}{2}}(2m-3)^2}\left(\frac{A_1}{\mu^{k\alpha}}+A_2\right). \qquad (3.23)
\end{aligned}
$$

Then,

$$
\begin{aligned}
(E^{(2)}_{\mu^k,M}(f))^2 &= \sum_{n=1}^{\mu^k}\sum_{m=M}^{\infty}|c_{n,m}|^2\\
&\leq \sum_{n=1}^{\mu^k}\sum_{m=M}^{\infty}\frac{3}{8\mu^{5k}}\frac{1}{(2m-3)^4}\left(\frac{A_1}{\mu^{k\alpha}}+A_2\right)^2, by(3.23)\\
&= \sum_{n=1}^{\mu^k}\left(\frac{A_1}{\mu^{k\alpha}}+A_2\right)^2\frac{3}{8\mu^{5k}}\sum_{m=M}^{\infty}\frac{1}{(2m-3)^4}
\end{aligned}
$$

$$(E^{(2)}_{\mu^k,M}(f))^2 = \frac{1}{16\mu^{4k}}\left(\frac{A_1}{\mu^{k\alpha}}+A_2\right)^2\int\limits_M^\infty \frac{dm}{(2m-3)^4}.$$

Thus,

$$\begin{aligned} E^{(2)}_{\mu^k,M}(f) &\le \frac{1}{4\mu^{2k}}\left(\frac{A_1}{\mu^{k\alpha}}+A_2\right)\frac{1}{(2m-3)^{\frac{3}{2}}} \\ &\le \frac{A}{4\mu^{2k}(2M-3)^{\frac{3}{2}}}\left(1+\frac{1}{\mu^{k\alpha}}\right), A = max[A_1, A_2]. \end{aligned}$$

Hence,

$$E^{(2)}_{\mu^k,M}(f) = O\left(\frac{1}{(2M-3)^{\frac{3}{2}}}\frac{1}{\mu^{2k}}\left(1+\frac{1}{\mu^{k\alpha}}\right)\right), M \ge 3.$$

Thus, the Theorem (3.2) is completely proved.

4 Corollaries

Following corollaries are deduced from Theorem (3.1) and (3.2)

Corollary 4.1. Let $f \in L^2[0,1)$ such that $f^{'} \in Lip_\alpha[0,1]$ and its Legendre wavelet expansion be

$$f(x) = \sum_{n=1}^{\infty}\sum_{m=0}^{\infty} c_{n,m}\Psi^{(L)}_{n,m}(x) = \sum_{n=1}^{\infty}\sum_{m=0}^{\infty} < f, \Psi^{(L)}_{n,m} > \Psi^{(L)}_{n,m}(x).$$

Then Legendre wavelet approximations satisfy:

(i)For $f(x) = \sum\limits_{n=1}^{\infty} c_{n,0}\Psi^{(L)}_{n,0}(x)$,

$E^{(1)}_{2^{k-1},0}(f) = \min\limits_{S_{2^{k-1},0}(f)} \|f - \sum\limits_{n=1}^{2^{k-1}} c_{n,0}\Psi^{(L)}_{n,0}\|_2 = O\left(\frac{1}{2^k}\left(1+\frac{1}{2^{k\alpha}}\right)\right).$

(ii)For $f(x) = \sum\limits_{n=1}^{\infty}\sum\limits_{m=0}^{\infty} c_{n,m}\Psi^{(L)}_{n,m}(x)$,

$E^{(1)}_{2^{k-1},M}(f) = \min\limits_{S_{2^{k-1},M}(f)} \|f - \sum\limits_{n=1}^{2^{k-1}}\sum\limits_{m=0}^{M-1} c_{n,m}\Psi^{(L)}_{n,m}\|_2$

$= O\left(\frac{1}{2^k(2M-1)^{\frac{1}{2}}}\left(1+\frac{1}{2^{k\alpha}}\right)\right), M \ge 2.$

Corollary 4.2. Let $f \in L^2[0,1)$ such that $f^{''} \in Lip_\alpha[0,1]$ and its extended Legendre wavelet expansion be

$$f(x) = \sum_{n=1}^{\infty}\sum_{m=0}^{\infty} c_{n,m}\Psi^{(L)}_{n,m}(x) = \sum_{n=1}^{\infty}\sum_{m=0}^{\infty} < f, \Psi^{(L)}_{n,m} > \Psi^{(L)}_{n,m}(x).$$

Then extended Legendre wavelet approximations satisfy:

(i)For $f(x) = \sum\limits_{n=1}^{\infty} c_{n,0}\Psi^{(L)}_{n,0}(x)$,

$E^{(2)}_{2^{k-1},0}(f) = \min\limits_{S_{2^{k-1},0}(f)} \|f - \sum\limits_{n=1}^{2^{k-1}} c_{n,0}\Psi^{(L)}_{n,0}\|_2 = O\left(\frac{1}{2^k}\left(1+\frac{1}{2^{k(\alpha+1)}}\right)\right).$

(ii)For $f(x) = \sum\limits_{n=1}^{\infty}\sum\limits_{m=0}^{1} c_{n,m}\Psi^{(L)}_{n,m}(x)$,

$E^{(2)}_{2^{k-1},1}(f) = \min\limits_{S_{2^{k-1},1}(f)} \|f - \sum\limits_{n=1}^{2^{k-1}}\sum\limits_{m=0}^{1} c_{n,m}\Psi^{(L)}_{n,m}\|_2 = O\left(\frac{1}{2^{2k}}\left(1+\frac{1}{2^{k\alpha}}\right)\right).$

(iii)For $f(x) = \sum\limits_{n=1}^{\infty}\sum\limits_{m=0}^{\infty} c_{n,m}\Psi_{n,m}^{(L)}(x)$,

$E_{2^{k-1},M}^{(2)}(f) = \min\limits_{S_{2^{k-1},M}(f)} \|f - \sum\limits_{n=1}^{2^{k-1}}\sum\limits_{m=0}^{M-1} c_{n,m}\Psi_{n,m}^{(L)}\|_2$

$= O\left(\frac{1}{(2M-3)^{\frac{3}{2}}}\frac{1}{2^{2k}}\left(1+\frac{1}{2^{k\alpha}}\right)\right), M \geq 3.$

Proofs of Corollaries (5.1) and (5.2)are followed by the proof of Theorem (3.1) and (3.2) respectively.

5 Conclusions

(1)The estimates of the Theorems are obtained as following

(i)$E_{\mu^k,0}^{(1)}(f) = O\left(\frac{1}{\mu^k}\left(1+\frac{1}{\mu^{k\alpha}}\right)\right).E_{\mu^k,0}^{(1)}(f) \to 0 \; as \; k \to \infty$

(ii)$E_{\mu^k,M}^{(1)}(f) = O\left(\frac{1}{(2M-1)^{\frac{1}{2}}\mu^k}\left(1+\frac{1}{\mu^{k\alpha}}\right)\right).E_{\mu^k,M}^{(1)}(f) \to 0 \; as \; k \to \infty, M \to \infty$

(iii)$E_{\mu^k,0}^{(2)}(f) = O\left(\frac{1}{\mu^k}\left(1+\frac{1}{\mu^{k(\alpha+1)}}\right)\right).E_{\mu^k,0}^{(2)}(f) \to 0 \; as \; k \to \infty$

(iv)$E_{\mu^k,1}^{(2)}(f) = O\left(\frac{1}{\mu^{2k}}\left(1+\frac{1}{\mu^{k\alpha}}\right)\right).E_{\mu^k,1}^{(2)}(f) \to 0 \; as \; k \to \infty$

(v)$E_{\mu^k,M}^{(2)}(f) = O\left(\frac{1}{(2M-3)^{\frac{3}{2}}}\frac{1}{\mu^{2k}}\left(1+\frac{1}{\mu^{k\alpha}}\right)\right).$

$E_{\mu^k,M}^{(2)}(f) \to 0 \; as \; k \to \infty, M \to \infty.$

Therefore $E_{\mu^k,0}^{(1)}(f), E_{\mu^k,M}^{(1)}(f), E_{\mu^k,0}^{(2)}(f), E_{\mu^k,1}^{(2)}(f) and E_{\mu^k,M}^{(2)}(f)$ are best possible errors of approximations in wavelet analysis.

(2) Generalised Legendre wavelet estimators of f'' belonging to $Lip_\alpha[0,1]$ is better and sharper than the estimator of f' belonging to $Lip_\alpha[0,1]$.

(3)$E_{\mu^k,0}^{(1)}(f)$ can not be obtained directly by $E_{\mu^k,M}^{(1)}(f)$ by taking $M = 0$.

(4)$E_{\mu^k,0}^{(2)}(f)$ and $E_{\mu^k,1}^{(2)}(f)$ also not obtained by $E_{\mu^k,M}^{(2)}(f)$ by taking $M = 0$ and $M = 1$ respectively. Hence,$E_{\mu^k,0}^{(1)}(f), E_{\mu^k,M}^{(1)}(f), E_{\mu^k,0}^{(2)}(f), E_{\mu^k,1}^{(2)}(f)$ and $E_{\mu^k,M}^{(2)}(f)$are estimated separately.

(5) Legendre wavelet errors of approximations $E_{2^{k-1},0}^{(1)}(f), E_{2^{k-1},M}^{(1)}(f), E_{2^{k-1},0}^{(2)}(f), E_{2^{k-1},1}^{(2)}(f)andE_{2^{k-1},M}^{(2)}(f)$ are obtained by $E_{\mu^k,0}^{(1)}(f), E_{\mu^k,M}^{(1)}(f), E_{\mu^k,0}^{(2)}(f), E_{\mu^k,1}^{(2)}(f)$and $E_{\mu^k,M}^{(2)}(f)$ respectively by taking $\mu = 2$.

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