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PRIME AND MAXIMAL FUZZY FILTERS OF ADLs

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Abstract: In this paper, we introduce the concepts of prime L-fuzzy filter and maximal L-fuzzy filter of an ADL A with truth values in a frame L. Also, we introduce the concepts of L-fuzzy prime filter and L-fuzzy maximal filter of A which are generalized than above concepts. Mainly, all prime (minimal prime, maximal) L-fuzzy filters of a given ADL A are determined by establishing a one-to-one correspondence between prime (minimal prime, maximal) L-fuzzy filters of A and the pairs (M, α), where M is a prime (minimal prime, maximal) filter of A and α is a prime element (minimal prime element, dual atom) in L.

1 Introduction

Ever since Zadeh [12] introduced the notion of a fuzzy subset of a set X as a function of X into [0, 1], and studied fuzzy substructures of many algebraic structures. J.A. Goguen [1] generalized and continued the work of L.A. Zadeh and realized that the interval $[0, 1]$ of real numbers is not sufficient to take the truth values of general fuzzy statements. Liu [2] introduced the notion of a fuzzy ideal of a ring with truth values in the interval $[0, 1]$ and T.K. Mukharjee and M.K. Sen [3] introduced the notion of a fuzzy prime ideals. However, Swamy and Swamy [10] introduced the concept of fuzzy prime ideal of a ring with truth values in a complete lattice satisfying the infinite meet distributive law.

U. M. Swamy and G. C. Rao [9] have introduced the notion of an Almost Distributive Lattice (ADL). An ADL $(A, \wedge, \vee, 0)$ satisfies all the axioms of distributive lattice, except possibly the commutativity of the operations $\wedge$ and $\vee$. It is known that, in any ADL the commutativity of $\vee$ is equivalent to that of $\wedge$ and also to the right distributivity of $\vee$ over $\wedge$. It is well known that, for any lattice $(L, \wedge, \vee)$, interchanging the operations $\wedge$ and $\vee$ again yields a lattice, known as the dual of L. An ideal of the dual $(L, \vee, \wedge)$ is known as a filter of a lattice $(L, \wedge, \vee)$. Unlike the case of a lattice, by interchanging the operations $\wedge$ and $\vee$ in an ADL $(A, \wedge, \vee, 0)$, we do not get an ADL again. Recently, Swamy, Raj and Natnael [5, 6, 7, 8] have introduced the notion of fuzzy ideals (filters) and fuzzy prime (maximal) ideals of an ADL A with the truth values in a Complete lattice $L = (L, \wedge, \vee)$ satisfying the infinite meet distributive law; that is

$$a \wedge \left(\bigvee_{s \in S} s \right) = \bigvee_{s \in S} (a \wedge s)$$

for all $a \in L$ and $S \subseteq L$ (such a lattice is called a frame).

The aim of this paper is extend the concept of fuzzy prime ideals and fuzzy maximal ideals of ADL's to the case filters of ADLs. Also, here we determine all prime (minimal prime, maximal) fuzzy filters of an ADL A by establishing a one-to-one correspondence between prime (minimal prime, maximal) L-fuzzy filters of A and the pairs (M, α), where M is a prime (minimal prime, maximal) filter of A and α is a prime element (minimal prime element, dual atom) in L.

2 Preliminaries

In this section, we recall some definitions and basic results mostly taken from [8] and [9].

Definition 2.1. An algebra $A = (A, \wedge, \vee, 0)$ of type $(2, 2, 0)$ is called an Almost Distributive Lattice (abbreviated as ADL) if it satisfies the following conditions for all a, b and $c \in A$.

(1) $0 \wedge a = 0$

(2) $a \vee 0 = a$

(3) $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$

(4) $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$

(5) $(a \vee b) \wedge c = (a \wedge c) \vee (b \wedge c)$

(6) $(a \vee b) \wedge b = b$

Any bounded below distributive lattice is an ADL. Any nonempty set X can be made into an ADL which is not a lattice by fixing an arbitrarily chosen element 0 in X and by defining the binary operations $\wedge$ and $\vee$ on X by

$$a \wedge b = \begin{cases} 0, & \text{if } a = 0 \\ b, & \text{if } a \neq 0 \end{cases} \quad \text{and} \quad a \vee b = \begin{cases} b, & \text{if } a = 0 \\ a, & \text{if } a \neq 0. \end{cases}$$

This ADL $(X, \wedge, \vee, 0)$ is called a discrete ADL.

Definition 2.2. Let $A = (A, \wedge, \vee, 0)$ be an ADL. For any a and $b \in A$, define

$$a \leq b \text{ if } a = a \wedge b \ (\Leftrightarrow a \vee b = b).$$

Then $\leq$ is a partial order on A with respect to which 0 is the smallest element in A.

An ADL $A = (A, \wedge, \vee, 0)$ is called an associative ADL if the operation $\vee$ associative; that is $(a \vee b) \vee c = a \vee (b \vee c)$ for all a, b and $c \in A$. Throughout this paper, by an ADL we mean an associative ADL only. We recall the following properties of ADL's.

Theorem 2.3. *The following hold for any a, b and c in an ADL A.*

(1) $a \wedge 0 = 0 = 0 \wedge a$ *and* $a \vee 0 = a = 0 \vee a$

(2) $a \wedge a = a = a \vee a$

(3) $a \wedge b \leq b \leq b \vee a$

(4) $a \wedge b = a \Leftrightarrow a \vee b = b$

(5) $a \wedge b = b \Leftrightarrow a \vee b = a$

(6) $(a \wedge b) \wedge c = a \wedge (b \wedge c)$ *(i.e., $\wedge$ is associative)*

(7) $a \vee (b \vee a) = a \vee b$

(8) $a \leq b \Rightarrow a \wedge b = a = b \wedge a \ \big(\Leftrightarrow a \vee b = b = b \vee a\big)$

(9) $(a \wedge b) \wedge c = (b \wedge a) \wedge c$

(10) $(a \vee b) \wedge c = (b \vee a) \wedge c$

(11) $a \wedge b = b \wedge a \Leftrightarrow a \vee b = b \vee a$

(12) $a \wedge b = \inf\{a, b\} \Leftrightarrow a \wedge b = b \wedge a \Leftrightarrow a \vee b = \sup\{a, b\}$.

An element $m \in A$ is said to be maximal if, for any $x \in A$, $m \leq x$ implies $m = x$. It can be easily observed that m is maximal if and only if $m \wedge x = x$ for all $x \in A$.

Definition 2.4. Let F be a non empty subset of an ADL A. Then F is called a filter of A if $a, b \in F \Rightarrow a \wedge b \in F$ and $x \vee a \in F$ for all $x \in A$.

As a consequence, for any filter F of A, $a \vee x \in F$ for all $a \in F$ and $x \in A$. For any $S \subseteq A$, the smallest filter of A containing S is called the filter generated by S in A and is denoted by $[S\rangle$. It is known that

$$[S\rangle = \Big\{ b \vee \Big(\bigwedge_{i=1}^{n} x_i \Big) : n \geq 0,\ x_i \in S \text{ and } b \in A \Big\}.$$

When $S = \{x\}$, we write $[x\rangle$ for $[\{x\}\rangle$. Note that, $[x\rangle = \{a \vee x \mid a \in A\}$.

Definition 2.5. An L-fuzzy subset λ of X is a mapping from X into L, where L is a frame. If L is the unit interval $[0, 1]$ of real numbers, then these are the usual fuzzy subsets of X.

Definition 2.6. Let λ be an L-fuzzy subset of A. For any $\alpha \in L$, we denote the level subset $\lambda^{-1}[\alpha, 1]$, by simply λ_α. That is;

$$\lambda_\alpha = \Big\{ x \in A : \alpha \leq \lambda(x) \Big\}.$$

λ is said to be an L-fuzzy filter of A if λ_α is a filter of A for all $\alpha \in L$.

Theorem 2.7. *Let λ be an L-fuzzy subset of A. Then the following are equivalent to each other.*
(1) *λ is an L-fuzzy filter of A*
(2) *$\lambda(m) = 1$ for all maximal elements $m \in A$ and $\lambda(x \wedge y) = \lambda(x) \wedge \lambda(y)$, for all $x, y \in A$*
(3) *$\lambda(m) = 1$ for all maximal elements $m \in A$ and $\lambda(x \vee y) \geq \lambda(x) \vee \lambda(y)$ and $\lambda(x \wedge y) \geq \lambda(x) \wedge \lambda(y)$, for all $x, y \in A$.*

Theorem 2.8. *Let λ be an L-fuzzy filter of A, S a non-empty subset of A and $x, y \in A$. Then we have the following.*
(1) $x \sim y \implies \lambda(x) = \lambda(y)$
(2) $\lambda(x \vee y) = \lambda(y \vee x)$
(3) $x \leq y \Longrightarrow \lambda(x) \leq \lambda(y)$ *(λ is an isotone mapping)*
(4) $x \in [S\rangle \Longrightarrow \lambda(x) \geq \bigwedge_{i=1}^{n} \lambda(a_i)$ *for some* $a_1, a_2, ..., a_n \in S$
(5) $x \in [y\rangle \Longrightarrow \lambda(x) \geq \lambda(y)$.

Theorem 2.9. *$\mathcal{F}_L F(A)$, the set of L-fuzzy filters of A is a complete lattice under the point-wise ordering, in which for any family $\{\lambda_i : i \in \Delta\}$ of L-fuzzy filters of A, the infimum $\bigwedge_{i \in \Delta} \lambda_i$ and supremum $\bigvee_{i \in \Delta} \lambda_i$ are given by*

$$\Big(\bigwedge_{i \in \Delta} \lambda_i \Big)(x) = \bigwedge_{i \in \Delta} \lambda_i(x)$$

$$\textit{and } \Big(\bigvee_{i \in \Delta} \lambda_i \Big)(x) = \bigvee \Big\{ \bigwedge_{a \in F} \Big(\bigvee_{i \in \Delta} \lambda_i(a) \Big) : x \in [F\rangle, F \subset\subset A \Big\}.$$

Theorem 2.10. *Let F be a proper filter of A and $\alpha \in L$. Then the L-fuzzy subset α^F of A defined by*

$$\alpha^F(x) = \begin{cases} 1 & \text{if } x \in F \\ \alpha & \text{if } x \notin F \end{cases}$$

is an L-fuzzy filter of A.

Theorem 2.11. *Let A be an ADL with a maximal element and L a frame and $1 \neq \alpha \in L$. Then*
(1) *$\alpha^F \wedge \alpha^G = \alpha^{F \cap G}$ and $\alpha^F \vee \alpha^G = \alpha^{F \vee G}$ for any M and $G \in \mathcal{F}(A)$, the lattice of filters of A*
(2) *$\{\alpha^F : F \in \mathcal{F}(A)\}$ is a sublattice of $\mathcal{F}_L F(A)$*
(3) *$F \longmapsto \alpha^F$ is an isomorphism of the lattice $\mathcal{F}(A)$ of filters of A onto the lattice of all α-level L-fuzzy filters of A corresponding to filters of A.*

3 Prime L-fuzzy filters

Let us recall from [9] that a proper filter P of an ADL A is said to be prime if for any $x, y \in A$,

$x \vee y \in P$ implies that $x \in P$ or $y \in P$;
(equivalently, for any filters G and H of A, $G \cap H \subseteq P \Rightarrow G \subseteq P$ or $H \subseteq P$).

An L-fuzzy filter λ of A is called proper if $\lambda(x) \neq 1$ for some $x \in A$.

Definition 3.1. A proper L-fuzzy filter λ of A is called a prime L-fuzzy filter of A if, for any L-fuzzy filters ν and μ of A, $\nu \wedge \mu \leq \lambda$ implies either $\nu \leq \lambda$ or $\mu \leq \lambda$.

Let us recall that an element $x \neq 1$ in a bounded lattice $(X, \wedge, \vee)$ is called prime if, for any $a, b \in X$, $a \wedge b \leq x$ implies either $a \leq x$ or $b \leq x$.

In this section, we determine all prime L-fuzzy filters of A by establishing a one-to-one correspondence between prime L-fuzzy filters and pairs (M, α), where M is a prime filter of A and α is a prime element in L.

Theorem 3.2. *Let M be a filter of an ADL A and $\alpha \in L$. Then α^M is a prime L-fuzzy filter of A if and only if M is a prime filter of A and α is a prime element in L.*

Proof. Suppose that α^M is a prime L-fuzzy filter of A. Since α^M is proper, $\alpha^M(x) \neq 1$, for some $x \in A$. Therefore $x \notin M$ and hence $M \subsetneqq A$. If G and H are filters of A such that $G \cap H \subseteq M$. Then $\alpha^G \wedge \alpha^H = \alpha^{G \cap H} \leq \alpha^M$ and hence $\alpha^G \leq \alpha^M$ or $\alpha^H \leq \alpha^M$, so that $G \subseteq M$ or $H \subseteq M$. Therefore, M is a prime filter of A. Also, for any $\gamma, \beta \in L$,
$\gamma \wedge \beta \leq \alpha \Rightarrow (\gamma \wedge \beta)^M \leq \alpha^M \Rightarrow \gamma^M \wedge \beta^M \leq \alpha^M \Rightarrow \gamma^M \leq \alpha^M$ or $\beta^M \leq \alpha^M \Rightarrow \gamma \leq \alpha$ or $\beta \leq \alpha$. Therefore, α is a prime element in L. Conversely, suppose that M is a prime filter of A and α is a prime element in L. Since M is proper and $\alpha \neq 1$, then α^M is a proper L-fuzzy filter of A. Let μ and λ be any L-fuzzy filters of A such that $\mu \nleq \alpha^M$ and $\lambda \nleq \alpha^M$. Then there exists $x, y \in A$ such that $\mu(x) \nleq \alpha^M(x)$ and $\lambda(y) \nleq \alpha^M(y)$. This implies that $\alpha^M(x) = \alpha = \alpha^M(y)$ (otherwise, $\alpha^M(x) = 1 \geq \mu(x)$ and $\alpha^M(y) = 1 \geq \lambda(y)$) and hence $x \notin M$ and $y \notin M$. Since M is a prime filter, $x \vee y \notin M$. Also, since α is prime and $\mu(x) \nleq \alpha$ and $\lambda(y) \nleq \alpha$, we have $\mu(x) \wedge \lambda(y) \nleq \alpha$. Now, $(\mu \wedge \lambda)(x \vee y) = \mu(x \vee y) \wedge \lambda(x \vee y) \geq \mu(x) \wedge \lambda(y)$ (since μ and λ are isotones) and hence $(\mu \wedge \lambda)(x \vee y) \nleq \alpha = \alpha^M(x \vee y)$, so that $(\mu \wedge \lambda) \nleq \alpha^M$. Hence, α^M is a prime L-fuzzy filter of A. □

Theorem 3.3. *A proper L-fuzzy filter λ of A is prime if and only if the following are satisfied.*
(1) λ *is two valued*
(2) $\lambda(0)$ *is a prime element in L*
(3) λ_1 *is a prime filter of A.*

Proof. Suppose that λ is a prime L-fuzzy filter of A.
(1): Suppose λ assumes more than two values. Then there exists $x, y \in A$ and $\alpha \neq \beta \in L - \{1\}$ such that $\lambda(x) = \alpha$, $\lambda(y) = \beta$ and $\lambda(m) = 1$, for any maximal m in A. Now, define L-fuzzy subsets μ and ν of A as follows:

$$\mu(z) = \begin{cases} 1 & \text{if } z \in [x\rangle \\ 0 & \text{if } z \notin [x\rangle \end{cases} \qquad \text{and} \quad \nu(z) = \begin{cases} 1 & \text{if } z \text{ is maximal} \\ \alpha & \text{otherwise.} \end{cases}$$

Then, clearly $\mu = 0^{[x\rangle}$ and $\nu = \alpha^M$, where M is the filter of all maximal elements of A and hence μ and ν are L-fuzzy filters of A. Now, for any $z \in A$

z is maximal $\Rightarrow \big(\mu \wedge \nu\big)(z) = \mu(z) \wedge \nu(z) = 1 \wedge 1 = 1 = \lambda(z)$
z is not maximal and $z \in [x\rangle \Rightarrow (\mu \wedge \nu)(z) = \mu(z) \wedge \nu(z) = 1 \wedge \alpha = \alpha = \lambda(x) \leq \lambda(z)$
(since $z \in [x\rangle \Rightarrow z = z \vee x \Rightarrow \lambda(z) = \lambda(z \vee x) \geq \lambda(z) \vee \lambda(x) \geq \lambda(x)$)
z is not maximal and $z \notin [x\rangle \Rightarrow \mu(z) \wedge \nu(z) = 0 \wedge \alpha = 0 \leq \lambda(z)$.

Therefore, $\mu \wedge \nu \leq \lambda$. Since λ is prime, $\mu \leq \lambda$ or $\nu \leq \lambda$. But $\mu \nleq \lambda$ (since $\mu(x) = 1, \lambda(x) = \alpha$ and $1 \neq \alpha$). Therefore, $\nu \leq \lambda$. In particular, $\nu(y) \leq \lambda(y) \neq \lambda(m)$, for any maximal element m in A, so that y is not maximal and hence $\alpha = \nu(y) = \beta$, which is a contradiction.

(2): Since λ is proper, then there exists $x \in A$ such that $\lambda(x) \neq 1$. Assume that $\lambda(0) = 1$. Then $1 = \lambda(0) = \lambda(x \wedge 0) \Rightarrow \lambda(x) \wedge \lambda(0) = 1 \Rightarrow \lambda(x) \wedge \lambda(m) = 1,$ for any maximal m in A
$\Rightarrow \lambda(m \wedge x) = 1 \Rightarrow \lambda(x) = 1.$
Which is a contradiction. Therefore, $\lambda(0) \neq 1$. Let α and $\beta \in L$ such that $\alpha \wedge \beta \leq \lambda(0)$. Define μ and ν of A as follows:

$$\mu(x) = \begin{cases} 1 & \text{if } x \text{ is maximal} \\ \alpha & \text{otherwise} \end{cases} \quad \text{and} \quad \nu(x) = \begin{cases} 1 & \text{if } x \text{ is maximal} \\ \beta & \text{otherwise} \end{cases}$$

Then, clearly μ and ν are L-fuzzy filters of A and $\mu \wedge \nu \leq \lambda$. Since λ is prime, $\mu \leq \lambda$ or $\nu \leq \lambda$, in particular, $\mu(0) \leq \lambda(0)$ or $\nu(0) \leq \lambda(0)$. Therefore, $\alpha \leq \lambda(0)$ or $\beta \leq \lambda(0)$ and hence $\lambda(0)$ is a prime element in L.
(3): Let $F = \{x \in A : \lambda(x) = 1\}$. Then clearly, F is a proper filter of A. Let α be the value of λ, then for any $x \in A$,

$$\lambda(x) = \begin{cases} 1 & \text{if } x \in F \\ \alpha & \text{if } x \notin F \end{cases}$$

and hence $\lambda = \alpha^F$. By theorem 3.2, F is prime. Conversely, we suppose that λ satisfying the conditions (1),(2) and (3). By (1), there exists $\alpha(\neq 1) \in L$ such that $\lambda(x) = \alpha$, for each $x \in A$ and x is not maximal. In particular, $\lambda(0) = \alpha$ and hence by (2), α is prime. Let $F = \{x \in A : \lambda(x) = 1\}$. By (3), F is a prime filter of A . Therefore, $\lambda = \alpha^F$ and hence by theorem 3.2, λ is a prime L-fuzzy filter of A. □

The theorems 3.2 and 3.3 give up the following result.

Theorem 3.4. *Let λ be an L-fuzzy subset of A. Then λ is a prime L-fuzzy filter of A if and only if there exists a prime filter M of A and a prime element α in L such that $\lambda = \alpha^M$.*

We write $\mathcal{P}(L)$, $\mathcal{P}(F(A))$ and $\mathcal{P}(\mathcal{F}_L F(A))$ for the sets of all prime elements of L, prime filters of A and prime L-fuzzy filters of A respectively. As usual, by a minimal prime element of L (minimal prime filter of A, minimal prime L-fuzzy filter of A) we mean a minimal element in the post $(\mathcal{P}(L), \leq)$ $\Big((\mathcal{P}(F(A)), \subseteq), (\mathcal{P}(\mathcal{F}_L F(A)), \leq)\Big)$.

By theorem 3.4, there is a one-to-one correspondence between $\mathcal{P}(\mathcal{F}_L F(A))$ and $\mathcal{P}(F(A)) \times \mathcal{P}(L)$. Also, the prime L-fuzzy filter of A corresponding to the pair (P, α) is α^P.

We write $m\mathcal{P}(L)$, $m\mathcal{P}(F(A))$ and $m\mathcal{P}(\mathcal{F}_L F(A))$ for the sets of all minimal prime elements of L, minimal prime filters of A and minimal prime L-fuzzy filters of A respectively.

The following theorem is straight forward verification and it establishes a one-to-one correspondence between the sets $m\mathcal{P}(\mathcal{F}_L F(A))$ and $m\mathcal{P}(F(A)) \times m\mathcal{P}(L)$.

Theorem 3.5. *Let λ be an L-fuzzy filter of A. Then λ is a minimal prime L-fuzzy filter of A if and only if $\lambda = \alpha^F$, for some minimal prime filter F of A and a minimal prime element α in L.*

4 *L*-fuzzy prime filters

In this section, we introduce the concept of an L-fuzzy prime filter of an ADL A which is weaker than that of a prime L-fuzzy filter of A.

Definition 4.1. A proper L-fuzzy filter λ of A is called an L-fuzzy prime filter of A if, for any $x, y \in A$, $\lambda(x \vee y) = \lambda(x)$ or $\lambda(y)$.

First we prove the following which gives a characterization of an L-fuzzy prime filter of A.

Theorem 4.2. *Let λ be a proper L-fuzzy filter of A. Then the following are equivalent to each other.*
(1) *for each $\alpha \in L$, $\lambda_\alpha = A$ or λ_α is a prime filter of A*

(2) *λ is an L-fuzzy prime filter of A*

(3) *for any $x, y \in A$, $\lambda(x \vee y) \leq \lambda(x) \vee \lambda(y)$, and either $\lambda(x) \leq \lambda(y)$ or $\lambda(y) \leq \lambda(x)$.*

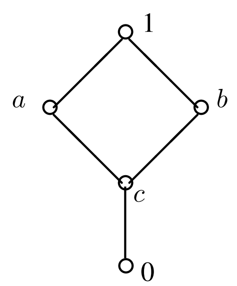
Proof. $(1) \Rightarrow (2)$: Let $x, y \in A$ and $\alpha = \lambda(x \vee y)$. Then $x \vee y \in \lambda_\alpha$ and hence $x \in \lambda_\alpha$ or $y \in \lambda_\alpha$. $x \in \lambda_\alpha \Rightarrow \lambda(x \vee y) = \alpha \leq \lambda(x) \leq \lambda(x \vee y)$ (since λ is an isotone) $\Rightarrow \lambda(x \vee y) = \lambda(x)$. Similarly, $y \in \lambda_\alpha \Rightarrow \lambda(x \vee y) = \lambda(y)$.
$(2) \Rightarrow (3)$: Let $x, y \in A$. Then, $\lambda(x \vee y) = \lambda(x)$ or $\lambda(y)$.
$\lambda(x \vee y) = \lambda(x) \Rightarrow \lambda(x \vee y) = \lambda(x) \leq \lambda(x) \vee \lambda(y)$ and $\lambda(y) \leq \lambda(x \vee y) = \lambda(x)$.
Similarly, $\lambda(x \vee y) = \lambda(y) \Rightarrow \lambda(x \vee y) \leq \lambda(x) \vee \lambda(y)$ and $\lambda(x) \leq \lambda(y)$.
$(3) \Rightarrow (1)$: Let $\alpha \in L$ be fixed. If $\lambda_\alpha \neq A$, then λ_α is a proper filter of A. Also, for any $x, y \in A$,
$x \vee y \in \lambda_\alpha \Rightarrow \alpha \leq \lambda(x \vee y) \leq \lambda(x) \vee \lambda(y) = \lambda(x)$ or $\lambda(y)$ (by (3))
$\Rightarrow \alpha \leq \lambda(x)$ or $\alpha \leq \lambda(y) \Rightarrow x \in \lambda_\alpha$ or $y \in \lambda_\alpha$
Therefore, λ_α is a prime filter of A. □

Theorem 4.3. *A prime L-fuzzy filter of A is an L-fuzzy prime filter of A.*

Proof. Let λ be a prime L-fuzzy filter of A. Then $\lambda = \alpha^F$, for some prime filter F of A and α a prime element in L. Since $\alpha < 1$, λ is a proper. Let $x, y \in A$. Then,
$x \vee y \in F \Rightarrow \lambda(x \vee y) = 1$ and $x \in F$ or $y \in F \Rightarrow \lambda(x \vee y) = 1 = \lambda(x)$ or $\lambda(y)$
and $x \vee y \notin F \Rightarrow x \notin F$ and $y \notin F \Rightarrow \lambda(x \vee y) = \alpha = \lambda(x) = \lambda(y)$.
Therefore, λ is an L-fuzzy prime filter of A. □

The converse of the above theorem is not true. For, consider the following example.

Example 4.4. Let $A = \{0, a, b, c, 1\}$ be a lattice represented by the Hasse diagram is given below:



Define $\lambda : A \to [0, 1]$ by $\lambda(a) = \lambda(1) = 1$, $\lambda(b) = \lambda(c) = 0.5$ and $\lambda(0) = 0$. Then, λ is an L-fuzzy prime filter and it satisfies the condition (2) and (3) of theorem 3.3 and λ is not two valued. Therefore, λ is not a prime L-fuzzy filter of A.

In the following theorem, we slightly generalize α-level fuzzy filters of A and identify general prime filters of A with L-fuzzy prime filters of A.

Theorem 4.5. *Let M a proper filter of A and $\alpha, \beta \in L$. Let $\langle \alpha, \beta \rangle^M$ be an $L-$fuzzy subset of A defined by*

$$\langle \alpha, \beta \rangle^M(x) = \begin{cases} 1 & \text{if } x \text{ is maximal} \\ \alpha & \text{if } x \text{ is not maximal and } x \in M \\ \beta & \text{if } x \text{ is not maximal and } x \notin M. \end{cases}$$

Then,

(1) *$\langle \alpha, \beta \rangle^M$ is an L-fuzzy filter of A if and only if $\beta \leq \alpha$ and, in this case $\langle \alpha, \beta \rangle^M$ is proper if and only if $\beta < 1$*

(2) *M is a prime filter of A if and only if the characteristic map χ_M is an L-fuzzy prime filter of A.*

Let us recall that an element a of a lattice $(X, \wedge, \vee)$ is called $\vee$-prime if, for any $x, y \in X$,

$$a \leq x \vee y \Rightarrow a \leq x \text{ or } a \leq y.$$

In this sense, the largest element 1 is called $\vee$-prime if, for any $x, y \in X$

$$x \vee y = 1 \Rightarrow x = 1 \text{ or } y = 1.$$

From [11], that an element p of an ADL $A = (A, \wedge, \vee, 0)$ is called $\vee$-irreducible if for any $a, b \in A$,

$$p = a \vee b = b \vee a \Rightarrow \text{either } p = a \text{ or } p = b.$$

Note that, for any maximal element m in A. m is $\vee$-irreducible in A if and only if m is $\vee$-prime in the lattice $[0, m]$, where $[0, m] = \{x \in A : x \leq m\}$.

Theorem 4.6. *Let A be an ADL in which every maximal element is $\vee$-irreducible and F be a proper filter of A. Then F is a prime filter of A if and only if $\langle \alpha, \beta \rangle^F$ is an L-fuzzy prime filter of A for all $1 \neq \beta \leq \alpha$ in L.*

Proof. Suppose that F is a prime filter of A and $1 \neq \beta \leq \alpha$ in L. Let $x, y \in F$. Then,
$x \vee y$ is maximal $\Rightarrow x$ or y is maximal in A
$\Rightarrow \langle \alpha, \beta \rangle^F(x \vee y) = 1 = \langle \alpha, \beta \rangle^F(x)$ or $\langle \alpha, \beta \rangle^F(y)$
$x \vee y$ is not maximal and $x \vee y \in F \Rightarrow x \in F$ or $y \in F$ and either x or y is not maximal
$\Rightarrow \langle \alpha, \beta \rangle^F(x \vee y) = \alpha = \langle \alpha, \beta \rangle^F(x)$ or $\langle \alpha, \beta \rangle^F(y)$
$x \vee y$ is not maximal and $x \vee y \notin M \Rightarrow x \notin F$ and $y \notin F$ and either x or y is not maximal
$\Rightarrow \langle \alpha, \beta \rangle^F(x \vee y) = \beta = \langle \alpha, \beta \rangle^F(x)$ or $\langle \alpha, \beta \rangle^F(y)$.
Converse follows from the fact that, $\chi_F = \langle 1, 0 \rangle^F$. □

It can be easily verified that for any L-fuzzy prime filter λ of A, λ_1 is a prime filter of A. By an L-fuzzy minimal prime filter of A we mean, as usual, a minimal element in the set of all L-fuzzy prime filters of A under the point-wise partial ordering.

Theorem 4.7. *Let λ be an L-fuzzy prime filter of A. If λ is an L-fuzzy minimal prime filter of A, then λ_1 is a minimal prime filter of A.*

Proof. Suppose that λ is an L-fuzzy minimal prime filter of A. Let M be a prime filter of A and $M \subset \lambda_1$. Then χ_M is an L-fuzzy prime filter of A and $\chi_M \lneqq \lambda$. This implies that λ is not an L-fuzzy minimal prime filter of A, which is a contradiction. Thus λ_1 is a minimal prime filter of A. □

The converse of the above theorem is not true. For in the example 4.4, if we define
$\lambda : A \to [0, 1]$ by $\lambda(a) = \lambda(1) = 1$ and $\lambda(c) = \lambda(b) = \lambda(0) = 0.5$. It can easily verified that, $\lambda_\alpha = A$ if $0 \leq \alpha \leq 0.5$ and $\lambda_\alpha = \{a, 1\}$ is a prime filter of A if $0.5 < \alpha \leq 1$. Therefore, λ is an L-fuzzy prime filter of A and $\lambda_1 = \{a, 1\}$ is a minimal prime filter of A. But λ is not an L-fuzzy minimal prime filter of A, since if we define $\nu(a) = \nu(1) = 1$ and $\nu(c) = \nu(b) = \nu(0) = 0.3$, then ν is an L-fuzzy prime filter of A and $\nu < \lambda$.

The following theorem is a characterization of L-fuzzy minimal prime filters of A.

Theorem 4.8. *Let A be an ADL in which every maximal is $\vee$-irreducible and λ be an L-fuzzy prime filter of A. Then λ is an L-fuzzy minimal prime filter of A if and only if λ_α is a minimal prime filter of A, for all $\alpha \in L$.*

Proof. Suppose λ is an L-fuzzy minimal prime filter of A and λ_α is not minimal prime filter of A, for some $0 < \alpha < 1$ in L. Then there exists a prime filter G of A such that $G \subset \lambda_\alpha$. Define $\nu : A \to L$ by

$$\nu(x) = \begin{cases} 1 & \text{if } x \text{ is maximal} \\ \alpha & \text{if } x \text{ is not maximal and } x \in G \\ 0 & \text{if } x \text{ is not maximal and } x \notin G. \end{cases}$$

Then, clearly $\nu = \langle \alpha, 0 \rangle^G$ and hence ν is an L-fuzzy prime filter of A (by theorem 4.6). Also, $\nu \leq \lambda$. Since $G \subset \lambda_\alpha$, there exists $y \in \lambda_\alpha$ such that $y \notin G$. Therefore, $\nu(y) = 0 < \alpha \leq \lambda(y)$. Therefore, $\nu \lneqq \lambda$, which is a contradiction. Thus for each $\alpha \in L$, λ_α is a minimal prime filter of A. Conversely, suppose for each $\alpha \in L$, λ_α is a minimal prime filter of A. Let ν be an L-fuzzy prime filter of A such that $\nu \leq \lambda$. Then for each $\alpha \in L$, $\nu_\alpha \subseteq \lambda_\alpha$. By the minimality of λ_α, we have $\nu_\alpha = \lambda_\alpha$ and hence $\nu = \lambda$. Therefore λ is an L-fuzzy minimal prime filter of A. □

5 Maximal L-fuzzy filters

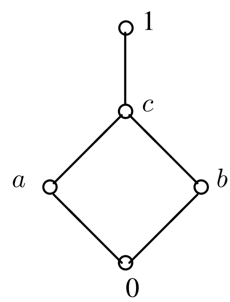
By a maximal L-fuzzy filter of an ADL A we mean a maximal element in the set of all proper L-fuzzy filter of A under the point-wise partial ordering. In ths section, we determine all maximal L-fuzzy filters of A by establishing a one-to-one correspondence between maximal L-fuzzy filters of A and the pairs (M, α), where M is a maximal filter of A and α is a dual atom of L.

Let us recall that an element $\alpha \neq 1$ in L is called a dual atom if there is no $\beta \in L$ such that $\alpha < \beta < 1$. Clearly, α is a dual atom if and only if α is a maximal element of $L - \{1\}$. In general, every maximal filter (maximal element) of any bounded distributive lattice is a prime filter (prime element). The following is an immediate consequence.

Theorem 5.1. *Every maximal L-fuzzy filter of A is a prime L-fuzzy filter of A.*

The converse of this is not true. For, consider the following example.

Example 5.2. Let $A = \{0, a, b, c, 1\}$ be a lattice represented by the Hasse diagram is given below:



Now, define $\nu : A \to [0, 1]$ by $\nu(1) = 1$ and $\nu(x) = 0.5$. Clearly, $\nu = \alpha^F$, where $\alpha = 0.5$ and $F = \{1\}$ is a prime filter of A. Therefore, ν is a prime L-fuzzy filter of A, but not maximal, since $0.5^F < 0.8^F$.

Theorem 5.3. *Let λ be an L-fuzzy subset of A. Then λ is a maximal L-fuzzy filter of A if and only if there exist a maximal filter M of A and dual atom α in L such that $\lambda = \alpha^M$.*

Proof. Suppose that λ is a maximal L-fuzzy filter of A. Then by theorem 5.1, λ is a prime L-fuzzy filter of A and hence $\lambda = \alpha^M$, for some prime filter M of A and prime element α in L. Let N be a proper filter of A containing M. Then $\lambda = \alpha^M \leq \alpha^N$ and α^N is a proper L-fuzzy filter of A. By the maximality of λ, $\alpha^M = \alpha^N$ and hence $M = N$. Thus M is a maximal filter of A. Also, if $\alpha \leq \beta < 1$ in L, then $\lambda = \alpha^M \leq \beta^M$ and β^M is a proper L-fuzzy filter of A. Again by the maximality of λ, $\alpha^M = \beta^M$ and hence $\alpha = \beta$. Thus α is a dual atom in L. Conversely suppose that M is a maximal filter of A and α is a dual atom in L such that $\lambda = \alpha^M$. Since M is proper, there exists $x \in A - M$ such that $\lambda(x) = \alpha < 1$. Therefore, λ is a proper L-fuzzy filter of A. Let ν be a proper L-fuzzy filter of A such that $\lambda \leq \nu$. Consider $N = \{x \in A : \nu(x) = 1\}$. Clearly, $N = \nu_1$ is an filter of A and

$$y \in M \Rightarrow 1 = \alpha^M(y) = \lambda(y) \leq \nu(y) \Rightarrow \nu(y) = 1 \Rightarrow y \in N.$$

Therefore, $M \subseteq N$. Also, since ν is proper, there exists $x \in A$ such that $\nu(x) \neq 1$ and hence $x \notin N$. Therefore, N is a proper filter of A. By the maximality of M, it follows that $M = N$. For any $x \in A$ with $\nu(x) = \beta < 1$, we get $x \notin N$ and $\alpha = \alpha^M(x) = \lambda(x) \leq \nu(x) = \beta$, and hence $\alpha = \beta$, since α is a dual atom. Now,

$$\begin{aligned} x \in M &\Rightarrow \lambda(x) = 1 = \nu(x) \\ \text{and } x \notin M &\Rightarrow \lambda(x) = \alpha = \beta = \nu(x). \end{aligned}$$

Thus, $\lambda = \nu$. Therefore, λ is a maximal L-fuzzy filter of A. □

6 L-fuzzy maximal filters

In this section, we introduce the concept of an L-fuzzy maximal filter of A which is weaker than that of a maximal L-fuzzy filter of A.

Definition 6.1. A proper L-fuzzy filter λ of A is called an L-fuzzy maximal filter of A if, for each $\alpha \in L$, either $\lambda_\alpha = A$ or λ_α is a maximal filter of A.

It can be easily verified that for any L-fuzzy maximal filter λ of A, λ_1 is a maximal filter of A. Mainly, in this section we give a characterization of an L-fuzzy maximal filter of A. First we prove the following.

Lemma 6.2. *Any L-fuzzy maximal filter of A attains exactly two values.*

Proof. Let λ be an L-fuzzy maximal filter of A. Suppose that λ assumes more than two values. Then there exists $x, y \in A$ and $\alpha \neq \beta$ in $L - \{1\}$ such that $\lambda(x) = \alpha$, $\lambda(y) = \beta$ and $\lambda(m) = 1$, for maximal $m \in A$. Then $x \in \lambda_\alpha$ and $x \notin \lambda_1$. Therefore, $\lambda_1 \subsetneq \lambda_\alpha \subseteq A$. By the maximality of λ_1, we get that $\lambda_\alpha = A$ and in particular, $y \in \lambda_\alpha$, so that $\alpha \leq \beta$. Similarly, $\beta \leq \alpha$. Therefore, $\alpha = \beta$. □

Theorem 6.3. *A proper L-fuzzy filter λ of A is an L-fuzzy maximal filter of A if and only if $\lambda = \alpha^M$, for some maximal filter M of A and $\alpha(\neq 1) \in L$.*

Proof. Suppose that λ is an L-fuzzy maximal filter of A. Then $M = \{x \in A : \lambda(x) = 1\}$ is a maximal filter of A. By lemma 6.2, $\lambda(x) = \alpha$, for each $x \in A - M$ and $\lambda(m) = 1$, for any maximal element m in A. This implies that $\lambda = \alpha^M$. Conversely, suppose that $\lambda = \alpha^M$, for some maximal filter M of A and $\alpha(\neq 1)$ in L. Then $M = \{x \in A : \lambda(x) = 1\}$ is a maximal filter of A. Since $\alpha < 1$ and M is proper, it follows that λ is a proper L-fuzzy filter of A. Also, for any $\beta \in L$,

$$\begin{aligned} &\beta \leq \alpha \Rightarrow A = \lambda_\alpha \subseteq \lambda_\beta \Rightarrow \lambda_\beta = A \\ \text{and } &\beta \nleq \alpha \Rightarrow \lambda_\beta \subseteq M \text{ (since, } x \notin M \Rightarrow \lambda(x) = \alpha \text{ and hence } x \notin \lambda_\beta) \\ &\Rightarrow \lambda_\beta = M \text{ (since, } M = \lambda_1 \subseteq \lambda_\beta). \end{aligned}$$

Thus $\lambda_\beta = A$ or M, for each $\beta \in L$. Therefore, λ is an L-fuzzy maximal filter of A. □

Corollary 6.4. *Let M be a proper filter of A. Then M is a maximal filter of A if and only if the characteristic map χ_M is an L-fuzzy maximal filter of A.*

Corollary 6.5. *Every maximal L-fuzzy filter of A is an L-fuzzy maximal filter of A.*

The converse of this is not true. In the example 4.4, if we define $\lambda : A \to L$ by $\lambda(x) = 1$ and $\lambda(0) = 0.5$. Clearly λ is an L-fuzzy filter of A. It can be easily verified that, $\lambda_\alpha = A$ if $0 \leq \alpha \leq 0.5$ and $\lambda_\alpha = \{a, b, c, 1\}$ is a maximal filter of A if $0.5 < \alpha \leq 1$. Therefore, λ is an L-fuzzy maximal filter of A but not maximal L-fuzzy filter of A, since $\alpha = 0.5$ is not a dual atom in L. In fact, $L = [0, 1]$ has no dual atoms.

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